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DECOMPOSITION OF VARIATIONAL MEASURE AND THE ARC-LENGTH OF A CURVE IN $\mathbb{R}^n$

Abstract

This paper discusses the decomposition of variational measures in $\mathbb{R}^n$ and, by using integral expressions of variational measure, gives an arc-length integral formula for the continuous curve in $\mathbb{R}^n$.

1 Introduction

Let $G = \{(f_1(t), f_2(t), \cdots, f_n(t)) : t \in [0, c]\}$, then G is a continuous curve whenever each f_i is a continuous function. We see that

$$L(x) = \sup \sum_j \sqrt{\sum_{i=1}^n |f_i(x_j) - f_i(x_{j-1})|^2}$$

is the arc-length of G from $t = 0$ to $t = x$, where the supremum is taken over all divisions of $[0, x]$. If $L(c) < \infty$, then we say that G is a rectifiable curve.

It is well known that G is a rectifiable curve if and only if each f_i is a bounded variation function on $[0, c]$ and the following inequality

$$L(x) \geq \int_0^x \sqrt{\sum_{i=1}^n [f_i{'}(x)]^2}\, dt$$

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holds. The equality holds if and only if each f_i is an absolutely continuous function (c.f. [6] p.122-124). We find that the curve G has a definite arc-length whenever each coordinate function f_i is a bounded variation function; but when we use the the arc-length integral formula to calculate the arc-length of G, it is required that the coordinate function f_i must be absolutely continuous. In fact, any bounded variation function often contains a singular part, but this singular part can not be expressed as a Lebesgue integral. It had been discussed and expressed with a general integral formula by M. J. Pelling in [1], however this formula can not be used in calculations of the arc length of a curve for concrete. Here we will give a better integral formula, and use it in the concrete calculation. This is achieved on the basis of the following facts:
(1) A function, being singular for some (e.g. Lebesgue) measure, may be absolutely continuous for another (e.g. Hausdorff) measure.
(2) The decomposition of the singular part may be an infinite process.
(3) There is a more clear and easier calculating formula to relate these measures than Radon-Nikodym Theorem.

2 Relations between Variational Measure and Hausdorff Measure

We assume the readers are familiar with the definition and the properties of the Hausdorff measure $\mathcal{H}^s$. If this is not the case, the details can be found in [2] and [5].

Let $[a, b]$ be a closed interval and $E \subset [a, b]$. A finite sequence of intervals $\{I_i\}$ is said to be a *cover* of E, if $\cup_i I_i \supset E$; and $\{I_i\}$ is said to be a *division* of $[a, b]$, if $I_i \cap I_j^o = \phi$ and $\cup_i I_i = [a, b]$, where E^o denotes the interior of set E. We also denote the diameter of E by $|E|$, and write $f(I) = f(v) - f(u)$, where $I = [u, v]$. Thus, the f can be regarded as an additive function of a linear interval $I \subset [a, b]$.

Let $f(x)$ be a continuous bounded variation function on $[0, c]$, write it as $f \in CBV$, and write the total variation of f on $[0, t]$ as $f^*(t) = \sup \sum_j |f(I_j)|$, where the supremum is taken over all divisions $\{I_j\}$ of $[0, t]$. Then we have $f^*(c) < \infty$ and f^* is a continuous monotone function. Let $E \subset [0, c]$, and a set function μ is defined as $\mu(E) = \inf \sum_j f^*(I_j)$, where the infimum is taken over all covers $\{I_j\}$ of E , it is easy to check that:
(1) μ is a Radon outer measure on $[0, c]$.
(2) $\mu(I) = f^*(I)$ whenever $E = I$ is a interval, so we write μ as f^* from now on.

In this paper, the outer measure and measure are regard as measure.

In this section, we always assume $f \in CBV$, $0 < s \le 1$, $E \subset 0, c]$.

The absolute upper s-derivative of f at t is defined to be

$$\overline{D}_s|f|(t) = \lim_{\delta \to 0} \sup_{t \in I, |I| < \delta} \frac{|f(I)|}{|I|^s},$$

we see that it is the absolute upper derivative of f at t whenever $s = 1$, writing it as $\overline{D}|f|(t)$. It is easy to check that $\overline{D}_s|f|$ is a Borel measurable function.

Lemma 1. *Let $f \in CBV$, then for any $\epsilon > 0$, there is $\delta > 0$ such that*

$$f^*(c) < \sum_j |f(I_j)| + \epsilon$$

whenever $\{I_j\}$ is a division of $[0, c]$ which satisfies $|I_j| < \delta$ for each j; therefore we have

$$\sum_j f^*(I_j) < \sum_j |f(I_j)| + \epsilon$$

whenever $\{I_j\}$ is a partial division of $[0, c]$ which satisfies $|I_j| < \delta$ for each j.

This is a classical conclusion, cf. [3], Chapter 8, Section 3.

Lemma 2. *Let $\lambda > 0$.*

(1) If $\overline{D}_s|f|(t) \le \lambda$ for every $t \in E$, then we have $f^*(E) \le \lambda \mathcal{H}^s(E)$;

(2) If $\overline{D}_s|f|(t) \ge \lambda$ for every $t \in E$, then we have $\mathcal{H}^s(E) \le \lambda^{-1} f^*(E)$;

(3) Let E be $\mathcal{H}^s - \sigma$ finite. If $\overline{D}_s|f|(t) = 0$ for every $t \in E$, then we have $f^*(E) = 0$;

(4) If $E_\infty = \{t \in E : \overline{D}_s|f|(t) = \infty\}$, then we have $\mathcal{H}^s(E_\infty) = 0$.

Proof. (1) Let $\eta > 0$. Since $\overline{D}_s|f|(t) \le \lambda < \lambda + \eta$ for every $t \in E$, there exists a positive function $\delta(t)$ on E, such that

$$\frac{|f(I)|}{|I|^s} < \lambda + \eta$$

for any I which satisfies $t \in I \subset (t - \delta(t), t + \delta(t))$. For $k = 1, 2, \cdots$, let

$$E_k = \{t \in E : \delta(t) \ge \frac{1}{k}\},$$

then we have $E_k \subset E_{k+1}$, $k = 1, 2, \cdots$, and $E = \cup_k E_k$.

Given $\epsilon > 0$. By Lemma 1, there is a corresponding $\delta > 0$. Let $N > \frac{1}{\delta}$, then for $k \geq N$, we have $\frac{1}{k} < \delta$. Take a $\frac{1}{k}$-cover $\{I_j\}$ of E_k, so that

$$\mathcal{H}^s(E_k) \geq \sum_j |I_j|^s - \epsilon,$$

it follows that

$$\begin{aligned}\mathcal{H}^s(E_k) + \epsilon \geq \sum_j |I_j|^s > (\lambda+\eta)^{-1} \sum_j |f(I_j)| \\ > (\lambda+\eta)^{-1}(\sum_j f^*(I_j) - \epsilon) > (\lambda+\eta)^{-1}(f^*(E_k) - \epsilon).\end{aligned}$$

Therefore, we have

$$\mathcal{H}^s(E) \geq \mathcal{H}^s(E_k) \geq (\lambda+\eta)^{-1} f^*(E_k)$$

for any $k \geq N$, by letting $\epsilon \to 0$, and then

$$\mathcal{H}^s(E) \geq (\lambda+\eta)^{-1} \lim_{k\to\infty} f^*(E_k) = (\lambda+\eta)^{-1} f^*(E),$$

hence (1) holds, by letting $\eta \to 0$.

(2) Given any $\epsilon > 0$. Since f^* is a Radon measure, there is an open set G such that $G \supset E$ and $f^*(G) < f^*(E) + \epsilon$. Let $\eta > 0$, so that $\lambda - \eta > 0$, and let

$$\mathcal{V} = \{I \subset [0,c] : I \subset G, |I|^s < (\lambda-\eta)^{-1}|f(I)|\}.$$

Since $\overline{D}_s|f|(t) > \lambda - \eta$ for every $t \in E$, we see that $\mathcal{V}$ is a Vitali covering class of E.

Let $\{I_j\} \subset \mathcal{V}$ be a non-overlapping intervals, we have

$$\begin{aligned}\sum_j |I_j|^s < \sum_j (\lambda-\eta)^{-1}|f(I_j)| \leq \sum_j (\lambda-\eta)^{-1} f^*(I_j) \\ \leq (\lambda-\eta)^{-1} f^*(G) \leq (\lambda-\eta)^{-1}[f^*(E)+\epsilon],\end{aligned}$$

it follows that

$$\mathcal{H}^s(E) < (\lambda-\eta)^{-1}[f^*(E)+\epsilon],$$

here, we have used the equivalent definitions of $\mathcal{H}^s$, see [5]. It follows, by letting $\epsilon \to 0$ and $\eta \to 0$, that

$$\mathcal{H}^s(E) < \lambda^{-1} f^*(E).$$

(3) Let $E = \cup_i E_i$, then every E_i is $\mathcal{H}^s-$ finite. Therefore, we have $f^*(E_i) \leq \lambda\mathcal{H}^s(E_i)$ for every $\lambda > 0$ by 1), it follows that $f^*(E_i) = 0$ for every i, and that $f^*(E) = 0$.

(4) Write $E_k = \{t \in E : \overline{D}_s|f|(t) > k\}$, then $E_\infty \subset \cap_k E_k$. Since 2), we have

$$\mathcal{H}^s(E_k) \leq k^{-1} f^*(E_k) \leq k^{-1} f^*(c)$$

for each k, hence $\mathcal{H}^s(E_\infty) = 0$, and the proof is complete. □

Theorem 1. *Let $f \in CBV$, $E \subset [0, c]$ be a $\mathcal{H}^s - \sigma$ finite set, and*

$$E_+ = \{t \in E : \overline{D}_s|f|(t) < \infty\},$$

then

$$f^*(E_+) = \int_E \overline{D}_s|f|\, d\mathcal{H}^s.$$

PROOF. Let $E_\infty = \{t \in E : \overline{D}_s|f|(t) = \infty\}$, we have $\mathcal{H}^s(E_\infty) = 0$ by Lemma 2(4), and we see that

$$\int_E \overline{D}_s|f|\, d\mathcal{H}^s = \int_{E_+} \overline{D}_s|f|\, d\mathcal{H}^s.$$

Let $E_0 = \{t \in E : \overline{D}_s|f|(t) = 0\}$, $1 < p < \infty$ and

$$E^{(k)} = \{t \in E : p^k \leq \overline{D}_s|f|(t) < p^{k+1}\}$$

for $k = 0, \pm 1, \pm 2, \cdots$, we see that $E_+ = E_0 \cup_{k=-\infty}^{+\infty} E^{(k)}$, and that $f^*(E_0) = 0$ by Lemma 2(3). It follows from Lemma 2(1) and 2(2) that

$$\begin{aligned} f^*(E_+) &= \sum_k f^*(E^{(k)}) \leq \sum_k p^{k+1}\mathcal{H}^s(E^{(k)}) = p \sum_k p^k \mathcal{H}^s(E^{(k)}) \\ &\leq p \sum_k \int_{E^{(k)}} \overline{D}_s|f|\, d\mathcal{H}^s = p \int_{E_+} \overline{D}_s|f|\, d\mathcal{H}^s, \end{aligned}$$

and

$$\begin{aligned} f^*(E_+) &= \sum_k f^*(E^{(k)}) \geq \sum_k p^k \mathcal{H}^s(E^{(k)}) = p^{-1} \sum_k p^{k+1} \mathcal{H}^s(E^{(k)}) \\ &\geq p^{-1} \sum_k \int_{E^{(k)}} \overline{D}_s|f|\, d\mathcal{H}^s = p^{-1} \int_{E_+} \overline{D}_s|f|\, d\mathcal{H}^s, \end{aligned}$$

respectively. By letting $p \to 1^+$, we see that

$$f^*(E_+) = \int_{E_+} \overline{D}_s |f| \, d\mathcal{H}^s = \int_E \overline{D}_s |f| \, d\mathcal{H}^s,$$

and the proof is complete. □

Remark 1. This theorem indicates that E can be decomposed into two parts E_+ and E_∞, where f^* is absolute continuous with respect to $\mathcal{H}^s$ over E_+, and is singular with respect to $\mathcal{H}^s$ over E_∞.

Theorem 2. *Let $f \in CBV$. If there exist s_k and E_k which satisfying: $1 = s_1 > s_2 > \cdots > s_q > 0$; $E_0 = [0, c]$, $E_k = \{t \in E_0 : \overline{D}_{s_k}|f|(t) = \infty\}(k = 1, 2, \cdots, q)$, and E_k is $\mathcal{H}^{s_{k+1}} - \sigma$ finite $(k = 1, 2, \cdots, q-1)$. Write $E^+_{k-1} = \{t \in E_{k-1} : \overline{D}_{s_k}|f|(t) < \infty\}$ $(k = 1,\ 2, \cdots, q)$, then we have*

$$f^*(E_0) = \sum_{k=1}^{q} f^*(E^+_{k-1}) + f^*(E_q) = \sum_{k=1}^{q} \int_{E_{k-1}} \overline{D}_{s_k}|f| \, d\mathcal{H}^{s_k} + f^*(E_q).$$

If we assume further that f^* is absolute continuous with respect to $\mathcal{H}^{s_q}$, then we have

$$f^*(E_0) = \sum_{k=1}^{q} \int_{E_{k-1}} \overline{D}_{s_k}|f| \, d\mathcal{H}^{s_k}.$$

PROOF. Noticing that if $\overline{D}_{s_k}|f|(t) = \infty$, then $\overline{D}_{s_{k-1}}|f|(t) = \infty$, so we have $E_k \subset E_{k-1}(k = 1, 2, \cdots, q)$. Since $E^+_{k-1} = E_{k-1} - E_k$, we can see that $E^+_0, E^+_1, \cdots, E^+_{q-1}$ and E_q are non-intersecting Borel sets, and also $E_0 = \bigcup_{k=1}^{q} E^+_{k-1} \cup E_q$. It follows from Theorem 1 that the first conclusion follows.

If f^* is absolute continuous with respect to $\mathcal{H}^{s_q}$, then $\mathcal{H}^{s_q}(E_q) = 0$ by the definition of E_q and Lemma 2(4), and then we have $f^*(E_q) = 0$, the second equality is proved. □

3 The Arc-Length of a Curve

Theorem 3. *Let $f_i \in CBV(i = 1, 2, \cdots, n)$. If, for each i, there exist $s_{i,k}$ and $E_{i,k}$ which satisfying: $1 = s_{i,1} > s_{i,2} > \cdots > s_{i,q(i)} > 0$; $E_{i,0} = [0, c]$, $E_{i,k} = \{t \in [0, c] : \overline{D}_{s_{i,k}}|f_i|(t) = \infty\}$ $(k = 1, 2, \cdots, q(i))$, $E_{i,k}$ is $\mathcal{H}^{s_{i,k+1}} - \sigma$ finite $(k = 1, 2, \cdots, q(i) - 1)$, and $f^*_i(E_{i,q(i)}) = 0$. Let us put the finite sequence $\{s_{i,k}\}$ in order as $1 = s_1 > s_2 > \cdots > s_q > 0$; and write $E_0 = [0, c]$, $E_j = \{t \in E_0 :$ there is f_i such that $\overline{D}_{s_j}|f_i|(t) = \infty\}(j = 1, 2, \cdots, q)$, then we have*

$$L(c) = \sum_{j=1}^{q} \int_{E_{j-1}} \sqrt{\sum_{i=1}^{n} (\overline{D}_{s_j}|f_i|)^2} \, d\mathcal{H}^{s_j}.$$

PROOF. Let $I \subset [0,c]$, write $P(I) = \sqrt{\sum_{i=1}^n (f_i(I))^2}$. By the fact that

$$L(c) = \sup \sum_j P(I_j),$$

where the supremum is taken over all divisions $\{I_j\}$ of $[0,c]$, we see that L is a total variation of P, because the relation between L and P is the same as f^* and f in the beginning of Section 2. Noticing that

$$\begin{aligned}\overline{D}_s P(t) &= \lim_{\delta\to 0} \sup_{t\in I, |I|<\delta} \frac{\sqrt{\sum_{i=1}^n (f_i(I))^2}}{|I|^s} \\ &= \lim_{\delta\to 0} \sup_{t\in I, |I|<\delta} \sqrt{\sum_{i=1}^n (\frac{f_i(I)}{|I|^s})^2} = \sqrt{\sum_{i=1}^n (\overline{D}_s |f_i|(t))^2},\end{aligned}$$

and Theorem 2, the conclusion follows. □

Here is an example about the arc-length of a curve.

(1) First, we construct three Cantor-like sets $E^{(j)} (j = 1,2,3)$. For each $j \in \{1, 2, 3\}$, by removing an open interval $\Delta_1^{(j)}$ from $[0,1]$, we obtain two closed intervals $\Delta_0^{(j)}$ and $\Delta_2^{(j)}$ which satisfying $|\Delta_0^{(j)}| = |\Delta_2^{(j)}| = \frac{1}{j+2}$; recursively, for closed intervals $\Delta_\sigma^{(j)}$, $\sigma = \epsilon_1\epsilon_2\cdots\epsilon_k$, $\epsilon_i = 0$ or 2 $(i = 1,2,\cdots,k)$, by removing an open interval $\Delta_{\sigma 1}^{(j)}$ from $\Delta_\sigma^{(j)}$, we obtain two closed intervals $\Delta_{\sigma 0}^{(j)}$ and $\Delta_{\sigma 2}^{(j)}$ which satisfying $|\Delta_{\sigma 0}^{(j)}| = |\Delta_{\sigma 2}^{(j)}| = \frac{|\Delta_\sigma^{(j)}|}{j+2}$. Let

$$E^{(j)} = \bigcap_{k=1}^{\infty} \bigcup_{\substack{\epsilon_i=0 \text{ or } 2\\ i=1,2,\cdots,k}} \Delta^{(j)}_{\epsilon_1\epsilon_2\cdots\epsilon_k},$$

$E^{(j)}$ is said to be a Cantor-like set, $E^{(1)}$ is especially the Cantor set. It is easy to check that their Hausdorff dimension is

$$\begin{aligned}\dim_{\mathcal{H}} E^{(1)} &= s_1 = \frac{\log 2}{\log 3},\\ \dim_{\mathcal{H}} E^{(2)} &= s_2 = \frac{1}{2},\\ \dim_{\mathcal{H}} E^{(3)} &= s_3 = \frac{\log 2}{\log 5}.\end{aligned}$$

For each j, define a Cantor-like function $g_j : [0,1] \to \mathbb{R}$ as follows:

$$g_j(t) = \begin{cases} \sum_{i=1}^k \epsilon_i 2^{-i-1} + 2^{-k-1}, & t \in \Delta^{(j)}_{\epsilon_1\epsilon_2\cdots\epsilon_k 1},\\ \sup\limits_{x\notin E^{(j)}, x\le t} g_j(x), & t \in E^{(j)}.\end{cases}$$

Clearly, these are continuous monotone increasing functions, $g_j(0) = 0, g_j(1) = 1$, and $\overline{D}g_j(t) = 0, t \in [0,1]\backslash E^{(j)}$.

We will compute $\overline{D}_{s_j}|g_j|(t) = \overline{D}_{s_j}g_j(t) = 1$, $t \in E^{(j)}$. Because of the same method, we will only compute $\overline{D}_{s_1}g_1(t)$, and for convenience, will omit the index $j = 1$.

Let $t \in E$. If $t \in \Delta_{\epsilon_1} \cap \Delta_{\epsilon_1\epsilon_2} \cap \cdots$, where $\epsilon_i = 0$ or 2 ($i = 1, 2, \cdots$), we write $t = 0.\epsilon_1\epsilon_2\cdots$, then $t = \sum_i \epsilon_i 3^{-i}$ and $g(t) = \sum_i \epsilon_i 2^{-i-1}$.

Since $t \in \Delta_{\epsilon_1\epsilon_2\cdots\epsilon_k}, |\Delta_{\epsilon_1\epsilon_2\cdots\epsilon_k}| = 3^{-k}$ and $g(\Delta_{\epsilon_1\epsilon_2\cdots\epsilon_k}) = 2^{-k}$, we have

$$\frac{g(\Delta_{\epsilon_1\epsilon_2\cdots\epsilon_k})}{|\Delta_{\epsilon_1\epsilon_2\cdots\epsilon_k}|^s} = \frac{2^{-k}}{(3^{-k})^{\frac{\log 2}{\log 3}}} = 1,$$

therefore $\overline{D}_s g(t) \geq 1$, $t \in E$.

In order to prove the inequality $\overline{D}_s g(t) \leq 1, t \in E$, let $t \in I = [t_1, t_2]$, we might as well assume that $t_1, t_2 \in E$, otherwise we can appropriately reduce I (this will not reduce $\frac{g(I)}{|I|^s}$), therefore $t_2 - t_1 = 0.0\cdots0\alpha_k\alpha_{k+1}\cdots$, where $\alpha_k = 2, \alpha_i = 0$ or $\pm 2, i \geq k+1$. Since $g(I) = g(t_2) - g(t_1) = \sum_{i\geq k} \alpha_i 2^{-i-1}$, $|I| = t_2 - t_1 = \sum_{i \geq k} \alpha_i 3^{-i}$, we shall only need to prove

$$(\sum_{i\geq k} \alpha_i 3^{-i})^s \geq \sum_{i \geq k} \alpha_i 2^{-i-1}. \tag{1}$$

Since the power function x^s is continuous, it suffices to show that

$$(\sum_{i=k}^{k+p} \alpha_i 3^{-i})^s \geq \sum_{i=k}^{k+p} \alpha_i 2^{-i-1} \tag{2}$$

holds for any non-negative integer p and $\sum_{i=k}^{k+p} \alpha_i 3^{-i} \geq 0$. We shall prove (2) by induction.

First, let $p = 0$, it is obvious that the inequality (2) holds when $\alpha_k = 0$; when $\alpha_k = 2$, by the fact that

$$(2 \cdot 3^{-k})^s = 2^s \cdot 2^{-k} \geq 2^{-k} = 2 \cdot 2^{-k-1},$$

the inequality (2) holds.

Next, assume the inequality (2) has been proved for $p - 1$. To obtain the inequality (2) for p, if $\alpha_k = 0$, notice that $\sum_{i=k+1}^{k+p} \alpha_i 3^{-i} = \sum_{i=k}^{k+p} \alpha_i 3^{-i} \geq 0$, the inequality (2) follows from the inequality

$$(\sum_{i=k+1}^{k+p} \alpha_i 3^{-i})^s = (3^{-1} \sum_{i=k}^{k+p-1} \alpha_{i+1} 3^{-i})^s = 2^{-1}(\sum_{i=k}^{k+p-1} \alpha_{i+1} 3^{-i})^s$$

$$\geq 2^{-1} \sum_{i=k}^{k+p-1} \alpha_{i+1} 2^{-i-1} = \sum_{i=k+1}^{k+p} \alpha_i 2^{-i-1};$$

if $\alpha_k = 2$, we need to check that

$$(2 \cdot 3^{-k} + \sum_{i=k+1}^{k+p} \alpha_i 3^{-i})^s \geq 2^{-k} + \sum_{i=k+1}^{k+p} \alpha_i 2^{-i-1}. \tag{3}$$

When $\sum_{i=k+1}^{k+p} \alpha_i 3^{-i} \geq 0$, we consider the function

$$h(t) = (2 \cdot 3^{-k} + t)^s - 2^{-k} - t^s, t \in [0, 3^{-k}].$$

Since $h'(t) < 0$, we see that $h(t)$ is decreasing on $[0, 3^{-k}]$, but $\sum_{i=k+1}^{k+p} \alpha_i 3^{-i} \leq 3^{-k}$ we have $h(\sum_{i=k+1}^{k+p} \alpha_i 3^{-i}) \geq h(3^{-k}) = 0$, the inequality (3) holds. When $\sum_{i=k+1}^{k+p} \alpha_i 3^{-i} < 0$, by considering the function

$$h(t) = (2 \cdot 3^{-k} - t)^s - 2^{-k} + t^s, t \in [0, 3^{-k}],$$

the inequality (3) can be easily proved in the same way. So the inequality (2) holds and $\overline{D}_s g(t) \leq 1, t \in E$.

By Lemma 2, we have $\mathcal{H}^s(E) = g^*([0,1]) = g(1) - g(0) = 1$.

Remark 2. Actually, the above procedure has given a method of calculating $\mathcal{H}^s(E)$.

(2) Let

$$f_1(t) = t + g_1(t) - g_2(t), f_2(t) = 2t - g_1(t) + g_3(t),$$

$$G = \{(f_1(t), f_2(t)) : t \in [0,1]\}.$$

We will calculate the arc-length of the curve which is generated by G. It suffices to check that

(a) $\overline{D}|f_1|(t) = 1$ *a.e.* on $[0,1]$;
(b) $\overline{D}|f_2|(t) = 2$ *a.e.* on $[0,1]$;
(c) $\overline{D}_{s_1}|f_1|(t) = \overline{D}_{s_1}|f_2|(t) = \overline{D}_{s_1} g_1(t)$ $\mathcal{H}^{s_1} - a.e.$ over $E^{(1)}$;
(d) $\overline{D}_{s_2}|f_1|(t) = \overline{D}_{s_2} g_2(t)$ $\mathcal{H}^{s_2} - a.e.$ over $E^{(2)}$;
(e) $\overline{D}_{s_3}|f_2|(t) = \overline{D}_{s_3} g_3(t)$ over $E^{(3)}$.

In fact, (a) and (b) are obvious. For (c), consider the equality

$$\overline{D}_{s_1}|f_1|(t) = \overline{D}_{s_1} g_1(t) \;\; \mathcal{H}^{s_1} - a.e. \; over \; E^{(1)} \tag{4}$$

first. For $I \subset [0,1]$, we clearly have

$$|f_1(I)| \leq |I| + |g_1(I)| + |g_2(I)|, \tag{5}$$

$$|g_1(I)| \leq |f_1(I)| + |I| + |g_2(I)|. \tag{6}$$

For any $t \in E^{(1)} \backslash E^{(2)}$, there is some $\Delta_{\sigma 1}^{(2)}$ such that $t \in I \subset \Delta_{\sigma 1}^{(2)}$, this gives $g_2(I) = 0$. By the fact $\dfrac{|I|}{|I|^{s_1}} \to 0(|I| \to 0)$, using (5) and (6), we obtain

$$\overline{D}_{s_1}|f_1|(t) = \overline{D}_{s_1}|g_1|(t) = \overline{D}_{s_1} g_1(t).$$

But $\mathcal{H}^{s_1}(E^{(2)}) = 0$, the equality (4) follows. The equality

$$\overline{D}_{s_1}|f_2|(t) = \overline{D}_{s_1} g_1(t) \;\; \mathcal{H}^{s_1} - a.e. \; over \; E^{(1)}$$

can be proved in the same way.

Similarly, we can prove the inequality (d) and (e). It follows that

$$\begin{aligned} L(1) &= \int_0^1 \sqrt{1+2^2}\, dt + \int_{E^{(1)}} \sqrt{1+1}\, d\mathcal{H}^{s_1} + \int_{E^{(2)}} d\mathcal{H}^{s_2} + \int_{E^{(3)}} d\mathcal{H}^{s_3} \\ &= \sqrt{5} + \sqrt{2} + 1 + 1 = 2 + \sqrt{5} + \sqrt{2}, \end{aligned}$$

which is the arc-length of the curve as required.

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