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MAXIMAL FAMILIES FOR THE CLASS OF UPPER AND LOWER SEMI-QUASICONTINUOUS FUNCTIONS

Abstract

In this article we investigate the maximal additive (multiplicative) [lattice] families for the class of real functions defined on topological spaces which are upper and lower semi-quasicontinuous at each point.

Let $\mathbb{R}$ be the set of all reals and let $(X, \mathcal{T})$ be a topological space. A function $h : X \to \mathbb{R}$ is quasicontinuous (resp. upper semi- quasicontinuous) [resp. lower semi-quasicontinuous] at a point $x \in X$ if for every positive real ε and for every set $U \in \mathcal{T}$ containing x there is a nonempty open set $V \subset U$ such that $h(V) \subset (h(x) - \varepsilon, h(x) + \varepsilon)$ (resp. $h(V) \subset (-\infty, h(x) + \varepsilon)$), [resp. $h(V) \subset (h(x) - \varepsilon, \infty)$], ([2,3]).

Let $Q(X)$ denote the family of all quasicontinuous functions $f : X \to \mathbb{R}$ and let $\mathcal{E}(X, x)$ (resp. $\mathcal{E}(X)$) be the family of all functions $g : X \to \mathbb{R}$ which are upper and lower semi-quasicontinuous at the point $x \in X$ (resp. at each point $t \in X$). It is obviously that if $f \in \mathcal{E}(X, x)$ then $-f \in \mathcal{E}(X, x)$.

Observe that for $X = \mathbb{R}$ with the Euclidean topology the family $Q(\mathbb{R})$ of all quasicontinuous functions $g : \mathbb{R} \to \mathbb{R}$ is a nowhere dense subset in the space $\mathcal{E}(\mathbb{R})$ with the metric $\rho_C(g, h) = \min(1, \sup_{x \in \mathbb{R}} |g(x) - h(x)|)$ of the uniform convergence ([4]).

Denote by $D(f)$ the set of all discontinuity points of a function $f : X \to \mathbb{R}$, by $C(f)$ the set of all continuity points of $f : X \to \mathbb{R}$ and by $C(X)$ the family of all continuous real functions on X.

Moreover the symbol $\mathbb{R}$ denotes the topological space with the Euclidean topology and nonempty subsets $A \subset \mathbb{R}$ are considered as topological subspaces of $\mathbb{R}$.

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1 Maximal Families

If Φ is a nonempty class of functions $f : X \to \mathbb{R}$, then

$$\mathrm{Max}_{\mathrm{add}}(\Phi) = \{g : X \to \mathbb{R}; \forall_{f\in\Phi} f + g \in \Phi\}$$

is called the maximal additive family for Φ,

$$\mathrm{Max}_{\mathrm{mult}}(\Phi) = \{g : X \to \mathbb{R}; \forall_{f\in\Phi} fg \in \Phi\}$$

is called the maximal multiplicative family for Φ,

$$\mathrm{Max}_{\mathrm{max}}(\Phi) = \{g : X \to \mathbb{R}; \forall_{f\in\Phi} \max(f, g) \in \Phi\}$$

is called the maximal family for Φ with respect to max; and

$$\mathrm{Max}_{\mathrm{min}}(\Phi) = \{g : X \to \mathbb{R}; \forall_{f\in\Phi} \min(f, g) \in \Phi\}$$

is called the maximal family for Φ with respect to min.

Let $\Phi = Q(X)$. In [3] it was proved that:

1. $\mathrm{Max}_{\mathrm{add}}(Q(X)) = C(X)$,
2. $\mathrm{Max}_{\mathrm{max}}(Q(X)) = \mathrm{Max}_{\mathrm{min}}(Q(X)) = C(X)$ and
3. if $f \in Q(X)$ is such that $f(x) \neq 0$ for every $x \in X$ then $\frac{1}{f} \in Q(X)$.

The concept of the family $\mathrm{Max}_{\mathrm{mult}}(Q(X))$ is more complicated. If

$$N(Q) = \{f \in Q(X); \text{if } x \in D(f) \text{ then } f(x) = 0 \text{ and } x \in cl(C(f) \cap f^{-1}(0))\}$$

then $\mathrm{Max}_{\mathrm{mult}}(Q(X)) = N(Q)$ for the complete metric space X,([2,3]).

In [2] it was also observed that if X is a topological space and a function $f : X \to \mathbb{R}$ is quasicontinuous at a point $x \in X$, then for every function $g : X \to \mathbb{R}$ which is continuous at x, the product fg is quasicontinuous at x. This last remark and above description are not true for $\mathcal{E}(X)$, (see example 1). The theorems in the last part of this paper are an attempt to describe the family $\mathrm{Max}_{\mathrm{mult}}(\mathcal{E}(X))$.

Remark 1. If $f \in \mathcal{E}(X)$, then there is the function $g \in C(X)$ such that the composition $g \circ f \notin \mathcal{E}(X)$.

For example, let $f \in \mathcal{E}(\mathbb{R})$ be such that

$$f(x) = \begin{cases} \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and let $g \in C(\mathbb{R})$ be $g(x) = |x|$. Then $g \circ f \notin \mathcal{E}(\mathbb{R}, 0)$.

2 The Results

Let $\Phi = \mathcal{E}(X)$, where $(X, \mathcal{T})$ is an arbitrary topological space.

Remark 2. The inclusions

$$C(X) \subset \mathrm{Max}_{\mathrm{add}}(\mathcal{E}(X)) \cap \mathrm{Max}_{\mathrm{max}}(\mathcal{E}(X)) \cap \mathrm{Max}_{\mathrm{min}}(\mathcal{E}(X)) \subset \mathcal{E}(X)$$

are true.

PROOF. If we prove five inclusions:

(i) $C(X) \subset \mathrm{Max}_{\mathrm{add}}(\mathcal{E}(X))$;

(ii) $\mathrm{Max}_{\mathrm{max}}(\mathcal{E}(X)) \subset \mathcal{E}(X)$;

(iii) $\mathrm{Max}_{\mathrm{min}}(\mathcal{E}(X)) \subset \mathcal{E}(X)$;

(iv) $C(X) \subset \mathrm{Max}_{\mathrm{max}}(\mathcal{E}(X)) \cap \mathrm{Max}_{\mathrm{min}}(\mathcal{E}(X))$;

(v) $\mathrm{Max}_{\mathrm{add}}(\mathcal{E}(X)) \subset \mathcal{E}(X)$,

then the proof will be complete.
PROOF OF (i). Let $f \in C(X)$, $g \in \mathcal{E}(X)$, $x \in X$, $U \in \mathcal{T}$ with $x \in U$ and let $\varepsilon > 0$. From the continuity of f at x it follows that there is an open set $W \subset U$ containing x such that $f(W) \subset (f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2})$. Since $g \in \mathcal{E}(X)$, there are nonempty open sets V, $S \subset W$ such that

$$g(V) \subset (-\infty, g(x) + \frac{\varepsilon}{2}) \text{ and } g(S) \subset (g(x) - \frac{\varepsilon}{2}, \infty).$$

Then for $u \in V$ and $v \in S$ we have

$$f(u) + g(u) < f(x) + \frac{\varepsilon}{2} + g(x) + \frac{\varepsilon}{2} = f(x) + g(x) + \varepsilon,$$

and

$$f(v) + g(v) > f(x) - \frac{\varepsilon}{2} + g(x) - \frac{\varepsilon}{2} = f(x) + g(x) - \varepsilon$$

Hence $f + g \in \mathcal{E}(X)$ and $f \in \mathrm{Max}_{\mathrm{add}}(\mathcal{E}(X))$.
PROOF OF (ii). Now, fix $f \in \mathrm{Max}_{\mathrm{max}}(\mathcal{E}(X))$, $x \in X$, $U \in \mathcal{T}$ with $x \in U$ and $\varepsilon > 0$. Then the function $h = \max(f, f(x) - 2\varepsilon) \in \mathcal{E}(X)$. Since $h(x) = f(x)$, there is a nonempty open set $W \subset U$ with

$$h(W) \subset (h(x) - \varepsilon, \infty) = (f(x) - \varepsilon, \infty).$$

So, $f(W) = h(W) \subset (f(x) - \varepsilon, \infty)$.

Similarly, since $h(x) = f(x)$, there is a nonempty open set $V \subset U$ such that

$$h(V) \subset (-\infty, h(x)+\varepsilon) = (-\infty, f(x)+\varepsilon).$$

Thus $f(V) \subset (-\infty, f(x)+\varepsilon)$. So, $f \in \mathcal{E}(X)$ and $\mathrm{Max}_{\max}(\mathcal{E}(X)) \subset \mathcal{E}(X)$.
PROOF OF (iii). From the equality $\max(-f, g) = -\min(f, -g)$ it follows that if $f \in \mathrm{Max}_{\min}(\mathcal{E}(X))$, then $-f \in \mathrm{Max}_{\max}(\mathcal{E}(X)) \subset \mathcal{E}(X)$ and consequently $f \in \mathcal{E}(X)$.
PROOF OF (iv). Let $f \in C(X)$, $g \in \mathcal{E}(X)$, $x \in X$, $U \in \mathcal{T}$ with $x \in U$ and $\varepsilon > 0$. Let $h = \max(f, g)$. Suppose that $W \subset U$ is an open set such that $x \in W$ and $f(W) \subset (f(x)-\varepsilon, f(x)+\varepsilon)$. Then, for $u \in W$ we have

$$h(u) \geq f(u) \geq f(x) - \varepsilon \text{ for } u \in W.$$

So, if $f(x) \geq g(x)$, then $h(W) \subset (h(x)-\varepsilon, \infty)$. If $f(x) < g(x)$, then, by the relation $g \in \mathcal{E}(X)$, there is a nonempty open set $V \subset W$ with $g(V) \subset (g(x)-\varepsilon, \infty)$. Hence $h(V) \subset (h(x)-\varepsilon, \infty)$. Now, let $S \subset W$ be a nonempty open set with $g(S) \subset (-\infty, g(x)+\varepsilon)$. So, on the set S we have

$$f(u) < f(x)+\varepsilon \leq h(x)+\varepsilon \text{ and } g(u) < g(x)+\varepsilon \leq h(x)+\varepsilon.$$

Therefore $h(S) \subset (-\infty, h(x)+\varepsilon)$.

For the proof of the inclusion $C(X) \subset \mathrm{Max}_{\min}(\mathcal{E}(X))$ fix $f \in C(X)$ and $g \in \mathcal{E}(X)$ and observe that $\min(f, g) = -\max(-f, -g) \in \mathcal{E}(X)$.
PROOF OF (v). Since the function $0 \in \mathcal{E}(X), \mathrm{Max}_{\mathrm{add}}(\mathcal{E}(X)) \subset \mathcal{E}(X)$.
This completes the proof of Remark 2. □

Theorem 1. *The equality* $C(X) = \mathrm{Max}_{\mathrm{add}}(\mathcal{E}(X))$ *holds.*

PROOF. Let $f \in \mathcal{E}(X) \setminus C(X)$ and let $x \in D(f)$. Then there is a positive real ε such that

$$x \in \mathrm{cl}(\mathrm{int}\{u; f(u) > f(x)+\varepsilon\}) \text{ or } x \in \mathrm{cl}(\mathrm{int}\{u; f(u) < f(x)-\varepsilon\}),$$

where cl and int denote the closure and the interior operations respectively.

Suppose that $x \in \mathrm{cl}(\mathrm{int}\{u; f(u) > f(x)+\varepsilon\})$. In the other case the reasoning is analogous. The function

$$g(u) = \begin{cases} f(u) & \text{if } f(u) \leq f(x)+\varepsilon \text{ and } u \neq x \\ f(x)+\varepsilon & \text{if } f(u) > f(x)+\varepsilon \text{ or } \ u = x \end{cases}$$

belongs to $\mathcal{E}(X)$. Observe that the function $-g \in \mathcal{E}(X)$ but the sum $f + (-g)$ does not belong to $\mathcal{E}(X)$ because

$$f(u) - g(u) \begin{cases} = 0 & \text{if } f(u) \leq f(x)+\varepsilon \text{ and } u \neq x \\ > 0 & \text{if } f(u) > f(x)+\varepsilon \text{ and } u \neq x \\ = -\varepsilon & \text{if } u = x. \end{cases}$$

Thus f does not belong to $\mathrm{Max}_{\mathrm{add}}(\mathcal{E}(X))$ and $\mathrm{Max}_{\mathrm{add}}(\mathcal{E}(X)) \subset C(X)$. Consequently, because we have (i) in the proof of Remark 2, $\mathrm{Max}_{\mathrm{add}}(\mathcal{E}(X)) = C(X)$ and the proof of Theorem 1 is complete. □

Theorem 2. *The equalities* $C(X) = \mathrm{Max}_{\max}(\mathcal{E}(X)) = \mathrm{Max}_{\min}(\mathcal{E}(X))$ *hold.*

PROOF. Let $f \in \mathcal{E}(X) \setminus C(X)$ be a function and let $x \in X$ be a point belonging to $D(f)$. Fix $\varepsilon > 0$ and let $U = \mathrm{int}\{u; f(u) > f(x) + \varepsilon\}$ be such that $x \in \mathrm{cl}\, U$. Observe that the function

$$g(u) = f(x) + \varepsilon \text{ for } u \in X \setminus \mathrm{cl} \text{ and } g(u) = f(x) - \varepsilon \text{ on } \mathrm{cl}\, U$$

belongs to $\mathcal{E}(X)$. Moreover

$$\max(f(u), g(u)) \geq f(x) + \varepsilon \text{ for } u \in X \setminus (\mathrm{cl}\, U \cap \{u; f(u) < f(x) + \varepsilon\})$$

$$\text{and } \max(f(u), g(u)) < f(x) + \varepsilon \text{ for } u \in N = \mathrm{cl}\, U \cap \{u; f(u) < f(x) + \varepsilon\},$$

where N is nowhere dense in X. In particular $\max(f(x), g(x)) = f(x)$; so, $\max(f, g) \notin \mathcal{E}(X, x)$ and consequently $\max(f, g) \notin \mathcal{E}(X)$.

In the opposite case, if we consider $V = \mathrm{int}\{u; f(u) \leq f(x) + \varepsilon\}$ such that $x \in \mathrm{cl}\, V$, the reasoning will be analogous. From these cases follows that $\mathrm{Max}_{\max}(\mathcal{E}(X)) \subset C(X)$ and, because we have (iv) in the proof of Remark 2, $\mathrm{Max}_{\max}(\mathcal{E}(X)) = C(X)$.

For the proof of the inclusion $\mathrm{Max}_{\min}(\mathcal{E}(X)) \subset C(X)$ observe that if $f \in \mathrm{Max}_{\min}(\mathcal{E}(X))$, then $-f \in \mathrm{Max}_{\max}(\mathcal{E}(X)) = C(X)$ and consequently $f \in C(X)$. The proof of theorem 2 is complete. □

For the investigation of the class $\mathrm{Max}_{\mathrm{mult}}(\mathcal{E}(X))$ we first consider the following example.

Example 1. Let $g(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{othervise on } \mathbb{R} \end{cases}$ and let $f(x) = x$ for $x \in \mathbb{R}$. Then $f \in C(\mathbb{R})$ and $g \in \mathcal{E}(\mathbb{R})$, but the product $fg \notin \mathcal{E}(\mathbb{R})$, since $f(0)g(0) = 0$ and $f(x)g(x) = 1$ for each $x \neq 0$.

Theorem 3. *If* $f, g \in \mathcal{E}(X)$, *if* $x \in C(f)$ *and if* $f(x) \neq 0$, *then* $fg \in \mathcal{E}(X, x)$.

PROOF. Let $U \in \mathcal{T}$ be a nonempty set containing a point x and $\varepsilon > 0$. We will consider the following cases:

(1) $f(x) > 0$ and $g(x) > 0$. Let $r > 0$ be a real such that

$$f(x) - r > 0 \text{ and } 0 < (f(x) + g(x))r - r^2 < \varepsilon, \text{ and } r^2 + (f(x) + g(x))r < \varepsilon.$$

There are a nonempty open set $P \subset U$ containing x such that $f(P) \subset (f(x) - r, f(x) + r))$ and a nonempty open set $V \subset P$ such that

$$g(u) > \max(g(x) - r, \frac{g(x)}{2}) \text{ for } u \in V.$$

Then for $u \in V$ we have

$$f(u)g(u) > (f(x)-r)(g(x)-r) = f(x)g(x)+r^2-(f(x)+g(x))r > f(x)g(x)-\varepsilon.$$

If there is a point $t \in P$ such that $g(t) < 0$, then there is a nonempty open set $W \subset P$ with $g(W) \subset (-\infty, 0)$ and consequently, $f(u)g(u) < 0 < f(x)g(x) + \varepsilon$ for $u \in W$. If $g(u) \geq 0$ for each point $u \in P$ then there is a nonempty open set $W \subset P$ such that $g(u) < g(x) + r$ for $u \in W$. Consequently, for $u \in W$ we have

$$f(u)g(u) < (f(x)+r)(g(x)+r) = f(x)g(x)+r^2+(f(x)+g(x))r < f(x)g(x)+\varepsilon.$$

(2) $f(x) < 0$ and $g(x) < 0$. This case may be reduced to case (1), since

$$fg = (-f)(-g) \text{ and } -f, -g \in \mathcal{E}(X).$$

(3) $f(x) < 0$ and $g(x) > 0$. This case may be reduced to case (1) since

$$-f(x) > 0, g(x) > 0, \text{ and } -f, g \in \mathcal{E}(X) \text{ imply}$$

$$-fg \in \mathcal{E}(X, x) \text{ and so } fg \in \mathcal{E}(X, x).$$

(4) $f(x) > 0$ and $g(x) < 0$. As in case (2) this case may be reduced to case (3).

(5) $f(x) > 0$ and $g(x) = 0$. If there is a nonempty open set $P \subset U \cap g^{-1}(0)$, then the proof is evident. So, we can suppose that $\operatorname{int}(U \cap g^{-1}(0)) = \emptyset$. Then let $P \subset U$ be an open nonempty set containing x such that $f(P) \subset (f(x) - r, f(x) + r)$, where $r > 0$ is a real such that $f(x) - r > 0$ and $(f(x) + r)r < \varepsilon$. If there is a point $t \in P$ with $g(t) < 0$, then there is a nonempty open set $W \subset P$ such that $g(W) \subset (-\infty, 0)$, and consequently for $u \in W$ we have

$$f(u)g(u) < 0 < f(x)g(x) + \varepsilon.$$

If $g(t) \geq 0$ for each point $t \in P$, then there is a nonempty open set $W \subset P$ such that $g(u) < r$ for $u \in W$. So, for $u \in W$ we obtain

$$f(u)g(u) < (f(x) + r)r < \varepsilon = f(x)g(x) + \varepsilon.$$

If there is a point $t \in P$ with $g(t) > 0$, then there is a nonempty open set $V \subset P$ such that $g(u) > 0$ for each point $u \in V$. Then for $u \in V$ we obtain

$$f(u)g(u) > 0 > -\varepsilon = f(x)g(x) - \varepsilon.$$

If $g(u) \leq 0$ for each point $u \in P$, then, by the equality $g(x) = 0$, there is a nonempty open set $V \subset P$ such that $g(u) > -r$ for $u \in V$. Then for each $u \in V$ we obtain

$$f(u)g(u) > -(f(x) + r)r > -\varepsilon = f(x)g(x) - \varepsilon.$$

(6) $f(x) < 0$ and $g(x) = 0$. This case can be reduced to case (5).
This completes the proof of Theorem 3. □

Theorem 4. *If $f : X \to \mathbb{R}$ and if $x \in X$ are such that $f(x) = 0$ and $x \in \mathrm{cl}(\mathrm{int}(f^{-1}(0)))$, then for each function $g : X \to \mathbb{R}$ the product $fg \in \mathcal{E}(X, x)$.*

PROOF. This theorem is evident.

Lemma 1. *If $f \in \mathcal{E}(X)$ and $f(u) > 0$ for all $u \in X$, then $\frac{1}{f} \in \mathcal{E}(X)$.*

PROOF. Fix a point $y \in X$, a set $U \in \mathcal{T}$ containing y and a positive real ε. Since $f \in \mathcal{E}(X, y)$ and $f(y) > \frac{f(y)}{1+\varepsilon f(y)}$ there is a nonempty set $W \subset U$ such that $f(u) > \frac{f(y)}{1+\varepsilon f(y)}$ for all $u \in W$. Consequently, for $u \in W$, we have

$$\frac{1}{f(u)} < \frac{1 + \varepsilon f(y)}{f(y)} = \frac{1}{f(y)} + \varepsilon.$$

So, the upper semi-quasicontinuous of $\frac{1}{f}$ is proved.

Now, let $\varepsilon_0 = \min(\varepsilon, \frac{1}{2f(y)})$. Since $f \in \mathcal{E}(X, y)$ and $f(y) < \frac{f(y)}{1-\varepsilon_0 f(y)}$, there is a nonempty set $V \subset U$ such that $f(u) < \frac{f(y)}{1-\varepsilon_0 f(y)}$ for all $u \in V$. Consequently, $\frac{1}{f}$ is lower semi-quasicontinuous, since

$$\frac{1}{f(u)} > \frac{1 - \varepsilon_0 f(y)}{f(y)} = \frac{1}{f(y)} - \varepsilon_0 \geq \frac{1}{f(y)} - \varepsilon \text{ for all } u \in V.$$

From $\frac{1}{f} \in \mathcal{E}(X, y)$ it follows that $\frac{1}{f} \in \mathcal{E}(X)$ and this completes the proof of Lemma 1. □

Remark 3. If $f \in \mathcal{E}(X)$ and $f(u) < 0$ for all $u \in X$, then $\frac{1}{f} \in \mathcal{E}(X)$.

PROOF. Apply Lemma 1 to $-f$.

Remark 4. There exists a function $f \in \mathcal{E}(X)$ such that if $f(u) > 0$ for some $u \in X$ and $f(v) < 0$ for some $v \in X$, then $\frac{1}{f} \notin \mathcal{E}(X)$.

Example 2. Let

$$f(x) = \begin{cases} 2 & \text{for } x \in (0, \infty) \\ 1 & \text{for } x = 0 \\ -1 & \text{for } x \in (-\infty, 0). \end{cases}$$

Then $f \in \mathcal{E}(X)$ but $\frac{1}{f} \notin \mathcal{E}(X)$.

Theorem 5. *If $f \in \mathcal{E}(X)$ and a point $x \in X$ are such that $f(x) = 0$, $f(u) \neq 0$ for $u \neq x$ and $x \in \mathrm{cl}(f^{-1}((-\infty, 0)))$, and $x \in \mathrm{cl}(f^{-1}((0, \infty)))$, then there is a function $g \in \mathcal{E}(X)$ such that $fg \notin \mathcal{E}(X)$.*

PROOF. Since the function $h(u) = \frac{1}{f(u)}$ for $u \in X \setminus \{x\}$ belongs to $\mathcal{E}(X \setminus \{x\})$, the function $g(x) = 0$ and $g(u) = h(u)$ for $u \neq x$ belongs to $\mathcal{E}(X)$. But the product $f(x)g(x) = 0$ and $f(u)g(u) = 1$ for $u \neq x$, is not in $\mathcal{E}(X)$. This completes the proof. □

Theorem 6. *If a function $f \in \mathcal{E}(X)$ is discontinuous at a point $x \in X$ and $f(x) \neq 0$, then there is a function $g \in \mathcal{E}(X)$ such that the product $fg \notin \mathcal{E}(X)$.*

PROOF. Suppose that $f(x) > 0$. If $f(x) < 0$, then we can consider the function $-f$. In this situation we have three cases:

(1) $l = \liminf f(x) < f(x) < \limsup f(x) = d$.

Let $U \in \mathcal{T}$ be a nonempty set containing x and let a be a real such that $l < f(x) < a < d$. Denote $V = \mathrm{int}\{u \in U; f(u) < a\}$ and suppose that $x \in \mathrm{cl}\, V$. Of course, because $f \in \mathcal{E}(X)$, such a choice is possible.

Let $g : X \to \mathbb{R}$ be a function such that

$$g(t) = \begin{cases} 1 & \text{for } t \in \mathrm{cl}\, V \\ 0 & \text{for } t \in X \setminus \mathrm{cl}\, V. \end{cases}$$

In particular $g(x) = 1$. Observe that $g \in Q(X)$; so $g \in \mathcal{E}(X)$. Moreover, the product $fg \notin \mathcal{E}(X, x)$ because

$$(fg)(t) \begin{cases} = f(x) < a & \text{for } t = x \\ = 0 < a & \text{for } t \in X \setminus \mathrm{cl}\, V \\ < a & \text{for } t \in \{u \in U; f(u) < a\}, \end{cases}$$

$$\text{but} \qquad (fg)(t) > a \text{ for } t \in \mathrm{cl}\, V \setminus \{u \in U; f(u) < a\},$$

where $\operatorname{cl} V \setminus \{u \in U; f(u) < a\}$ is nowhere dense in X, so $fg \notin \mathcal{E}(X,t)$ for $t \in \operatorname{cl} V \setminus \{u \in U; f(u) < a\}$.

(2) $l = \liminf f(x) = f(x) < \limsup f(x) = d$.

Fix reals a, b such that $0 < \frac{f(x)}{2} < a < l = f(x) < b < d$. Let $U \in \mathcal{T}$ be a nonempty set containing x such that $f(U) \subset (a, \infty)$ and $V = \operatorname{int}(\operatorname{cl} U)$). Observe that $f(V) \subset [a, \infty)$. In addition, since $U \subset V$, the set V is a neighborhood of x and if $u \in X \setminus V$, then $u \in \operatorname{cl}(\operatorname{int}(X \setminus V))$.

Let $A = \{u \in V; f(u) \geq b\}$. Then obviously $x \in \operatorname{cl}(\operatorname{int} A))$. Put

$$g(u) = \begin{cases} -1 & \text{for } u \in V \cap clA \\ -\frac{2f(x)}{f(u)} & \text{for } u \in V \setminus clA \\ 0 & \text{for } u \in X \setminus V. \end{cases}$$

Observe that $g \in \mathcal{E}(X,u)$ for $u \in (X \setminus V) \cup (V \cap \operatorname{cl} A)$ because g is quasicontinuous at this points. From Lemma 1 it follows that $g \in \mathcal{E}(X,u)$ for each $u \in V \setminus \operatorname{cl} A$. But

$$(fg)(u) \begin{cases} = -f(x) & \text{for } u = x \\ \leq -b < -f(x) & \text{for } u \in A \\ = -2f(x) < -f(x) & \text{for } u \in V \setminus \operatorname{cl} A \\ = 0 & \text{for } u \in X \setminus V, \end{cases}$$

and $(fg)(u) \in \mathbb{R}$ for $u \in N = (\operatorname{cl} A) \setminus A$, where N is nowhere dense in X, which means that $fg \notin \mathcal{E}(X,x)$.

(3) $l = \liminf f(x) < f(x) = \limsup f(x) = d$.

Fix reals a, b such that $l < \frac{f(x)}{2} < a < f(x) = d < b$. Let $U \in \mathcal{T}$ be a nonempty set containing x such that $f(U) \subset (-\infty, b)$. If $V = \operatorname{int}(\operatorname{cl} U)$, then $f(V) \subset (-\infty, b]$ and in addition, since $U \subset V$, the set V is a neighborhood of x. Moreover, if $u \in X \setminus V$, then $u \in \operatorname{cl}(\operatorname{int}(X \setminus V))$.

Let $A = \{u \in V; f(u) \leq a\}$. Obviously in this case $x \in \operatorname{cl}(\operatorname{int} A)$. Put

$$g(u) = \begin{cases} 1 & \text{for } u \in V \cap \operatorname{cl} A \\ \frac{f(x)}{2f(u)} & \text{for } u \in V \setminus \operatorname{cl} A \\ 0 & \text{for } u \in X \setminus V. \end{cases}$$

As in case (2), the function $g \in \mathcal{E}(X)$. Observe that

$$(fg)(u) = \begin{cases} f(x) & \text{for } u = x \\ 0 < a & \text{for } u \in X \setminus V \\ \frac{1}{2}f(x) < a & \text{for } u \in V \setminus \operatorname{cl} A \\ f(u) \leq a & \text{for } u \in A, \end{cases}$$

and $(fg)(u) \in \mathbb{R}$ for $u \in N = \operatorname{cl} A \setminus A$, where N is nowhere dense in X. Thus $fg \notin \mathcal{E}(X, x)$.

From the cases (1), (2), (3) follows that $fg \notin \mathcal{E}(X)$ and this completes the proof of Theorem 6. $\square$

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