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HAUSDORFF DIMENSION OF CENTERED SETS

Abstract

In this paper we will consider nowhere dense perfect subsets of $[0,1]$ that arise as a natural generalization of symmetric perfect sets in a paper of Humke [3], called *centered* sets. These sets have relatively large basic intervals but none of the standard methods used in determining the dimension of a generalized Cantor set could be used to show that *centered* sets can not have zero Hausdorff dimension. The aim of this paper is to prove that and to give lower bound for this dimension.

1 Introduction

This paper will study the Hausdorff dimension of nowhere dense perfect subsets of $[0,1]$ that are *centered*. The term *centered* was first used by Humke in [3] and it is defined as follows.

Let E be a nowhere dense perfect set in $[0,1]$. Let Σ_n denote the set of all binary sequences of length n and let $\Sigma = \cup_{n=0}^{\infty}\Sigma_n$.

(a) $|\sigma|$ denotes the length of the binary sequence σ and it is called the order of σ.

(b) $\sigma_{|m}$ denotes the first m digits of σ.

Define an order on $\sum$ by $\sigma < \tau$ if $d(\sigma) < d(\tau)$, where $d : \Sigma \to [0,1]$, $d(\emptyset) = \frac{1}{2}$, and if $|\sigma| = N$,

$$d(\sigma) = \frac{1}{2} + \sum_{n=1}^{N} \frac{2\sigma(n) - 1}{2^{n+1}}.$$

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This is the usual order on the indices of the intervals forming a perfect set. Consider the intervals that are contiguous to E in $[0,1]$. To label these open intervals we can use any order preserving surjection $g : \Sigma \to E^c$, by choosing J_σ to be equal to $g(\sigma)$. The noncontiguous intervals of E can be then labeled using Σ as the index set by choosing I_σ to denote that unique closed component of $[0,1] \setminus \cup_{|\tau|<|\sigma|} I_\sigma$ which contains J_σ. It is customary to refer to these I_σ's as the *basic intervals* or *fundamental intervals* of E.

Definition 1. A nowhere dense perfect set E is called centered if there exists a binary labeling of E such that the centering constant γ is positive:

$$\gamma = \lim_{N\to\infty} \inf \left\{ \frac{|I_{\sigma i}|}{|I_\sigma|} : i = 0, 1 \text{ and } |\sigma| \geq N \right\}.$$

Note that considering a given nowhere dense perfect set one can always label the contiguous intervals such that the resulting γ is zero. The definition of a set being centered simply requires the existence of one labeling, such that the ratio of the resulting basic intervals at consecutive stages stays away from zero eventually.

As in Beardon [1] and Humke [3] we use the following terminology:

(c) any cover of E formed by finitely many pairwise disjoint fundamental intervals is called a *fundamental cover* of E.

(d) a fundamental cover formed by fundamental intervals of the same order is called a *natural cover*.

(e) we will frequently use the term *holes* in place of the proper term *contiguous intervals* of E for the sake of brevity.

A number of dimension calculations for different types of generalized Cantor sets were summarized by Falconer in [2, pp.54-59], but in all of the listed examples "the upper estimates for the dimension depend on the number and size of the basic intervals, whilst the lower estimates depend on their spacing". Requiring that a set is centered poses a condition on the size of the basic intervals but clearly gives us no control over their spacing. Therefore, although it is expected that these sets can not have very small dimension, it remains our problem to prove that. The aim of this article is in fact to show this by proving that the dimension of a centered set with centering constant γ is at least $-\log 2/\log\gamma$.

2 Replacement Lemmas

The s-dimensional Hausdorff measure of E and the Hausdorff dimension of E will be denoted by $m^s(E)$ and $\dim(E)$ respectively. $M^s(E)$ stands for the *fundamental s-measure* of E and it is defined below as the approximation to the measure with the added restriction that the covers used to estimate the measure are fundamental systems of E.

Definition 2. Define

$$M_\delta^s(E) = \inf\left\{\sum_{n=1}^{\infty} |I_n|^s : E \subset \cup I_n, |I_n| \leq \delta\right\}$$

where each I_n is a fundamental interval of E.

Definition 3. $M^s(E) = \lim_{\delta\to 0} M_\delta^s(E) = \sup_{\delta>0} M_\delta^s(E)$.

In our first lemma we prove that fundamental covers determine the dimension of a centered set.

Lemma 1. *Let E be a nowhere dense perfect set that is centered with centering constant γ. Then*

$$\frac{\gamma^s}{2} M^s(E) \leq m^s(E) \leq M^s(E).$$

PROOF. Without loss of generality we may consider finite open coverings of E, since E is compact. Let $\epsilon > 0$, $\eta > 0$ with $\eta < \gamma$ be given and choose N such that

$$\frac{|I_{\sigma i}|}{|I_\sigma|} > \gamma - \eta \text{ for } n \geq N. \tag{1}$$

Let $\delta = \{\min|J_\sigma| : |\sigma| < N\}$ where J_σ, $|\sigma| = n$, are the contiguous intervals of order n to E. Let $C^\delta(E)$ be the set of all finite covers of E consisting of intervals whose lengths are less than δ. Let $\{F_1, F_2, \ldots, F_p\} \in C^\delta(E)$, such that

$$m_\delta^s(E) + \epsilon > \sum_{i=1}^{p} |F_i|^s. \tag{2}$$

Fix an index i and find the hole of lowest order among those contained by F_i. (The hole of lowest order contained by F_i is unique since if there were two such holes, then there would be one between them of lower order.) Denote this contiguous interval by J_τ. Then $|\tau| \geq N$ since $|F_i| < |J_\sigma|$ for $\sigma < N$. Now find one hole on the left and one on the right hand side of least order exceeding the order of τ contained in F_i on the left and right side of J_τ. Denote the index of these contiguous intervals by λ for the one on the left side and by ρ on the right. Consider next the two fundamental intervals, $I_{\lambda 1}$ and $I_{\rho 0}$ between the three selected holes. Clearly $|\lambda|, |\rho| > N$ since $|\tau| \geq N$. and $|I_{\lambda 1}|$, $|I_{\rho 0}| < \delta$ since each is contained in F_i. Thus, by inequality (1) it follows that

$$|I_\lambda| < \frac{1}{\gamma - \eta} |I_{\lambda 1}| < \frac{1}{\gamma - \eta} \delta \tag{3}$$

and similarly

$$|I_\rho| < \frac{1}{\gamma - \eta} |I_{\rho 0}| < \frac{1}{\gamma - \eta} \delta. \tag{4}$$

Also, $I_\lambda \cup I_\rho$ covers the same part of E that F_i does, since otherwise a hole of order less than that of λ or ρ would necessarily be contained in F_i contradicting the way they have been selected. Replacing F_i by $I_\lambda \cup I_\rho$ and repeating the above replacement process for each i yields a fundamental cover $\bigcup_{i=1}^p \left(I_\lambda^{(i)} \cup I_\rho^{(i)} \right)$ in $C^{\frac{\delta}{\gamma - \eta}}(E)$ by inequalities (3) and (4). Therefore,

$$M^s_{\frac{\delta}{\gamma - \eta}}(E) < \sum_{i=1}^p \left| I_\lambda^i \right|^s + \sum_{i=1}^p \left| I_\rho^i \right|^s$$

and it follows from (1) and (2) and from the fact that both $I_{\lambda 1}$ and $I_{\rho 0}$ are contained in F_i that

$$\begin{aligned}(\gamma - \eta)^s M^s_{\frac{\delta}{\gamma - \eta}}(E) &< \left[\sum_{i=1}^p \left| I_\lambda^i \right|^s (\gamma - \eta)^s + \sum_{i=1}^p \left| I_\rho^i \right|^s (\gamma - \eta)^s \right] \\ &< \left[\sum_{i=1}^p \left| I_{\lambda 1}^i \right|^s + \sum_{i=1}^p \left| I_{\rho 0}^i \right|^s \right] < 2 \sum_{i=1}^p |F_i|^s < 2(m^s_\delta(E) + \varepsilon).\end{aligned}$$

Let $\epsilon \to 0$, $\eta \to 0$ in this last inequality to get $\frac{\gamma^s}{2} M^s_{\frac{\delta}{\gamma}}(E) \leq m^s_\delta(E)$. Finally letting $\delta \to 0$, yields $\frac{\gamma^s}{2} M^s(E) \leq m^s(E) \leq M^s(E)$. We will proceed now to prove our main theorem using another replacement argument.

Theorem 1. *Let E be centered with centering constant γ. Then*

$$\dim(E) \geq -\frac{\log 2}{\log \gamma}.$$

In particular, every centered perfect set has Hausdorff dimension greater than zero.

PROOF. The idea of our proof is to define a replacement process that proves that for some δ, any fundamental cover in $C^{\delta}(E)$ gives a worse approximation to the s-measure than a specific natural cover if $0 < s < -\log 2/\log \gamma$. Let $\gamma > \epsilon > 0$ and $s = -\log 2/\log(\gamma - \epsilon)$. Find N so that if $n > N$ and $|\sigma| = n$, then

$$\frac{|I_{\sigma i}|}{|I_{\sigma}|} > \gamma - \epsilon. \tag{5}$$

Furthermore, let $\delta = \min\{|I_{\tau}| : |\tau| \leq N\}$. Let $\Phi = \{I_1, I_2, \ldots, I_p\} \in C^{\delta}$ be a fundamental system. (Since Lemma 1 proved that fundamental systems determine the dimension of centered sets it suffices to consider such systems only.) Let u be the maximum order and let l denote the minimal order within the orders of the intervals in Φ. Then Φ must contain a fundamental interval of order u which will be denoted by I_{λ}. Since the intervals in Φ are disjoint, $I_{\lambda|_1}, I_{\lambda|_2}, \ldots, I_{\lambda|_{u-1}} \notin \Phi$. Therefore, by the covering property of Φ, it follows that either $I_{\lambda|_{u-1}0}$ or $I_{\lambda|_{u-1}1} \in \Phi$ depending on the last digit of λ. Replace these two intervals of order u by $I_{\lambda|_{u-1}}$ and similarly replace all pairs of fundamental intervals of order u by intervals whose order is $(u-1)$, forming a new cover Φ^*. It follows from inequality (5) that

$$I_{\lambda|_{u-1}0}, I_{\lambda|_{u-1}1} > (\gamma - \epsilon)\, I_{\lambda|_{u-1}}$$

and since $s = -\log 2/\log(\gamma - \epsilon)$,

$$\left|I_{\lambda|_{u-1}0}\right|^s, \left|I_{\lambda|_{u-1}1}\right|^s > (\gamma - \epsilon)^s \left|I_{\lambda|_{u-1}}\right|^s = \frac{1}{2}\left|I_{\lambda|_{u-1}}\right|^s.$$

Therefore, $\left|I_{\lambda|_{u-1}0}\right|^s + \left|I_{\lambda|_{u-1}1},\right|^s > \left|I_{\lambda|_{u-1}}\right|^s$. This proves that by making the above replacement we do not increase the approximation to the s-measure; i.e., $\sum_{I\in\Phi} |I|^s > \sum_{I\in\Phi^*} |I|^s$. If we repeat this replacement process for the new cover Φ^*, and then again for the cover replacing Φ^*, then after at most $u - l$ replacements we will arrive to a natural cover. But we can continue the replacement even further as long as (5) holds to reach the natural cover of order N. Thus we proved that for such s $\sum_{I\in\Phi} |I|^s > \sum_{I_\tau, |\tau|=N} |I_\tau|^s$. This proves that the approximation to the s-measure by any fundamental cover is uniformly bounded away from zero and therefore

$$m^s(E) \geq m^s_{\delta}(E) \geq \sum_{I\in\Phi} |I|^s > \sum_{I_\tau, |\tau|=N} |I_\tau|^s > 0.$$

Hence $\dim(E) \geq s = -\frac{\log 2}{\log(\gamma-\epsilon)}$. Since this holds for every ϵ, $\dim(E) \geq -\frac{\log 2}{\log \gamma}$.

3 m-Generalized Centered Sets

It is natural to consider next the following generalized version of centered sets.

Let $[0,1] = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_k \supset \cdots$ be a decreasing sequence of closed sets, with each E_k a finite union of closed intervals with each interval of E_k containing exactly m intervals of E_{k+1}. We may use m-ary sequences to label the intervals of E_k. Let $[0,1] = I_\emptyset$. Label the intervals of E_1 as $I_0, I_1, I_2, \ldots, I_{m-1}$ and, in general, the subintervals of I_σ (where σ is any m-ary sequence) as $I_{\sigma 0}, I_{\sigma 1}, \ldots, I_{\sigma(m-1)}$. Consider the set $E = \cap_{k=1}^{\infty} E_k$.

Definition 4. A set E is called "generalized centered" if it is constructible by the above method with the added restriction that

$$\gamma = \lim_{N\to\infty} \inf \left\{ \frac{|I_{\sigma i}|}{|I_\sigma|} : i = 0,1,\ldots,m-1 \text{ and } |\sigma| \geq N \right\} > 0.$$

Results similar to that of Lemma 1 and Theorem 1 are true when E is m-generalized centered.

Lemma 2. *Let E be m-generalized centered with generalized centering constant γ. Then* $\frac{\gamma^s}{m} M^s(E) \leq m^s(E) \leq M^s(E)$.

Theorem 2. *Let E be generalized centered with generalized centering constant γ. Then* $\dim(E) \geq -\frac{\log m}{\log \gamma}$.

The proofs of Lemma 2 and Theorem 2 are similar to the proofs of Lemma 1 and Theorem 1 and are omitted.

Note that this lower bound for the dimension is exact for a special class of generalized centered sets called "uniform Cantor sets" that are studied in Falconer [2, pp. 58–59.]. The construction of a uniform Cantor set is a natural generalization of the middle third Cantor set. First, we fix an s, $0 < s < 1$, and $m \geq 2$, m a positive integer. Then we construct a set with the following property. At each stage k and for each basic interval I, at that stage, the intervals $I_1, I_2, \ldots, I_m$ of order $k+1$ are equally spaced and of equal length. Their length is determined by the equation

$$|I_i|^s = \frac{1}{m} |I|^s, \ 0 < s < 1. \tag{6}$$

It is known that a uniform Cantor set that is constructed for a given s has dimension s. (See Falconer [2, p. 57].) It follows from equation (6) that $|I_i|/|I| = m^{-1/s}$ and as a consequence these sets are generalized centered with centering constant $\gamma = m^{-1/s}$. Consequently, the lower bound for the dimension from Theorem 2 in this particular case is exact since

$$-\frac{\log m}{\log \gamma} = -\frac{\log m}{\log m^{-1/s}} = \frac{\log m}{\frac{1}{s}\log m} = s.$$

It is easy to show using Proposition 4.1 in [2] that if the limit of the supremum of the ratios of the basic intervals is γ, then the dimension is less than $-\log 2/\log \gamma$. Therefore the lower bound in Theorem 2 is also exact if in fact the *limit* of the ratios of the consecutive intervals is γ.

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