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ON SOME SUBCLASSES OF BAIRE 1 FUNCTIONS

Abstract

We will consider the classes of first return continuous, weakly first return continuous with respect to some trajectory and almost continuous functions in the sense of Stallings and we will show that for the Baire 1 functions the classes of functions mentioned above are equal. Moreover we will define the class $\mathcal{B}_1^{\mathcal{F}}$ of strongly $\mathcal{F}$-almost everywhere first return recoverable functions and shall present the relation between family $\mathcal{B}_1^{\mathcal{F}}$ and Baire 1 functions.

It is well known that almost continuous functions (in the sense of Stallings) which belongs to the first class of Baire satisfy the Young condition and a real valued function f defined on $[0,1]$ is an almost continuous, Baire 1 function if and only if f is first return continuous with respect to some trajectory. Moreover the class of first return recoverable functions with respect to some trajectory defined on the interval $[0,1]$ and the class of Baire 1 function on $[0,1]$ are equal.

In this paper we will define weakly first return continuous function and moreover we will define $\mathcal{F}$ - almost everywhere first return recoverable and strongly $\mathcal{F}$ - almost everywhere first return recoverable functions but in our considerations we shall use only the second class. This class we denote by $\mathcal{B}_1^{\mathcal{F}}$. It is worth noting that each Baire 1 function f defined on the interval $[0,1]$ belongs to the family $\mathcal{B}_1^{\mathcal{F}}$.

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In this paper we will also show that for a Baire 1 function the conditions of almost continuity (in the sense of Stallings), first return continuity and weakly first return continuity are equivalent (see, for example, [2], [1] or [6]).

We apply the classical symbols and notation. By $\mathbb{R}$ ($\mathbb{N}$) we denote the set of real (natural) numbers. Let $\overline{A}$ ($\mathrm{int}(A)$) denote the closure (the interior) of A, where $A \subset [0,1]$. By $\mathcal{B}_1$ ($\mathcal{A}$) we denote the family of all Baire 1 (almost continuous in the sense of Stallings) functions $f : [0,1] \to \mathbb{R}$.

We say that a function $f : [0,1] \to \mathbb{R}$ satisfies the Young condition if for every $x \in [0,1]$ there exist sequences $x_n \nearrow x$ and $y_n \searrow x$ such that both $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to $\{f(x)\}$.

By a trajectory we mean any sequence $\{x_n\}_{n=0}^{\infty}$ of distinct points in $[0,1]$, which is dense in $[0,1]$. Let $\{x_n\}$ be a fixed trajectory. For a given interval, or finite union of intervals, $H \subseteq [0,1]$, $r(H)$ will be the first element of the trajectory $\{x_n\}$ in H. First, for $x \in [0,1]$ we define what we shall mean by the first return route to x based on the trajectory $\{x_n\}$. If $\rho > 0$, we use $B_\rho(x)$ to denote $\{y \in [0,1] : |x-y| < \rho\}$. The first return route to x, $R_x = \{y_k\}_{k=1}^{\infty}$, is defined recursively via

$$y_1 = x_0,$$
$$y_{k+1} = \begin{cases} r(B_{|x-y_k|}(x)) & \text{if } x \neq y_k \\ y_k & \text{if } x = y_k. \end{cases}$$

We say that $f : [0,1] \to \mathbb{R}$ is first return recoverable with respect to $\{x_n\}$ at x provided that $\lim_{k\to\infty} f(y_k) = f(x)$, and if this happens for each $x \in [0,1]$, we say that $f : [0,1] \to \mathbb{R}$ is first return recoverable with respect to $\{x_n\}$.

We say that $x \in [0,1]$ is a Baire one point of $f : [0,1] \to \mathbb{R}$ with respect to $\{x_n\}$ if f is first return recoverable with respect to $\{x_n\}$ at x. For a fixed function f let $\mathcal{B}_1(f, \{x_n\})$ denote the set of all Baire one points of f with respect to $\{x_n\}$. Moreover let $\mathcal{B}_1^{\wedge}(f, \{x_n\}) = [0,1] \setminus \mathcal{B}_1(f, \{x_n\})$.

It is known that $f : [0,1] \to \mathbb{R}$ is a Baire 1 function iff there exists a trajectory $\{x_n\}$ such that $\mathcal{B}_1^{\wedge}(f, \{x_n\}) = \emptyset$ (see [2]).

For $0 < x \leq 1$, the left first return path to x based on $\{x_n\}$, $P_x^l = \{t_k\}$, is defined recursively via $t_1 = r(0,x)$ and $t_{k+1} = r(t_k, x)$. For $0 \leq x < 1$, the right first return path to x based on $\{x_n\}$, $P_x^r = \{s_k\}$, is defined analogously.

A function $f : [0,1] \to \mathbb{R}$ is first return continuous from the left [right] at x with respect to the trajectory $\{x_n\}$ provided that

$$\lim_{t\to x, t\in P_x^l} f(t) = f(x) \ \ [\lim_{s\to x, s\in P_x^r} f(s) = f(x)].$$

We say that for any $x \in (0,1)$, $f : [0,1] \to \mathbb{R}$ is first return continuous at

x with respect to the trajectory $\{x_n\}$ provided it is both left and right first return continuous at x with respect to the trajectory $\{x_n\}$.

We say that $x \in [0,1]$ is a first return continuity (from the left, from the right) point of $f : [0,1] \to \mathbb{R}$ with respect to $\{x_n\}$ if f is first return continuous (from the left, from the right) with respect to $\{x_n\}$ at x. For a fixed function f let $\mathcal{C}(f, \{x_n\})$ denote the set of all first return continuity points of f with respect to $\{x_n\}$. Moreover let $\mathcal{C}^\wedge(f, \{x_n\}) = [0,1] \setminus \mathcal{C}(f, \{x_n\})$. It is known that $f : [0,1] \to \mathbb{R}$ is an almost continuous Baire 1 function iff there exists a trajectory $\{x_n\}$ such that $\mathcal{C}^\wedge(f, \{x_n\}) = \emptyset$ (see [3]).

Let $\mathcal{F}$ be an ideal of subsets of real line such that if $A \in \mathcal{F}$, then $\mathrm{int}(A) = \emptyset$. A function $f : [0,1] \to \mathbb{R}$ is $\mathcal{F}$-almost everywhere first return recoverable with respect to the trajectory $\{x_n\}$ if $\mathcal{B}_1^\wedge(f, \{x_n\}) \in \mathcal{F}$. A function $f : [0,1] \to \mathbb{R}$ is strongly $\mathcal{F}$-almost everywhere first return recoverable with respect to the trajectory $\{x_n\}$ if $\overline{\mathcal{B}_1^\wedge(f, \{x_n\})} \in \mathcal{F}$.

In this paper we will examine the second class. Let us denote this class (of strongly $\mathcal{F}$-almost everywhere first return recoverable with respect to the trajectory $\{x_n\}$) by the symbol $\mathcal{B}_1^{\mathcal{F}}(\{x_n\})$. We will say that a function $f : [0,1] \to \mathbb{R}$ is strongly $\mathcal{F}$-almost everywhere first return recoverable if there exists a trajectory $\{x_n\}$ such that $f \in \mathcal{B}_1^{\mathcal{F}}(\{x_n\})$. Let $\mathcal{B}_1^{\mathcal{F}}$ denote the set of all strongly $\mathcal{F}$- almost everywhere first return recoverable functions $f : [0,1] \to \mathbb{R}$.

A function $f : [0,1] \to \mathbb{R}$ is nearly first return continuous ($f \in \mathcal{C}^\star(\{x_n\})$) with respect to the trajectory $\{x_n\}$ if $f \in \mathcal{B}_1^{\mathcal{F}}(\{x_n\})$ and for each component (a,b) of the interior of the set $\mathcal{B}_1(f, \{x_n\})$ there exists a trajectory $\{y_n\} \subset (a,b)$ such that $(a,b) \subset \mathcal{C}(f, \{y_n\})$ and a and b are unilateral first return continuity point with respect to $\{y_n\}$.

A nearly first return continuous function $f : [0,1] \to \mathbb{R}$ is weakly first return continuous with respect to the trajectory $\{x_n\}$ provided for each $x \in \mathcal{B}_1^\wedge(f, \{x_n\})$ there exists an open set $G \subset \mathcal{B}_1(f, \{x_n\})$ such that x is a bilateral accumulation point of the set G and there exists a trajectory $\{y_n\} \subset G$ such that $f \restriction_{\overline{G}}$ is first return continuous with respect to $\{y_n\}$.

Proposition 1. *Let $\mathcal{F}$ be an arbitrary ideal. A function $f : [0,1] \to \mathbb{R}$ is Baire 1 function iff there exists a trajectory $\{x_n\}$ such that $f \in \mathcal{B}_1^{\mathcal{F}}(\{x_n\})$ and $\mathcal{B}_1^\wedge(f, \{x_n\})$ is a countable closed set.*

Proof. If $f \in \mathcal{B}_1$, there exists a trajectory $\{x_n\}$ such that $\mathcal{B}_1^\wedge(f, \{x_n\}) = \emptyset$. Suppose now that there exists a trajectory $\{x_n\}$ such that $f \in \mathcal{B}_1^{\mathcal{F}}(\{x_n\})$ and $\mathcal{B}_1^\wedge(f, \{x_n\})$ is a countable closed set. Let $\{\alpha_n\}$ be a sequence of all elements of $\mathcal{B}_1^\wedge(f, \{x_n\})$. We define a trajectory $\{z_n\}$ by $z_{2n} = \alpha_n$, $z_{2n+1} = x_n$ for $n \in \mathbb{N}$. We shall prove that $x \in \mathcal{B}_1(f, \{z_n\})$ for each $x \in [0,1]$. To this purpose let us consider the following cases:

1) $x \in \mathcal{B}_1^{\wedge}(f, \{x_n\})$

Then there exists $n_0 \in \mathbb{N}$ such that $x = \alpha_{n_0} = z_{2n_0}$. Hence $y_k = z_{2n_0}$ for $k \geq 2n_0$, where $R_x = \{y_k\}_{k=1}^{\infty}$ is the first return route to x based on the trajectory $\{z_n\}$. Therefore $\lim_{k\to\infty} f(y_k) = f(z_{2n_0}) = f(x)$; so $x \in \mathcal{B}_1(f, \{z_n\})$.

2) $x \in \mathcal{B}_1(f, \{x_n\})$

By the assumption $\mathcal{B}_1(f, \{x_n\})$ is an open set, so there exists a neighborhood U of point x such that $U \subset \mathcal{B}_1(f, \{x_n\})$. Hence $\{z_n\} \cap U \subset \{x_n\}$. Therefore $\{y_k\}$ is a subsequence of a sequence $R_x^* = \{y_n^*\}$, where R_x^* is the first return route to x based on the trajectory $\{x_n\}$. Since $x \in \mathcal{B}_1(f, \{x_n\})$, $\lim_{k\to\infty} f(y_k) = f(x)$. Hence $x \in \mathcal{B}_1(f, \{z_n\})$. Therefore $\mathcal{B}_1^{\wedge}(f, \{z_n\}) = \emptyset$ and $f \in \mathcal{B}_1$. □

Remark. *In the previous proposition the assumption that $\mathcal{B}_1^{\wedge}(f, \{x_n\})$ is closed can not be omitted.*

Proof. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \setminus C^\star \\ 1 & \text{if } x \in C^\star, \end{cases}$$

where $C^\star$ is a set of all endpoints of components of complement of Cantor set.

Let $\{x_n\}$ be a sequence which is dense in $[0, 1]$ and contained in $[0, 1] \setminus C^\star$. Then $\mathcal{B}_1^{\wedge}(f, \{x_n\}) = C^\star$ is countable and $\overline{\mathcal{B}_1^{\wedge}(f, \{x_n\})} = C$, where C is Cantor set. Hence $f \in \mathcal{B}_1^{\mathcal{F}}(\{x_n\})$. On the other hand $f \notin \mathcal{B}_1$.

Theorem 1. *If $f : [0, 1] \to \mathbb{R}$ is Baire 1 function the following conditions are equivalent:*

(i) f is weakly first return continuous with respect to some trajectory $\{x_n\}$

(ii) f is first return continuous.

(iii) $f \in \mathcal{A}$.

Proof. It is well known that (ii) $\iff$ (iii). So, it is sufficient to prove implications (i) $\implies$ (iii) and (ii) $\implies$ (i). We start with the proof of the implication (i) $\implies$ (iii). Let f be weakly first return continuous with respect to some trajectory $\{x_n\}$. It is sufficient to prove that f satisfies the Young condition. So let $x_0 \in [0, 1]$. If $x_0 \in \mathcal{B}_1(f, \{x_n\})$, there exists an component (a, b) of the set $\mathcal{B}_1(f, \{x_n\})$ such that $x_0 \in (a, b)$. Since $f \in \mathcal{C}^\star(\{x_n\})$ we can infer that there exists a trajectory $\{y_n\} \subset (a, b)$ such that $(a, b) \subset \mathcal{C}(f, \{y_n\})$. Hence

$$x_0 \in (a, b) \subset \mathcal{C}(f, \{y_n\}) = \mathcal{C}^l(f, \{y_n\}) \cap \mathcal{C}^r(f, \{y_n\}).$$

So

$$\lim_{\substack{t\to x_0\\ t\in P^l_{x_0}(y_n)}} f(t)=f(x_0) \text{ and } \lim_{\substack{s\to x_0\\ s\in P^r_{x_0}(y_n)}} f(s)=f(x_0),$$

and consequently the Young condition holds.

If $x_0\in\mathcal{B}_1^{\wedge}(f,\{x_n\})$, then there exists an open set $G\subset\mathcal{B}_1(f,\{x_n\})$ such that x_0 is a bilateral accumulation point of G and there exists a trajectory $\{y_n\}\subset G$ such that $f\restriction_{\overline{G}}$ is first return continuous with respect to $\{y_n\}$. In particular, $f\restriction_{\overline{G}}$ is bilateral first return continuous with respect to $\{y_n\}$ at the point x_0. Therefore it is easy to conclude that the Young condition holds.

Now we shall prove (ii) $\Longrightarrow$ (i). From our assumption, we may infer that there exists a trajectory $\{z_n\}$ such that $[0,1]=\mathcal{C}(f,\{z_n\})$. Note that f is nearly first return continuous with respect to $\{x_n\}$. By assumption $f\in\mathcal{B}_1^{\mathcal{F}}(\{x_n\})$. Let (a,b) be an arbitrary component of the set $\mathcal{B}_1(f,\{x_n\})$. Let $\{y_n\}$ be a sequence such that

$$y_1=z_{\min\{k\in\mathbb{N}:z_k\in(a,b)\}},$$
$$y_{n+1}=z_{\min\{k\in\mathbb{N}:z_k\in(a,b)\setminus\{y_1,y_2,\dots,y_n\}\}}.$$

Then $\{y_n\}$ is a subsequence of the sequence $\{z_n\}$ such that

$$\{y_n\}\cap(c,d)=\{z_n\}\cap(c,d), \text{ for each } (c,d)\subset(a,b).$$

Therefore $\{y_n\}$ is a dense subset of (a,b), so it is a trajectory in (a,b). Moreover $(a,b)\subset\mathcal{C}(f\restriction_{(a,b)},\{y_n\})$, $a\in\mathcal{C}^r(f\restriction_{(a,b)},\{y_n\})$ and $b\in\mathcal{C}^l(f\restriction_{(a,b)},\{y_n\})$.

Indeed, let $x_0\in(a,b)$. We will prove that $x_0\in\mathcal{C}^l(f\restriction_{(a,b)},\{y_n\})$ (the proofs that $x_0\in\mathcal{C}^r(f\restriction_{(a,b)},\{y_n\})$, $a\in\mathcal{C}^r(f\restriction_{(a,b)},\{y_n\})$ and $b\in\mathcal{C}^l(f\restriction_{(a,b)},\{y_n\})$ are similar). Let $P^l_{x_0}(y_n)=\{t_n\}$ be a left first return path to x_0 based on $\{y_n\}$ and let ${P'}^l_{x_0}(z_n)=\{t'_n\}$ be a left first return path to x_0 based on $\{z_n\}$. Put $n_0=\min\{k\in\mathbb{N}:z_k\in(a,x_0)\}$.

If $n_0=1$, then $y_1=z_1$ and $t_1=r(0,x_0)=y_1=z_1=t'_1$. Next $t_2=r(t_1,x_0)=r(t'_1,x_0)=t'_2$ and $t_k=r(t_{k-1},x_0)=r(t'_{k-1},x_0)=t'_k$.

If $n_0>1$, then $z_k\in[0,a]\cup[x_0,1]$ for each $k<n_0$. Let $z_{k_1},\dots,z_{k_l}$ be a sequence of all elements of the sequence $\{z_n\}$ from the interval $[0,a]$, where $k_i<k_{i+1}$ for $i\in\{1,\dots,l-1\}$. Then $t'_1=r(0,x_0)=z_{k_1}$, $t'_2=r(z_{k_1},x_0)=z_{k_{i_2}}$, $\dots$, $t'_m=r(t'_{m-1},x_0)=r(z_{k_{i_{m-1}}},x_0)=z_{k_{i_m}}$ for some $i_2,\dots,i_m\in\{2,3,\dots,l\}$. Moreover let t'_m be the last element of a sequence ${P'}^l_{x_0}=\{t'_k\}$ in $[0,a]$. Then $t'_{m+1}=r(t'_m,x_0)=r(z_{k_{i_m}},x_0)=z_{n_0}=y_{j_0}$, for some $j_0\in\mathbb{N}$. Hence, by definition of the trajectory $\{y_n\}$ and n_0, y_{j_0} is the first element of the sequence $\{y_n\}$ from an interval (a,x_0). Hence

$$y_{j_0}=r(0,x_0)=t_1 \text{ and } t'_{m+1}=t_1.$$

So by induction we can show that $t'_{m+j} = t_j$ for each $j \geq 1$. Therefore in both considered cases $P^l_{x_0}(y_n) = P^l_{x_0}(z_n)$ except finite number of elements. Hence

$$\lim_{\substack{t \to x_0 \\ t \in P^l_{x_0}(y_n)}} f(t) = \lim_{\substack{t \to x_0 \\ t \in P^r_{x_0}(z_n)}} f(t) = f(x_0).$$

So $x_0 \in \mathcal{C}^l(f_{|(a,b)}, \{y_n\})$.

Now let $x_0 \in \mathcal{B}_1^{\wedge}(f, \{x_n\})$ and let $P^l_{x_0}(z_n) = \{t_k\}$ be the left first return path to x_0 based on $\{z_n\}$. We define a sequence $\{t''_k\}$ in the following way:

- if $t_k \in \mathcal{B}_1(f, \{x_n\})$ let $t''_k = t_k$
- if $t_k \in \mathcal{B}_1^{\wedge}(f, \{x_n\})$ let $t''_k = y_0$, where $y_0 \in \mathcal{B}_1(f, \{x_n\}) \cap (t_k, x_0)$ and $f(y_0) \in (f(t_k) - \frac{1}{k}, f(t_k) + \frac{1}{k})$.

Note that such a point y_0 in the second case above exists. Indeed in the opposite case $f(y) \in (-\infty, f(t_k) - \frac{1}{k}] \cup [f(t_k) + \frac{1}{k}, +\infty)$ for each $y \in \mathcal{B}_1(f, \{x_n\}) \cap (t_k, x_0)$. Obviously $\mathcal{B}_1(f, \{x_n\}) \cap (t_k, x_0) \neq \emptyset$ (because $\mathcal{B}_1(f, \{x_n\})$ is a dense set). Then

$$f\Bigg([t_k, x_0) \cap \Big((-\infty, f(t_k) - \frac{1}{k}\Big] \cup [f(t_k) + \frac{1}{k}, +\infty)\Big)\Bigg) \neq \emptyset$$

and $f(t_k) \in f([t_k, x_0))$. Hence, by the fact that $f \in \mathcal{C}^\star(\{z_n\}) = \mathcal{B}_1 \cap \mathcal{A} = \mathcal{B}_1 \cap \mathcal{D}$,

$$\big[f(t_k) - \frac{1}{k}, f(t_k)\big] \subset f([t_k, x_0)) \text{ or } \big[f(t_k), f(t_k) + \frac{1}{k}\big] \subset f([t_k, x_0)).$$

But $f((t_k, x_0) \cap \mathcal{B}_1(f, \{x_n\})) \cap (f(t_k) - \frac{1}{k}, f(t_k) + \frac{1}{k}) = \emptyset$; so

$$\begin{gathered} (f(t_k) - \frac{1}{k}, f(t_k)) \subset f([t_k, x_0) \cap \mathcal{B}_1^{\wedge}(f, \{x_n\})) \text{ or} \\ (f(t_k), f(t_k) + \frac{1}{k}) \subset f([t_k, x_0) \cap \mathcal{B}_1^{\wedge}(f, \{x_n\})), \end{gathered}$$

which is impossible because from our assumption $\mathcal{B}_1^{\wedge}(f, \{x_n\}))$ is countable.

In the analogous way we can define a sequence $\{s''_k\}$:

- if $s_k \in \mathcal{B}_1(f, \{x_n\})$, let $s''_k = s_k$
- if $s_k \in \mathcal{B}_1^{\wedge}(f, \{x_n\})$, let $s''_k = y_1$, where $y_1 \in \mathcal{B}_1(f, \{x_n\}) \cap (x_0, s_k)$ and $f(y_1) \in (f(s_k) - \frac{1}{k}, f(s_k) + \frac{1}{k})$.

Now, let $d_n \in (t''_n, t''_{n+1})$ and $p_n \in (s''_n, s''_{n+1})$, for each $n \in \mathbb{N}$. Put

$$G = \bigcup_{n=1}^{+\infty} ([(d_{n-1}, d_n) \cap S_{t''_n}] \cup [(p_{n-1}, p_n) \cap S_{s''_n}]),$$

where for each $n \in \mathbb{N}$, $S_{t''_n}$ ($S_{s''_n}$) is a component of the set $\mathcal{B}_1(f, \{x_n\})$ containing t''_n (s''_n). Obviously G is an open set contained in $\mathcal{B}_1(f, \{x_n\})$. Moreover x_0 is the bilateral accumulation point of the set G, because $t_k \nearrow x_0$, $t''_k \in [t_k, x_0)$ and $t''_k \in G$ for each $k \in \mathbb{N}$ and $s_k \searrow x_0$, $s''_k \in (x_0, s_k]$ and $s''_k \in G$ for each $k \in \mathbb{N}$. Put

$$y_n = \begin{cases} z_n & \text{if } z_n \notin P^l_{x_0}(z_n) \cup P^r_{x_0}(z_n) \text{ and } z_n \in G \\ t''_n & \text{if } z_n = t_n \\ s''_n & \text{if } z_n = s_n. \end{cases}$$

Then obviously $\{y_n\} \subset G$. We will show that $f \restriction_{\overline{G}}$ is first return continuous with respect to the trajectory $\{y_n\}$. Let $x \in \overline{G}$. Consider the following cases:

1^0 $x \in G \setminus \{x_0\}$.

Hence, if $P^l_x(y_n) = \{a_k\}$ and $P^r_x(y_n) = \{b_k\}$, then for sufficiently large k, $a_k \notin P^l_{x_0}(z_n) \cup P^r_{x_0}(z_n)$ and $b_k \notin P^l_{x_0}(z_n) \cup P^r_{x_0}(z_n)$. Hence

$$P^l_x(z_n) = P^l_x(y_n) \text{ and } P^r_x(z_n) = P^r_x(y_n)$$

except fo a finite number of elements. Hence

$$\lim_{\substack{t \to x \\ t \in P^l_x(y_n)}} f(t) = \lim_{t \to x t \in P^l_x(z_n)} f(t) = f(x_0)$$

and

$$\lim_{\substack{t \to x \\ t \in P^r_{x_0}(y_n)}} f(t) = \lim_{\substack{t \to x \\ t \in P^r_x(z_n)}} f(t) = f(x).$$

So $x \in \mathcal{C}(f \restriction_{\overline{G}}, \{y_n\})$.

2^0 $x = x_0$.

Then $P^l_{x_0}(y_n) = \{t''_n\}$ and $P^r_{x_0}(y_n) = \{s''_n\}$. Hence

$$\lim_{\substack{t \to x_0 \\ t \in P^l_{x_0}(y_n)}} f(t) = f(x) = \lim_{\substack{t \to x_0 \\ t \in P^r_{x_0}(y_n)}} f(t),$$

because $f(t''_n) \in (f(t_n) - \frac{1}{n}, f(t_n) + \frac{1}{n})$ and $f(s''_n) \in (f(s_n) - \frac{1}{n}, f(s_n) + \frac{1}{n})$ for each $n \in \mathbb{N}$. Hence $x_0 \in \mathcal{C}(f \restriction_{\overline{G}}, \{y_n\})$. $\square$

References

[1] J. B. Brown, *Almost continuous Darboux functions and Reed's pointwise convergence criteria*, Fundamenta Math., **86** (1974), 1–7.

[2] U. B. Darji, M. J. Evans, R. J. O'Malley, *A first return characterization of Baire one functions*, Real Analysis Exchange, **19** (1993-94), 510–515.

[3] U. B. Darji, M. J. Evans, R. J. O'Malley, *First return path systems: differentiability, continuity, and orderings*, Acta Math. Hungar., **66 (1-2)** (1995), 83–103.

[4] U. B. Darji, M. J. Evans, R. J. O'Malley, *Some interesting small subclasses of the Darboux Baire 1 functions*, Real Analysis Exchange, **19(1)** (1993-94), 328–331.

[5] M. J. Evans, R. J. O'Malley, *Fine tuning the recoverability of Baire one functions*, Real Analysis Exchange, **21(1)** (1995-96), 165–174.

[6] R. J. Pawlak, B. Świątek, *On Baire one point of functions*, preprint.