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FUBINI TYPE THEOREMS FOR THE BV INTEGRAL

Abstract

In this paper, we show that if an R-integrable (or BV integrable function) defined over a compact interval $[a, b] \times [c, d] \subset \mathbb{R}^2$ satisfies certain conditions, then over any subinterval of $[a, b] \times [c, d]$, the iterated integral exists and equals the double integral. We also present some examples relevant to our theory.

1 Introduction.

The R-integral (also known as the BV integral), was introduced by Pfeffer in [6] (the so called "v-integral" in that paper). It extends the Lebesgue integral in $\mathbb{R}^m$, and integrates the divergence of a bounded vector field under very general conditions (see [8], Theorem 5.1.12). However, Fubini's Theorem is false for the R-integral (see Example 5.1.14 in [8]). We present that example at the end of Section 4 (Proposition 4.4). In [11], the following problems for the R-integral over the unit interval $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ were posed:

1. Assuming that the double integral and the iterated integrals exist, do they have the same value?
2. Assuming that the double integral exists and the iterated integrals are equal, is their value equal to the double integral?

In Section 3, we show that if an R-integrable function on $[a, b] \times [c, d]$ satisfies certain conditions, then an iterated integral exists, and equals the double integral over any subinterval of $[a, b] \times [c, d]$. All examples of R-integrable functions on $[a, b] \times [c, d]$, known to the authors, for which over any subinterval,

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one iterated integral exists and equals the double integral, satisfy some set of conditions mentioned in Section 3.

We begin by giving some preliminary definitions and results in Section 2. In Section 3, we prove some results leading up to Fubini-type theorems for the R-integral. We also present some examples satisfying those conditions in Section 4.

2 Preliminaries.

Throughout this paper, we shall follow the notations and terminologies used in [8]. Also, all the results in this section, unless mentioned otherwise are from [8].

We give the following definition from [13], (Definition 5.6.4). See also Sections 5.7 and 5.8 of [3].

Definition 2.1. Let $E \subset \mathbb{R}^m$ be a measurable set. A unit vector n is called the measure-theoretic normal to E at x if

$$\lim_{r \to 0} r^{-m} |B(x,r) \cap \{y : (y-x) \cdot n < 0, y \notin E\}| = 0$$

and

$$\lim_{r \to 0} r^{-m} |B(x,r) \cap \{y : (y-x) \cdot n > 0, y \in E\}| = 0.$$

The measure-theoretic normal to E at x will be denoted by $\nu_E(x)$.

Note 2.1. *Let E be a locally BV set. Then*

(i) The measure-theoretic normal is defined $\mathcal{H}^{m-1}$-almost everywhere on $\partial_* E$. (See Lemma 5.9.5 of [13]).

(ii) The unit exterior normal, defined on page 34 of [8], equals the measure-theoretic normal $\mathcal{H}^{m-1}$-almost everywhere on $\partial_* E$. For details, refer to [13](chapter 5) or [3](chapter 5).

Theorem 2.1. *(see corollary 9.1.2 of [6] or chapter 1 of [8]) Let $E \subset \mathbb{R}$ be a BV set. Then E is equivalent to a unique figure $\bigcup_{i=1}^{p} [a_i, b_i]$.*

Theorem 2.2. *(Proposition 2.1.2 of [8]) Let F be an additive function on $\mathcal{BV}$. Then the following are equivalent:*

(i) F is a charge.

(ii) Given $\epsilon > 0$, there is an $\eta > 0$ such that $|F(C)| < \epsilon$ for each BV set $C \subset B(1/\epsilon)$ with $||C|| < 1/\epsilon$ and $|C| < \eta$.

(iii) $\lim F(A_i) = 0$ *for each sequence* $\{A_i\}$ *in* $\mathcal{BV}$ *with* $\{A_i\} \to \emptyset$.

Theorem 2.3. *(Remark 2.1.5 of [8]) Let* $F : \mathcal{BV}(\mathbb{R}) \to \mathbb{R}$, *be an additive function defined by*

$$F(A) = F(\cup_{i=1}^{p}[a_i, b_i]),$$

where A *is equivalent to the unique figure* $\cup_{i=1}^{p}[a_i, b_i]$. *Then,*

(i) There exists a vector field $v : \mathbb{R} \to \mathbb{R}$, *such that,*

$$F(A) = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^0 = \sum_{i=1}^{p}[v(b_i) - v(a_i)].$$

(ii) F *is a charge if and only if* v *is continuous.*

Derivates. We use the definitions and notations $\underline{D}_\eta F$, $\overline{D}_\eta F$, $\underline{D}F$, $\overline{D}F$ and DF as given in Section 2.3 of [8].

Definition 2.2. Let $n \geq 1$ be an integer, and let $E \subset \mathbb{R}^m$. A map $\phi : E \to \mathbb{R}^n$ is called Lipschitz at a point $x \in E$ if there are real numbers c_x and $\delta_x > 0$ such that $|\phi(x) - \phi(y)| \leq c_x|x - y|$ for each $y \in E \cap B(x, \delta_x)$.

Theorem 2.4. *(pages 62 and 63 of [8]) Let* $x \in \mathbb{R}^m$, *and let* $F := \int v \cdot \nu \, d\mathcal{H}^{m-1}$ *be the flux of a locally bounded Borel measurable vector field* $v : \mathbb{R}^m \to \mathbb{R}^m$.

(i) If v *is Lipschitz at* x, *then* F *is almost derivable at* x.

(ii) If v *is differentiable at* x, *then* F *is derivable at* x *and*

$$DF(x) = div\, v(x).$$

(iii) In $\mathbb{R}$, *the sufficient conditions given above in (i) and (ii) are also necessary.*

Note 2.2. *It follows easily from Theorem 2.3(i) and Theorem 2.4(iii) that if* F *is a charge in* $\mathbb{R}$, *which is derivable at* x, *then given* $\epsilon > 0$, *there exists* $\delta > 0$, *such that*

$$\left| \frac{F(E)}{|E|} - DF(x) \right| < \epsilon,$$

where $E = [y, z]$ *is such that* $x \in [y, z]$, *and* $0 < z - y < \delta$.

Additive Function of an Interval. See [9] (page 61) for the definition.

Note 2.3. *The additive function of an interval extends uniquely to an additive function defined over figures (see page 212 of [7] or page 61 of [9]).*

Definition 2.3. (page 106 of [9]) A sequence of sets (E_n) in $\mathbb{R}^m$ is said to tend to a point x if $x \in E_n$ for every n and $d(E_n) \to 0$ as $n \to \infty$.

Ordinary and Strong Derivates. See page 106 of [9] for the definitions of ordinary and strong derivates. We shall denote the ordinary upper derivate, ordinary lower derivate and the ordinary derivate (also referred to as ordinary derivative in [9]) of a set function F at $x \in \mathbb{R}^m$ by $\overline{F}(x)$, $\underline{F}(x)$ and $F'(x)$ respectively. Similarly, we shall denote the strong upper derivate, strong lower derivate and strong derivate (also referred to as strong derivative in [9]) of F at x by $\overline{F}_s(x)$, $\underline{F}_s(x)$ and $F'_s(x)$, respectively.

We state some theorems concerning these derivates.

Theorem 2.5. *([9], page 107) Let f be a Lebesgue integrable function in $\mathbb{R}^m$ and F be its indefinite integral. Then the following hold:*

(i) $\overline{F}_s(x) \leq f(x)$ at any point x at which the function f is upper semi-continuous.

(ii) $\underline{F}_s(x) \geq f(x)$ at any point x at which the function f is lower semi-continuous.

(iii) In particular therefore, $F'(x) = F'_s(x) = f(x)$ at any point x at which f is continuous.

The following two theorems are known as Ward's theorems, (see [9], Theorems 11.15 and 11.21).

Theorem 2.6. *Any additive function of an interval F is derivable (in the ordinary sense) at almost all points x at which either $\underline{F}(x) > -\infty$, or $\overline{F}(x) < \infty$.*

Theorem 2.7. *If F is an additive function of an interval, then*

(i) $F'(x) = \underline{F}_s(x) \neq \infty$ at almost all the points at which $\underline{F}_s(x) > -\infty$.

(ii) $F'(x) = \overline{F}_s(x) \neq -\infty$ at almost all the points at which $\overline{F}_s(x) < \infty$.

(iii) Thus, in particular, the function F is derivable in the strong sense at almost all the points at which both the extreme strong derivates $\underline{F}_s(x)$ and $\overline{F}_s(x)$ are finite.

The following result is stated in [8](page 61).

Lemma 2.1. *If F is derivable at a point $x \in \mathbb{R}^m$, then it is also derivable in the ordinary sense at that point, and the two derivates are equal.*

The HK-Integral. We briefly describe the Henstock-Kurzweil Integral over the real line. For details, refer to [1], [4], [10] or [12].

Definition 2.4. Let $I = [a, b]$ be a nondegenerate compact interval in $\mathbb{R}$.

(i) A partition (or a division) of I is a finite family of intervals $I_i := [x_i, x_{i+1}]$ such that
$$a = x_0 < x_1 < x_2 \cdots < x_n = b.$$

(ii) If $\mathcal{P} = \{I_i : i = 1, \dots n\}$ is a partition of an interval I such that for each subinterval I_i there is assigned a point $t_i \in I_i$, then we call t_i a tag of I_i. In this case we say that $\mathcal{P}$ is tagged and we write it as
$$\dot{\mathcal{P}} := \{(I_i, t_i) : i = 1, \dots n\} = \{(I_i, t_i)\}_{i=1}^{n}.$$

(iii) A function $\delta : I \to \mathbb{R}$ is said to be a gauge on I if $\delta(t) > 0$ for all $t \in I$.

(iv) Let $f : I \to \mathbb{R}$. For $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^{n}$, any tagged partition of I, the sum
$$S(f; \dot{\mathcal{P}}) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$
is called the Riemann sum of f corresponding to $\dot{\mathcal{P}}$.

(v) Let $\dot{\mathcal{P}} := \{(I_i, t_i) : i = 1, \dots, n\}$ be a tagged partition. If δ is a gauge on I, we say that $\dot{\mathcal{P}}$ is δ-fine if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for all $i = 1, \dots, n$.

(vi) A subpartition of I is a collection $\{J_j\}_{j=1}^{s}$ of nonoverlapping closed intervals in I. A tagged subpartition of I is a collection $\dot{\mathcal{P}}_0 := \{(J_j, t_j)\}_{j=1}^{s}$ of ordered pairs, consisting of intervals $\{J_j\}_{j=1}^{s}$ that form a subpartition of I, and tags $t_j \in J_j$ for $j = 1, \dots, s$. If δ is a gauge on I, we say that $\dot{\mathcal{P}}_0$ is δ-fine if $J_j \subset (t_j - \delta(t_j), t_j + \delta(t_j))$ for $j = 1, \dots, s$.

Lemma 2.2. *(Cousin's Lemma) If $I := [a, b]$ is a nondegenerate compact interval, and if δ is a gauge on I, then there exists a tagged partition of I that is δ-fine.*

Remark 2.1. Only a tagged partition can be δ-fine; hence it is not necessary to employ the word "tagged" while referring to δ-fine tagged partitions.

Definition 2.5. Let I be a nondegenerate compact interval in $\mathbb{R}$. Then a function $f : I \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable (or HK-integrable) on I if there exists a real number C such that for every $\epsilon > 0$, there exists a gauge δ_ϵ on I such that for every tagged partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ of I that is δ_ϵ-fine, $|S(f; \dot{\mathcal{P}}) - C| < \epsilon$. The family of all HK-integrable functions over $[a, b]$ is denoted by $\mathrm{HK}([a, b])$.

We now state and prove a lemma, which will be used in the next section.

Lemma 2.3. *Let* $P = \{([x_{i-1}, x_i], t_i) : i = 1, \ldots, p\}$ *be a* δ*-fine tagged partition of an interval* $[a, b]$*. Suppose* $\cup_{i=1}^{p-1}\{x_i\} \not\subset \mathbb{Q}$*. Then from* P*, we can construct another* δ*-fine tagged partition, say* Q*, such that each* x_i*, except possibly* a *or* b *is a rational number.*

Proof. We give the proof in steps.
Step 1. From the given δ-fine partition, we first construct a δ-fine partition where no point is the tag of two distinct subintervals. For this, if $t_i = t_{i+1}$ for some i, then as $t_i \in [x_{i-1}, x_i]$ and $t_i = t_{i+1} \in [x_i, x_{i+1}]$, we have $t_i = x_i = t_{i+1}$. Thus,

$$[x_{i-1}, x_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i)) \text{ and } [x_i, x_{i+1}] \subset (t_i - \delta(t_i), t_i + \delta(t_i)).$$

Hence,

$$[x_{i-1}, x_{i+1}] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$$

So we can consolidate the two abutting intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ in the partition to get one interval $[x_{i-1}, x_{i+1}]$ having t_i as the tag. Applying the same procedure to any two (consecutive) intervals having the same tag, we get a δ-fine tagged partition P' where no point is the tag of two distinct subintervals.

Step 2. Next, from the partition P', obtained in Step 1, we construct another δ-fine partition where each tag, except possibly a or b lies in the interior of each subinterval $[x_{i-1}, x_i]$. For this, suppose t_i, the tag of $[x_{i-1}, x_i]$, is the endpoint x_i. As no point is the tag of two distinct subintervals in P', we must have $t_{i+1} > t_i = x_i$. And

$$[x_i, x_{i+1}] \subset (t_{i+1} - \delta(t_{i+1}), t_{i+1} + \delta(t_{i+1})).$$

Choose a point

$$y \in (t_i, t_i + \delta(t_i)) \cap (t_i, t_{i+1}).$$

Then $\{([x_{i-1}, y], t_i), ([y, x_{i+1}], t_{i+1})\}$ is a δ-fine subpartition. Relabel y as x_i. Similarly, if t_i, the tag of $[x_{i-1}, x_i]$, is the endpoint x_{i-1}, then as $t_{i-1} < t_i$, we can choose some

$$y \in (t_i - \delta(t_i), t_i) \cap (t_{i-1}, t_i),$$

and relabel it as x_{i-1}. Applying this procedure to any tag (except possibly a and b), which is the endpoint of some subinterval, we construct a δ-fine partition P'' where each tag, except possibly a or b lies in the interior of each subinterval $[x_{i-1}, x_i]$.

Step 3. Finally, from P'', (as constructed in Step 2), we construct the partition Q with the required properties. Consider any x_i, where $x_i \notin \{a, b\}$. Then

$$x_i \in (t_i, t_i + \delta(t_i)) \cap (t_{i+1} - \delta(t_{i+1}), t_{i+1}) = I_i, \text{ say.}$$

If x_i is not a rational number, then we choose some rational number lying in the open interval I_i and replace x_i with this. Thus, we construct another δ-fine partition Q from P'' having the desired property. Note that the tags remain unchanged throughout the above three steps. □

The R-Integral. The R-integral was introduced by Pfeffer in [6] (called "v-integral" in that paper). It is also known as the BV-integral. The R-integral, defined on $\mathcal{BV}(\mathbb{R}^m)$, extends the Lebesgue integral in $\mathbb{R}^m$, and also, there is a very general Gauss-Green Theorem for this integral. Refer to [8] for details. To make the distinction between the R-integral and Lebesgue integral more apparent, we indicate the Lebesgue integral by employing the symbol $(L)\int$.

3 Fubini Type Theorems for the BV Integral.

In this section, we show that if an R-integrable function on $[a, b] \times [c, d]$ satisfies certain conditions, then an iterated integral exists over any subinterval of $[a, b] \times [c, d]$, and equals the double integral. We also present some examples satisfying those conditions in the next section.

Definition 3.1. Let F be a charge in $[a, b] \times [c, d]$. We say that F satisfies condition of X-type over $[a, b] \times [c, d]$, if there exists a countable subset C of $[c, d]$, such that, whenever $(x, y) \in [a, b] \times \{[c, d] \setminus C\}$,

$$\overline{F}_s(x, y) < \infty, \text{ and } \underline{F}_s(x, y) > -\infty. \tag{3.1}$$

Similarly, we say that F satisfies condition of Y-type over $[a, b] \times [c, d]$, if there exists a countable subset C of $[a, b]$, such that, whenever $(x, y) \in \{[a, b] \setminus C\} \times [c, d]$,

$$\overline{F}_s(x, y) < \infty, \text{ and } \underline{F}_s(x, y) > -\infty.$$

Note 3.1. *If $A := [a_1, b_1] \times [c_1, d_1] \subset [a, b] \times [c, d]$, then it can be easily shown that $F \llcorner A$ also satisfies (3.1) over $[a_1, b_1] \times \{[c_1, d_1] \setminus C\}$.*

We now state one of the main theorems of this section.

Theorem 3.1. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be R-integrable. Let F be the charge in $[a, b] \times [c, d]$, which is the indefinite R-integral of f.*

(i) *If the R-integral $\int_a^b f(x,y)\,dx$ exists for almost all $y \in [c,d]$, and F satisfies condition of X-type over $[a,b]\times[c,d]$, then the iterated R-integral $\int_c^d \int_a^b f\,dx\,dy$ exists, and*

$$\int_{[a,b]\times[c,d]} f = \int_c^d \int_a^b f\,dx\,dy.$$

(ii) *If the R-integral $\int_c^d f(x,y)dy$ exists for almost all $x \in [a,b]$, and F satisfies condition of Y-type over $[a,b]\times[c,d]$, then the iterated R-integral $\int_a^b \int_c^d f\,dy\,dx$ exists, and*

$$\int_{[a,b]\times[c,d]} f = \int_a^b \int_c^d f\,dy\,dx.$$

We first prove some results needed for the proof of the theorem.

Lemma 3.1. *Let F be a charge in $[a,b]\times[c,d]$. Define a function of an interval G in $\mathbb{R}$ by*

$$G([c_1,d_1]) := F([a,b]\times[c_1,d_1]),$$

for any subinterval $[c_1,d_1]\subset\mathbb{R}$. Then, G can be extended to a charge in $[c,d]$.

Proof. By additivity of F, it follows that G is an additive function of an interval in $\mathbb{R}$. Hence, G extends uniquely to an additive function over figures, which we again denote by G (see Note 2.3). By Theorem 2.1, any BV subset of $\mathbb{R}$ is equivalent to a unique figure. Define $G : \mathcal{BV}(\mathbb{R}) \to \mathbb{R}$ by

$$G(B) = G(\cup_{i=1}^k [c_i,d_i]),$$

if B is equivalent to $\cup_{i=1}^k [c_i,d_i]$. Thus, G is an additive function on $\mathcal{BV}(\mathbb{R})$. Now we prove that G is a charge in $[c,d]$. For this, choose $\epsilon > 0$, small enough so that

$$[a,b]\times[c,d] \subset B(1/\epsilon) \subset \mathbb{R}^2.$$

Then $[c,d] \subset (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$. Choose $\epsilon_1 > 0$, such that

$$\frac{1}{\epsilon_1} > \frac{\max\{1, b-a\}}{\epsilon} + 2. \tag{3.2}$$

It is clear that, $\epsilon_1 < \epsilon$. As F is a charge, by Theorem 2.2, given $\epsilon_1 > 0$, there exists $\eta_1 \in (0, b-a)$ such that

$$|F(E)| < \epsilon_1 \;\text{ whenever }\; |E| < \eta_1,\;\; ||E|| < \frac{1}{\epsilon_1},\; \text{and } E \in B(1/\epsilon_1).$$

Let $\eta = \frac{\eta_1}{b-a}$. Then $\eta \in (0,1)$. Let $B \subset (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ be any BV set such that $|B| < \eta$, and $||B|| < 1/\epsilon$. Then B is equivalent to a unique figure, say B_1; i.e., $|B_1| = |B|$. Also, $||B_1|| = ||B|| < \frac{1}{\epsilon}$. Now, $[c,d] \cap B_1$ is equivalent to a unique figure, say B_2. Let $B_3 :=$ cl $(B_1 \setminus B_2)$. Then, B_2 and B_3 are nonoverlapping figures. Let $B_2 = \cup_{i=1}^n [c_i, d_i]$. It is easy to see that $||B_2|| \leq ||B_1||$. Thus,

$$||B_2|| = 2n < \frac{1}{\epsilon}.$$

Let $E = [a,b] \times B_2$. Then

$$E \subset B(1/\epsilon) \subset B(1/\epsilon_1), \quad \text{and} \quad |E| = (b-a)|B_2| < (b-a)\eta = \eta_1.$$

Further,

$$\begin{aligned} ||E|| = \mathcal{H}^1(\partial_* E) &= 2n(b-a) + 2\sum_{i=1}^n (d_i - c_i) = 2n(b-a) + 2|B_2| \\ &< 2n(b-a) + 2\eta < 2n(b-a) + 2 < \frac{b-a}{\epsilon} + 2 \\ &\leq \frac{\max\{1, b-a\}}{\epsilon} + 2 < \frac{1}{\epsilon_1}, \end{aligned}$$

by (3.2). Thus, by Theorem 2.2, $|F(E)| < \epsilon_1$. Also, as F is a charge in $[a,b] \times [c,d]$, $F([a,b] \times B_3) = 0$; i.e., $G(B_3) = 0$.Thus,

$$G(B_1) = G(B_2 \cup B_3) = G(B_2) + G(B_3) = G(B_2).$$

Hence,

$$|G(B)| = |G(B_1)| = |G(B_2)| = |F(E)| < \epsilon_1 < \epsilon.$$

Thus, given $\epsilon > 0$, there exists $\eta > 0$ such that

$$|G(B)| < \epsilon \quad \text{whenever} \quad |B| < \eta, \quad ||B|| < \frac{1}{\epsilon}, \text{ and } B \in (-1/\epsilon, 1/\epsilon).$$

Hence, by Theorem 2.2, G is a charge. Also, it is clear that G is a charge in $[c,d]$. □

Definition 3.2. For a given charge F in $[a,b] \times [c,d]$, the charge G in $[c,d]$, as given by Lemma 3.1, is called the y-charge associated to F. Similarly, we can also define a charge H in $[a,b]$ and call it the x-charge associated to F.

Lemma 3.2. *Let F be a charge in $[a,b] \times [c,d]$. Suppose it satisfies condition of X-type over $[a,b] \times [c,d]$, and C is the set mentioned in Definition 3.1. Then for $y_0 \in (c,d) \setminus C$, there exist positive real numbers k and δ such that*

$$|F([a,b] \times [y_0, y_0 + y])| \leq ky$$

and

$$|F([a,b]\times[y_0-y,y_0])|\le ky$$

for every $y\in(0,\delta)$.

PROOF. Choose any $y_0\in(c,d)\backslash C$. Then for any $x\in[a,b]$, we have $\overline{F}_s(x,y_0)<\infty$, and $\underline{F}_s(x,y_0)>-\infty$; i.e.,

$$\begin{aligned}-\infty<\underline{F}_s(x,y_0)&=\sup_{\alpha>0}\left\{\inf_{0<d(B)<\alpha}\frac{F(B)}{|B|}\right\}\\ &\le\overline{F}_s(x,y_0)=\inf_{\alpha>0}\left\{\sup_{0<d(B)<\alpha}\frac{F(B)}{|B|}\right\}<\infty,\end{aligned}$$

where B is any cell in $\mathbb{R}^2$ containing the point (x,y_0). Thus, given $\epsilon>0$, there exists $\eta_x>0$, such that

$$\inf_{0<d(B)<\alpha}\frac{F(B)}{|B|}>\underline{F}_s(x,y_0)-\epsilon,\ \text{and}\ \sup_{0<d(B)<\alpha}\frac{F(B)}{|B|}<\overline{F}_s(x,y_0)+\epsilon,$$

whenever $0<\alpha\le\eta_x$. Hence, given $\epsilon>0$, there exists $\eta_x>0$, such that

$$\frac{F(B)}{|B|}>\underline{F}_s(x,y_0)-\epsilon,\ \text{and}\ \frac{F(B)}{|B|}<\overline{F}_s(x,y_0)+\epsilon, \tag{3.3}$$

whenever $0<d(B)<\eta_x$ and $(x,y_0)\in B$. We may assume that $\eta_x\le\min\{d-y_0,y_0-c\}$. Define a gauge γ on $[a,b]$, by

$$\gamma(x)=\frac{\eta_x}{\sqrt{5}}. \tag{3.4}$$

By Lemma 2.2, there exists a γ-fine partition $\{([x_{i-1},x_i],t_i):i=1,\dots,p\}$ of $[a,b]$. Let $\delta=\min\{\gamma(t_i):i=1,\dots,p\}$. Let

$$k_1=\max\{|\underline{F}_s(t_i,y_0)|,|\overline{F}_s(t_i,y_0)|:i=1,\dots,p\}+\epsilon. \tag{3.5}$$

Let $y\in(0,\delta)$, and $i\in\{1,\dots,p\}$. If

$$\frac{F([x_{i-1},x_i]\times[y_0,y_0+y])}{|[x_{i-1},x_i]\times[y_0,y_0+y]|}\ge 0,$$

using (3.3) and (3.4), we get

$$\frac{|F([x_{i-1},x_i]\times[y_0,y_0+y])|}{|[x_{i-1},x_i]\times[y_0,y_0+y]|}<\overline{F}_s(t_i,y_0)+\epsilon. \tag{3.6}$$

And if

$$\frac{F([x_{i-1},x_i]\times[y_0,y_0+y])}{|[x_{i-1},x_i]\times[y_0,y_0+y]|}<0,$$

using (3.3) and (3.4), we get

$$\frac{|F([x_{i-1},x_i]\times[y_0,y_0+y])|}{|[x_{i-1},x_i]\times[y_0,y_0+y]|}<|\underline{F}_s(t_i,y_0)|+\epsilon. \tag{3.7}$$

Thus,

$$\begin{aligned}\frac{|F([a,b]\times[y_0,y_0+y])|}{|[a,b]\times[y_0,y_0+y]|}&=\frac{|\sum_{i=1}^{p}F([x_{i-1},x_i]\times[y_0,y_0+y])|}{\sum_{i=1}^{p}|[x_{i-1},x_i]\times[y_0,y_0+y]|}\\&\le\frac{\sum_{i=1}^{p}|F([x_{i-1},x_i]\times[y_0,y_0+y])|}{\sum_{i=1}^{p}|[x_{i-1},x_i]\times[y_0,y_0+y]|}\\&\le\max\left\{\frac{|F([x_{i-1},x_i]\times[y_0,y_0+y])|}{|[x_{i-1},x_i]\times[y_0,y_0+y]|}:i=1,\ldots,p\right\}\\&<k_1,\ \text{using (3.5), (3.6) and (3.7);}\end{aligned}$$

i.e.,

$$\frac{|F([a,b]\times[y_0,y_0+y])|}{(b-a)y}<k_1;$$

i.e.,

$$|F([a,b]\times[y_0,y_0+y])|<(b-a)k_1y.$$

Let $k=(b-a)k_1$. Then the real numbers k and $\delta>0$ are such that

$$|F([a,b]\times[y_0,y_0+y])|<ky$$

whenever $y\in(0,\delta)$. Similarly,

$$|F([a,b]\times[y_0-y,y_0])|<ky,$$

whenever $y\in(0,\delta)$. □

Lemma 3.3. *Let F be as given in Lemma 3.2. Let G be the y-charge associated to F. Let $v:\mathbb{R}\to\mathbb{R}$ be a vector field associated to G (as given by Theorem 2.3). Then,*

(i) v is Lipschitz at y_0 for every $y_0\in(c,d)\setminus C$.

(ii) The charge G in $[c,d]$ is derivable a.e. over $[c,d]$, and DG is R-integrable over $[c,d]$, and

$$(R)\int_{[c,d]}DG=G([c,d]).$$

PROOF. (i) Let $y_0 \in (c,d) \setminus C$. Let k and δ be as in Lemma 3.2. We have

$$|v(y_0+y)-v(y_0)| = |G([y_0,y_0+y])| = |F([a,b]\times[y_0,y_0+y])| < ky$$

whenever $y \in (0,\delta)$. Similarly, $|v(y_0)-v(y_0-y)| < ky$ whenever $y \in (0,\delta)$. Thus, v is Lipschitz at y_0.

(ii) By part (i) and Theorem 2.4(i), G is almost derivable at each $y \in (c,d)\setminus C$. By Theorem 5.1.12 of [8], $\mathfrak{div}\ v(y)$; i.e., $DG(y)$ is defined a.e. over [c,d], and is R-integrable over [c,d], and

$$(R)\int_c^d DG = (L)\int_{\partial_*([c,d])} v\cdot \nu_{[c,d]}\, d\mathcal{H}^0.$$

Now, $\partial_*([c,d]) = \{c,d\}$, and using Definition 2.1, we get

$$\nu_{[c,d]}(c) = -1 \text{ and } \nu_{[c,d]}(d) = 1.$$

Hence,

$$(R)\int_c^d DG = v(d)-v(c) = G([c,d]). \qquad \square$$

Lemma 3.4. *Let F, G and v be as in Lemma 3.3. Let $e \in (a,b)$. Let F_1 be the reduction of F to $[a,e]\times[c,d]$, and F_2 be the reduction of F to $[e,b]\times[c,d]$. Let G_1 be the y-charge associated to the charge F_1, and let G_2 be the y-charge associated to F_2. Then there exists a null subset N of $[c,d]$, such that if $y \in (c,d)\setminus N$, then $DG_1(y), DG_2(y)$ and $DG(y)$ exist, and $DG(y) = DG_1(y) + DG_2(y)$.*

PROOF. As F satisfies condition of X-type over $[a,b]\times[c,d]$, it follows that F_1 and F_2 also satisfy condition of X-type over $[a,e]\times[c,d]$ and $[e,b]\times[c,d]$, respectively (see Note 3.1). Hence by Lemma 3.3, there exist null sets N_1 and N_2 such that $DG_1(y)$ exists over $[c,d]\setminus N_1$, and $DG_2(y)$ exists over $[c,d]\setminus N_2$. Let $N := N_1 \cup N_2$. Choose any $y_0 \in (c,d)\setminus N$. Let v_1 and v_2 be continuous functions associated to the charges G_1 and G_2 respectively. Then by Theorem 2.4 (iii), v_1 and v_2 are differentiable at y_0, and

$$v_1'(y_0) = DG_1(y_0) \text{ and } v_2'(y_0) = DG_2(y_0).$$

So given $\epsilon > 0$, there exists $\delta_1 > 0$ such that

$$\left|\frac{v_1(y_0+y)-v_1(y_0)}{y} - v_1'(y_0)\right| < \frac{\epsilon}{2}\ \forall\, y \in (0,\delta_1),$$

and there exists $\delta_2 > 0$ such that

$$\left|\frac{v_2(y_0+y)-v_2(y_0)}{y} - v_2'(y_0)\right| < \frac{\epsilon}{2}\ \forall\ y \in (0,\delta_2).$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for all $y \in (0,\delta)$,

$$\begin{aligned}
&\left|\frac{v(y_0+y)-v(y_0)}{y} - \{v_1'(y_0)+v_2'(y_0)\}\right| \\
=&\left|\frac{G([y_0,y_0+y])}{y} - \{v_1'(y_0)+v_2'(y_0)\}\right| \\
=&\left|\frac{F([a,b]\times[y_0,y_0+y])}{y} - \{v_1'(y_0)+v_2'(y_0)\}\right| \\
=&\left|\frac{F([a,e]\times[y_0,y_0+y])+F([e,b]\times[y_0,y_0+y])}{y} - \{v_1'(y_0)+v_2'(y_0)\}\right| \\
=&\left|\frac{G_1([y_0,y_0+y])+G_2([y_0,y_0+y])}{y} - \{v_1'(y_0)+v_2'(y_0)\}\right| \\
\leq&\left|\frac{v_1(y_0+y)-v_1(y_0)}{y} - v_1'(y_0)\right| + \left|\frac{v_2(y_0+y)-v_2(y_0)}{y} - v_2'(y_0)\right| \\
<&\frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, the right hand derivative of v exists at y_0, and equals $v_1'(y_0)+v_2'(y_0)$. Similarly, it can be shown that the left hand derivative of v also exists at y_0, and has the same value. Thus, v is differentiable at y_0,

$$v'(y_0) = v_1'(y_0)+v_2'(y_0).$$

Hence by Theorem 2.4(ii), $DG(y_0)$ exists and equals $v'(y_0)$, i.e., $DG_1(y_0) + DG_2(y_0)$. □

Theorem 3.2. *Let f satisfy the conditions of part (i) of Theorem 3.1. Let G be the y-charge associated to F. Then*

(i) $F_s'(x)$ exists a.e. over $[a,b]\times[c,d]$, and equals $f(x)$ a.e. over $[a,b]\times[c,d]$.

(ii)

$$\int_a^b f(x,y)\,dx = DG(y) \text{ for almost all } y \in [c,d].$$

PROOF. (i) As F is the indefinite R-integral of f, so almost everywhere over $[a,b]\times[c,d]$, we have

$$DF(x) = f(x). \tag{3.8}$$

By Lemma 2.1, F is derivable in the ordinary sense whenever $DF(x)$ exists, and

$$F'(x) = DF(x). \tag{3.9}$$

Also, $\overline{F}_s(x,y) < \infty$, and $\underline{F}_s(x,y) > -\infty$ whenever $(x,y) \in [a,b] \times \{[c,d] \setminus C\}$, where C is a countable subset of $[c,d]$. Hence, by Ward's Second Theorem (Theorem 2.7), F is derivable in the strong sense a.e. over $[a,b] \times [c,d]$. Also, whenever $F'_s(x)$ exists, then $F'(x)$ also exists, and $F'_s(x) = F'(x)$ (see page 106 of [8]). Now, it follows from (3.8) and (3.9), that

$$F'_s(x) = f(x) \text{ a.e. over } [a,b] \times [c,d].$$

(ii) First, we construct a null set $N \subset [c,d]$. For any $\{e,f\} \subset [a,b]$, $\quad e \neq f$, let $F_{\{e,f\}}$ denote the reduction of F to $[e,f] \times [c,d]$ or $[f,e] \times [c,d]$, depending on whether $e < f$ or $e > f$. Let $G_{\{e,f\}}$ be the y-charge associated to $F_{\{e,f\}}$. By Lemma 3.3(ii), $DG_{\{e,f\}}(x)$ exists almost everywhere over $[c,d]$. Let $N_{\{e,f\}}$ denote the null set such that $DG_{\{e,f\}}(x)$ exists over $[c,d] \setminus N_{\{e,f\}}$. If $e = f$, we define $N_{\{e,f\}} := \emptyset$. Let $\{r_1, r_2, \dots\}$ be an enumeration of all the rationals contained in $[a,b]$. For all $m = 1, 2, \dots,$ let

$$N_m := N_{\{a,r_m\}} \cup N_{\{b,r_m\}} \cup \{\cup_{n=1}^{\infty} N_{\{r_n,r_m\}}\}.$$

Let N' be the null set $\{y \in [c,d] : \int_a^b f(x,y)\,dx \text{ does not exist }\}$. Given $y \in [c,d]$, let

$$A_y := \{x \in [a,b] : F'_s(x,y) \neq f(x,y)\}.$$

In view of part (i), there exists a null set, say $N'' \subset [c,d]$, such that $|A_y| = 0$ whenever $y \in [c,d] \setminus N''$. (See [5], Theorem 14.2). And C is the countable set mentioned in part (i) above. Define

$$N := N_{\{a,b\}} \cup N' \cup N'' \cup C \cup \{\cup_{m=1}^{\infty} N_m\}.$$

We shall show that for every $y_0 \in (c,d) \setminus N$,

$$h(y_0) := \int_a^b f(x,y_0)\,dx = DG(y_0).$$

Since $N' \subset N$, and $N_{\{a,b\}} \subset N$, both $h(y_0)$ and $DG(y_0)$ exist. If possible, suppose $DG(y_0) > h(y_0)$. Then there exists $\epsilon > 0$, such that

$$DG(y_0) > h(y_0) + \epsilon(1 + 2b - 2a). \tag{3.10}$$

Since $N'' \subset N$,

$$|\{x \in [a,b] : f(x,y_0) \neq F'_s(x,y_0)\}| = 0.$$

By definition, whenever it exists, $F'_s(x, y_0) = \overline{F}_s(x, y_0)$. Also, as $C \subset N$, for all $x \in [a, b]$,

$$-\infty < \underline{F}_s(x, y_0) \leq \overline{F}_s(x, y_0) < \infty.$$

Wherever $f(x, y_0) \neq \overline{F}_s(x, y_0)$, we redefine $f(x, y_0) := \overline{F}_s(x, y_0)$. Note that redefining f over this subset of $[a, b] \times \{y_0\}$ does not change F (see page 186 of [8]), and hence the values of the various derivates remain unchanged over $[a, b] \times [c, d]$. Similarly, the value of $h(y_0)$ remains unchanged. Choose any $(x, y_0) \in [a, b] \times \{y_0\}$. Then as in the proof of Lemma 3.2, given $\epsilon > 0$, there exists $\gamma_x > 0$, such that

$$\frac{F(A)}{|A|} < \overline{F}_s(x, y_0) + \epsilon = f(x, y_0) + \epsilon \tag{3.11}$$

whenever A is a cell such that $0 < d(A) < \gamma_x$ and $(x, y_0) \in A$.

In view of Proposition 6.8 of [6], $f(x, y_0)$, as a function of x, is HK-integrable on $[a, b]$. Thus, given $\epsilon > 0$, there exists a gauge δ over $[a, b]$ such that if P is any δ-fine partition of $[a, b]$, then $|S(f; P) - h(y_0)| < \epsilon$. We may assume that

$$\delta(x) \leq \frac{\gamma_x}{\sqrt{5}} \ \ \forall\, x \in [a, b]. \tag{3.12}$$

By Lemma 2.2, there exists a δ-fine partition, say $P := \{([x_{i-1}, x_i], t_i) : i = 1, \ldots, p\}$, of $[a, b]$. Hence,

$$\Big|\sum_{i=1}^{p} f(t_i, y_0)(x_i - x_{i-1}) - h(y_0)\Big| < \epsilon;$$

i.e.,

$$\sum_{i=1}^{p} f(t_i, y_0)(x_i - x_{i-1}) < h(y_0) + \epsilon. \tag{3.13}$$

Also, in view of Lemma 2.3, we may assume that each x_i, except possibly a or b is a rational number. Thus, for any $i \in \{1, \ldots, p\}$, $DG_{\{x_{i-1}, x_i\}}(y_0)$ exists, as $y_0 \notin N$. Hence, by Note 2.2, given $\epsilon > 0$, there exists $r_i \in (0, \delta(t_i))$ such that

$$\Big|\frac{G_{\{x_{i-1}, x_i\}}([y_0, y_0 + r_i])}{r_i} - DG_{\{x_{i-1}, x_i\}}(y_0)\Big| < \epsilon(x_i - x_{i-1});$$

i.e.,

$$\Big|\frac{F([x_{i-1}, x_i] \times [y_0, y_0 + r_i])}{r_i} - DG_{\{x_{i-1}, x_i\}}(y_0)\Big| < \epsilon(x_i - x_{i-1}),$$

implying

$$DG_{\{x_{i-1},x_i\}}(y_0) < \frac{F([x_{i-1},x_i]\times[y_0,y_0+r_i])}{r_i} + \epsilon(x_i - x_{i-1}) \ \forall\ i = 1,\dots,p.$$

Thus,

$$\sum_{i=1}^{p} DG_{\{x_{i-1},x_i\}}(y_0) < \sum_{i=1}^{p}\left(\frac{F([x_{i-1},x_i]\times[y_0,y_0+r_i])}{r_i}\right) + \epsilon(b-a).$$

Since, by Lemma 3.4,

$$\sum_{i=1}^{p} DG_{\{x_{i-1},x_i\}}(y_0) = DG(y_0),$$

the above inequality gives,

$$DG(y_0) < \sum_{i=1}^{p}\left(\frac{F([x_{i-1},x_i]\times[y_0,y_0+r_i])}{r_i}\right) + \epsilon(b-a). \tag{3.14}$$

Since $r_i < \delta(t_i)$, using (3.11) and (3.12) we get,

$$\frac{F([x_{i-1},x_i]\times[y_0,y_0+r_i])}{|[x_{i-1},x_i]\times[y_0,y_0+r_i]|} < f(t_i,y_0) + \epsilon;$$

i.e.,

$$\frac{F([x_{i-1},x_i]\times[y_0,y_0+r_i])}{r_i} < f(t_i,y_0)(x_i - x_{i-1}) + \epsilon(x_i - x_{i-1}).$$

Thus,

$$\begin{aligned}\sum_{i=1}^{p}\Big(\frac{F([x_{i-1},x_i]\times[y_0,y_0+r_i])}{r_i}\Big) &< \sum_{i=1}^{p} f(t_i,y_0)(x_i - x_{i-1}) + \epsilon(b-a)\\ &< h(y_0) + \epsilon + \epsilon(b-a),\end{aligned} \tag{3.15}$$

using (3.13). Finally, by (3.14) and (3.15), we get

$$DG(y_0) < h(y_0) + \epsilon(1 + 2b - 2a),$$

which contradicts (3.10). Hence it is not possible that $DG(y_0) > h(y_0)$. Similarly, it can be shown that it is not possible to have $DG(y_0) < h(y_0)$. We give a brief sketch of the proof. Suppose, if possible, that $DG(y_0) < h(y_0)$. Then there exists $\epsilon > 0$, such that

$$DG(y_0) < h(y_0) - \epsilon(1 + 2b - 2a).$$

Similarly as above, whenever $f(x, y_0) \neq \underline{F}_s(x, y_0)$, redefine f by

$$f(x, y_0) := \underline{F}_s(x, y_0).$$

Proceeding further, we can obtain the following inequalities in place of (3.11), (3.13), (3.14), and (3.15).

$$\begin{aligned}
\frac{F(A)}{|A|} &> f(x, y_0) - \epsilon, \\
\sum_{i=1}^{p} f(t_i, y_0)(x_i - x_{i-1}) &> h(y_0) - \epsilon, \\
DG(y_0) &> \sum_{i=1}^{p} \Big(\frac{F([x_{i-1}, x_i] \times [y_0, y_0 + r_i])}{r_i} \Big) - \epsilon(b - a), \\
\sum_{i=1}^{p} \frac{F([x_{i-1}, x_i] \times [y_0, y_0 + r_i])}{r_i} &> h(y_0) - \epsilon - \epsilon(b - a).
\end{aligned}$$

The last two inequalities imply that

$$DG(y_0) > h(y_0) - \epsilon(1 + 2b - 2a).$$

So we again get a contradiction. Hence, $DG(y_0) = h(y_0)$. □

Proof of Theorem 3.1. We give the proof of part (i). The proof of part (ii) is similar. By Lemma 3.3(ii), $DG(y)$ exists a.e. over $[c, d]$, is R-integrable over $[c, d]$, and

$$G([c, d]) = \int_c^d DG(y)\, dy = \int_c^d \int_a^b f(x, y)\, dx\, dy,$$

by Theorem 3.2(ii); i.e., the iterated R-integral $\int_c^d \int_a^b f\, dx\, dy$ exists. And by definition, $G([c, d]) = F([a, b] \times [c, d])$. Hence,

$$\int_{[a,b]\times[c,d]} f = \int_c^d \int_a^b f\, dx\, dy. \qquad \square$$

We give next an extension of Theorem 3.1(i).

Theorem 3.3. *Suppose a function $f : [a,b] \times [c,d] \to \mathbb{R}$ is R-integrable. Let the charge F be the indefinite R-integral of f. Suppose the conditions of part (i) of Theorem 3.1 are satisfied. Then over any subinterval $[a_1, b_1] \times [c_1, d_1]$ of $[a,b] \times [c,d]$, the iterated R-integral $\int_{c_1}^{d_1} \int_{a_1}^{b_1} f\,dx\,dy$ exists, and*

$$\int_{[a_1,b_1]\times[c_1,d_1]} f = \int_{c_1}^{d_1} \int_{a_1}^{b_1} f\,dx\,dy.$$

PROOF. Let $A = [a_1, b_1] \times [c_1, d_1]$. By Proposition 5.1.6 of [8], the charge $F \text{L} A$ is the indefinite R-integral of $f|_A$. Now, $F \text{L} A$ satisfies condition of X-type over $[a_1, b_1] \times [c_1, d_1]$ (see Note 3.1). Again, by Proposition 5.1.6 of [8], if a function is R-integrable over the interval $[a,b]$, then it is also R-integrable over the subinterval $[a_1, b_1]$. Thus, conditions of part (i) of Theorem 3.1 are also satisfied over $[a_1, b_1] \times [c_1, d_1]$. Hence, the result follows by Theorem 3.1(i). □

Note 3.2. *Similarly, we also have the corresponding extension of Theorem 3.1(ii).*

Next we give another condition under which the iterated integral equals the double integral.

Theorem 3.4. *Suppose a function $f : [a,b] \times [c,d] \to \mathbb{R}$ is R-integrable. Let the charge F be the indefinite R-integral of f.*

(i) Suppose the following conditions are satisfied:

(a) The R-integral $\int_a^b f(x,y)\,dx$ exists for almost all $y \in [c,d]$.

(b) There exists a countable subset C of $[c,d]$, such that, whenever $(x,y) \in [a,b) \times \{[c,d] \setminus C\}$,

$$\overline{F}_s(x,y) < \infty, \text{ and } \underline{F}_s(x,y) > -\infty.$$

(c) There exists a sequence $(\alpha_n)_{n=1}^{\infty}$ in (a,b) such that $\alpha_n \to b$ and there exist functions $\kappa_1,\ \kappa_2 \in R([c,d])$, such that

$$\kappa_1(y) \le \int_a^{\alpha_n} f(x,y)\,dx \le \kappa_2(y) \ \ \forall\, n = 1, 2, \ldots, \text{ for almost all } y \in [c,d].$$

Then the iterated R-integral $\int_c^d \int_a^b f\,dx\,dy$ exists, and

$$\int_{[a,b]\times[c,d]} f = \int_c^d \int_a^b f\,dx\,dy.$$

(ii) *Under analogous conditions, the iterated R-integral* $\int_a^b \int_c^d f\,dy\,dx$ *exists, and*

$$\int_{[a,b]\times[c,d]} f = \int_a^b \int_c^d f\,dy\,dx.$$

PROOF. We prove part (i). Define $h,\ h_n : [c,d] \to \mathbb{R}$ by

$$h(y) := \begin{cases} \int_a^b f(x,y)\,dx, & \text{whenever the integral exists,} \\ 0, & \text{otherwise.} \end{cases}$$

and

$$h_n(y) := \begin{cases} \int_a^{\alpha_n} f(x,y)\,dx, & \text{whenever } \int_a^b f(x,y)\,dx \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

It follows from Theorem 2.3 (ii) that for a fixed $y \in [c,d]$ for which $\int_a^b f(x,y)\,dx$ exists, $\int_a^x f(x,y)\,dx$ is a continuous function of x. Hence, as $\alpha_n \to b$, we have

$$h_n \to h \text{ over } [c,d].$$

Let $A_n := [a,\alpha_n] \times [c,d]$, and $A := [a,b] \times [c,d]$. By Theorem 3.1(i), $h_n \in R([c,d])$ for all $n = 1, 2, \ldots$, and

$$F(A_n) = \int_c^d h_n(y)dy.$$

Since $\{A_n\} \to A$, and F is $\mathfrak{T}$-continuous,

$$\lim_{n\to\infty} F(A_n) = F(A).$$

Thus, it follows from Corollary 5.1.5(ii) of [8] that $h \in R([c,d])$ and

$$F(A) = \int_c^d h(y)dy. \qquad \square$$

We now give an extension of Theorem 3.4(i).

Theorem 3.5. *Suppose* f *and* F *satisfy the conditions of Theorem 3.4(i). Then over any subinterval* $[a_1,b_1] \times [c_1,d_1]$ *of* $[a,b] \times [c,d]$, *the iterated R-integral* $\int_{c_1}^{d_1} \int_{a_1}^{b_1} f\,dx\,dy$ *exists, and*

$$\int_{[a_1,b_1]\times[c_1,d_1]} f = \int_{c_1}^{d_1} \int_{a_1}^{b_1} f\,dx\,dy.$$

PROOF. Consider any nondegenerate subinterval $[a_1, b_1] \times [c_1, d_1]$. If $b_1 < b$, the conditions of Theorem 3.1(i) are satisfied. If $a_1 = a$, and $b_1 = b$, then the conditions of Theorem 3.4(i) are satisfied. Thus in both the above cases, the iterated integral exists and equals the double integral. Now suppose $a_1 > a$, and $b_1 = b$. Then over $[a, a_1] \times [c_1, d_1]$, the iterated integral exists and equals the double integral (by Theorem 3.1(i)), i.e.,

$$\int_{[a,a_1]\times[c_1,d_1]} f = \int_{c_1}^{d_1} \int_a^{a_1} f\,dx\,dy.$$

And by Theorem 3.4(i), $\int_{c_1}^{d_1} \int_a^b f\,dx\,dy$ exists, and

$$\int_{[a,b]\times[c_1,d_1]} f = \int_{c_1}^{d_1} \int_a^{b} f\,dx\,dy.$$

It follows from the above two equations that $\int_{a_1}^{b} f\,dx = \int_a^b f\,dx - \int_a^{a_1} f\,dx$ is integrable over $[c_1, d_1]$ (see Proposition 5.1.3 of [8]), and

$$\int_{[a_1,b]\times[c_1,d_1]} f = \int_{c_1}^{d_1} \int_{a_1}^{b} f\,dx\,dy,$$

in view of Proposition 5.1.8 of [8]. $\square$

We now state another theorem which can obviously be proved in the same way as Theorem 3.4(i) and Theorem 3.5.

Theorem 3.6. *Suppose a function $f : [a, b] \times [c, d] \to \mathbb{R}$ is R-integrable. Let F be the indefinite R-integral of f. Suppose the following conditions are satisfied:*

(a) The R-integral $\int_a^b f(x, y)\,dx$ exists for almost all $y \in [c, d]$.

(b) There exists a countable subset C of $[c, d]$, such that, whenever $(x, y) \in (a, b] \times \{[c, d] \setminus C\}$,

$$\overline{F}_s(x, y) < \infty, \text{ and } \underline{F}_s(x, y) > -\infty.$$

(c) There exists a sequence $(\alpha_n)_{n=1}^{\infty}$ in (a, b) such that $\alpha_n \to a$ and there exist functions $\kappa_1,\ \kappa_2 \in R([c, d])$, such that

$$\kappa_1(y) \le \int_{\alpha_n}^{b} f(x, y)\,dx \le \kappa_2(y) \ \ \forall\, n = 1, 2, \ldots, \text{ for almost all } y \in [c, d].$$

Then over any subinterval $[a_1, b_1] \times [c_1, d_1]$ *of* $[a, b] \times [c, d]$, *the iterated R-integral* $\int_{c_1}^{d_1} \int_{a_1}^{b_1} f \, dx \, dy$ *exists, and*

$$\int_{[a_1,b_1]\times[c_1,d_1]} f = \int_{c_1}^{d_1} \int_{a_1}^{b_1} f \, dx \, dy.$$

Another condition under which the iterated integral equals the double integral is given in the next theorem. The proof is similar to that of Theorem 3.4(i).

Theorem 3.7. *Suppose a function* $f : [a, b] \times [c, d] \to \mathbb{R}$ *is R-integrable. Let* F *be the indefinite R-integral of* f. *Suppose the following conditions are satisfied:*

(a) The R-integral $\int_a^b f(x, y) \, dx$ *exists for almost all* $y \in [c, d]$.

(b) There exists a countable subset C *of* $[c, d]$, *such that, whenever* $(x, y) \in (a, b) \times \{[c, d] \setminus C\}$,

$$\overline{F}_s(x, y) < \infty, \text{ and } \underline{F}_s(x, y) > -\infty.$$

(c) There exist sequences $(\alpha_n)_{n=1}^{\infty}$ *and* $(\beta_n)_{n=1}^{\infty}$ *in* (a, b) *such that* $\alpha_n \to b$ *and* $\beta_n \to a$ *and there exist functions* $\kappa_1, \kappa_2 \in R([c, d])$, *such that*

$$\kappa_1(y) \le \int_{\beta_n}^{\alpha_n} f(x, y) \, dx \le \kappa_2(y) \ \ \forall \, n = 1, 2, \ldots, \text{ for almost all } y \in [c, d].$$

Then the iterated R-integral $\int_c^d \int_a^b f \, dx \, dy$ *exists, and*

$$\int_{[a,b]\times[c,d]} f = \int_c^d \int_a^b f \, dx \, dy.$$

Remark 3.1. We do not know whether the iterated integral $\int_c^d \int_{a_1}^b f \, dx \, dy$, where $a_1 \in (a, b)$, exists for the function f given in the above Theorem 3.7. However, by applying Theorem 3.1(i), we can see that for each $x \in (a_1, b)$, the iterated R-integral $\int_c^d \int_{a_1}^x f \, dx \, dy$ exists, and

$$\int_{[a_1,x]\times[c,d]} f = \int_c^d \int_{a_1}^x f \, dx \, dy.$$

Hence it follows that $\lim_{x\to b-} \int_c^d \int_{a_1}^x f \, dx \, dy$ exists, and equals $\int_{[a_1,b]\times[c,d]} f$.

We now present the theorem on the existence of the iterated R-integral and its equality with the double R-integral over any subinterval in the most general form.

Theorem 3.8. *Suppose a function $f : [a, b] \times [c, d] \to \mathbb{R}$ is R-integrable. Let the charge F be the indefinite R-integral of f. Suppose*

$$[a, b] \times [c, d] = \cup_{i=1}^{n} A_i,$$

where A_i's are nonoverlapping cells, such that over each A_i, the conditions of either Theorem 3.1(i) or Theorem 3.4(i) or Theorem 3.6 are satisfied. Then over any subinterval $[\alpha, \beta] \times [\gamma, \delta]$ of $[a, b] \times [c, d]$, the iterated R-integral $\int_\gamma^\delta \int_\alpha^\beta f\, dx\, dy$ exists, and

$$\int_{[\alpha,\beta]\times[\gamma,\delta]} f = \int_\gamma^\delta \int_\alpha^\beta f\, dx\, dy.$$

PROOF. Given any subinterval $B := [\alpha, \beta] \times [\gamma, \delta]$ of $[a, b] \times [c, d]$, we have

$$B = \cup_{i=1}^{n} B \cap A_i = \cup_{j=1}^{p} B_j,$$

where $B_j = [a_j, b_j] \times [c_j, d_j]$ is a cell for all $j = 1, \ldots p$. The $c_j' s$ and $d_j' s$ form a partition of $[\gamma, \delta]$, say $P = \{\gamma = e_0, \ldots, e_q = \delta\}$. Let

$$C_r := [\alpha, \beta] \times [e_{r-1}, e_r] \ \ \forall\, r = 1, \ldots, q.$$

Clearly, $C_r = \cup_{k=1}^{s_r} \{[\alpha_{r_k}, \beta_{r_k}] \times [e_{r-1}, e_r]\}$, where $[\alpha_{r_k}, \beta_{r_k}] \times [e_{r-1}, e_r]$ is a subset of some A_i, for each $k = 1, \ldots, s_r$. Thus, over $[\alpha_{r_k}, \beta_{r_k}] \times [e_{r-1}, e_r]$, the iterated integral exists and equals the double integral in view of Theorems 3.3, 3.5 and 3.6. Hence, over each C_r, the iterated integral exists (by Proposition 5.1.3 of [8]) and equals the double integral (by Proposition 5.1.8 of [8]). Again, it follows from Proposition 5.1.8 of [8] that over $[\alpha, \beta] \times [\gamma, \delta]$, the iterated integral exists and equals the double integral. $\square$

At this stage we would like to pose the following questions.

Question 1. *Suppose a function $f : [a, b] \times [c, d] \to \mathbb{R}$ is R-integrable. If over any subinterval $[\alpha, \beta] \times [\gamma, \delta]$ of $[a, b] \times [c, d]$, the iterated R-integral $\int_\gamma^\delta \int_\alpha^\beta f\, dx\, dy$ exists, and equals the double integral over that subinterval, then does f necessarily satisfy conditions of Theorem 3.8?*

Question 2. *(Suggested by Pfeffer) Does the following hold true?*
Let a real-valued function f defined on $A := [a, b] \times [c, d]$ satisfy the following conditions:

(i) The double R-integral $\int_A f(z)dz$ *exists;*

(ii) The R-integral $\int_a^b f(x,y)\,dx$ *exists for almost all* $y \in [c,d]$*;*

(iii) $\underline{F}_s(z) > -\infty$ *and* $\overline{F}_s(z) < \infty$ *for each* $z \in A \setminus E$ *where* E *is a set of* σ*-finite measure* $\mathcal{H}^1$*.*

Then the R-integral $\int_c^d [\int_a^b f(x,y)\,dx]dy$ *exists and*

$$\int_A f(z)dz = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

4 Some Examples.

In this section, we present some examples relevant to the above theory. We give an example of a function over the unit square, which is R-integrable, but not Lebesgue integrable, and which also satisfies the conditions of Theorems 3.1 and 3.4. We also give an example of a function, where we apply Theorem 3.1 to show that the function is not R-integrable. Finally, we present an important example, for which only one iterated integral exists.

The following example is from [1](page 36).

Proposition 4.1. *Let* $c_n = 1 - \frac{1}{2^n}$, $n = 0,1,2,\ldots$ *Define* $g : [0,1] \to \mathbb{R}$ *by*

$$g(x) = \begin{cases} \frac{(-1)^{k+1}2^k}{k} & \text{for } x \in [c_{k-1}, c_k), \quad k = 1,2,\ldots \\ 0 & \text{for } x = 1. \end{cases}$$

Then the following hold:

(i) g *is not Lebesgue integrable over* $[0,1]$*.*

(ii) g *is R-integrable over* $[0,1]$*, and*

$$\int_0^1 g(x)\,dx = \log 2.$$

(iii) If the charge G *is the indefinite R-integral of* g*, then*

$$\overline{G}_s(1) = \infty \text{ and } \underline{G}_s(1) = -\infty.$$

Proof. (i) We have,

$$\int_0^1 |g(x)|dx = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

(ii) For each $\beta \in [0,1)$, g is clearly Lebesgue integrable and hence R-integrable. Also,

$$\lim_{\beta \to 1-} \int_0^{\beta} g(x)\,dx = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log 2.$$

Hence, by Corollary 6.3.3 of [8], g is R-integrable over $[0,1]$, and

$$\int_0^1 g(x)\,dx = \log 2.$$

(iii) First we show that $\overline{G}_s(1) = \infty$. We have,

$$\begin{aligned} G([c_n,1]) &= G([0,1]) - G([0,c_n] \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \\ &= \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k} \ \forall\, n = 1,2,\dots \end{aligned} \tag{4.1}$$

Let $d_n = \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k}$ $\forall\, n = 1,2,\dots$ Then

$$\begin{aligned} d_{2n} &= \sum_{k=2n+1}^{\infty} \frac{(-1)^{k+1}}{k} \\ &= \frac{1}{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} - \frac{1}{2n+4} + \dots \\ &= \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) + \left(\frac{1}{2n+3} - \frac{1}{2n+4}\right) + \dots \\ &> 0,\ \forall\, n = 1,2,\dots \end{aligned} \tag{4.2}$$

as the expression inside each bracket is positive. Similarly,

$$\begin{aligned} d_{2n+2} &= \left(\frac{1}{2n+3} - \frac{1}{2n+4}\right) + \left(\frac{1}{2n+5} - \frac{1}{2n+6}\right) + \dots \\ &> \left(\frac{1}{2n+3} - \frac{1}{2n+4}\right), \end{aligned} \tag{4.3}$$

as the expression inside each bracket is positive. Let

$$A_n = [c_{2n}, 1] \; \forall \; n = 1, 2, \ldots$$

Then (A_n) is a sequence tending to 1 according to Definition 2.3. Let

$$a_n = \frac{G(A_n)}{|A_n|} \; \forall \; n = 1, 2, \ldots$$

Then by (4.1),

$$a_n = 2^{2n} d_{2n} \text{ and } a_{n+1} = 2^{2n+2} d_{2n+2}.$$

Hence,

$$\begin{aligned}
\frac{a_n}{a_{n+1}} &= \frac{2^{2n}}{2^{2n+2}} \left(\frac{\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) + \left(\frac{1}{2n+3} - \frac{1}{2n+4}\right) + \ldots}{d_{2n+2}} \right) \\
&= \frac{1}{4} \left(\frac{\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) + d_{2n+2}}{d_{2n+2}} \right) \\
&= \frac{1}{4} \left(\frac{\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right)}{d_{2n+2}} + 1 \right) \\
&< \frac{1}{4} \left(\frac{\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right)}{\left(\frac{1}{2n+3} - \frac{1}{2n+4}\right)} + 1 \right) \qquad \text{by (4.3)} \\
&= \frac{1}{4} \left(\frac{(2n+3)(2n+4)}{(2n+1)(2n+2)} + 1 \right)
\end{aligned}$$

Thus, for all $n \geq 2$,

$$\frac{a_n}{a_{n+1}} < \frac{1}{4}(2+1) = \frac{3}{4}.$$

Hence, for $n > 2$,

$$a_{n+1} > \frac{4}{3} a_n > \cdots > \left(\frac{4}{3}\right)^{n-1} a_2,$$

where $a_2 > 0$ in view of (4.2). This implies that

$$a_n \to \infty \text{ as } n \to \infty. \tag{4.4}$$

Hence, $\overline{G}_s(1) = \infty$. Now we show that $\underline{G}_s(1) = -\infty$. We have,

$$\begin{aligned} d_{2n-1} &= \sum_{k=2n}^{\infty} \frac{(-1)^{k+1}}{k} \\ &= -\frac{1}{2n} + \frac{1}{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} + \dots \\ &= \left(-\frac{1}{2n} + \frac{1}{2n+1}\right) + \left(-\frac{1}{2n+2} + \frac{1}{2n+3}\right) + \dots \\ &< 0, \ \forall\, n = 1, 2, \dots \end{aligned} \tag{4.5}$$

as the expression inside each bracket is negative. Let

$$B_n = [c_{2n-1}, 1] \ \forall\, n = 1, 2, \dots$$

The sequence (B_n) also tends to 1. Let

$$b_n = \frac{G(B_n)}{|B_n|} \ \forall\, n = 1, 2, \dots$$

Then by (4.1),

$$b_n = 2^{2n-1} d_{2n-1} \text{ and } b_{n+1} = 2^{2n+1} d_{2n+1}.$$

Hence,

$$\begin{aligned} \frac{b_n}{b_{n+1}} &= \frac{2^{2n-1}}{2^{2n+1}} \left(\frac{\left(-\frac{1}{2n} + \frac{1}{2n+1}\right) + \left(-\frac{1}{2n+2} + \frac{1}{2n+3}\right) + \dots}{d_{2n+1}} \right) \\ &= \frac{1}{4} \left(\frac{\left(-\frac{1}{2n} + \frac{1}{2n+1}\right) + d_{2n+1}}{d_{2n+1}} \right) \\ &= \frac{1}{4} \left(\frac{\left(-\frac{1}{2n} + \frac{1}{2n+1}\right)}{d_{2n+1}} + 1 \right). \end{aligned}$$

Since $d_{2n+1} < 0$ by (4.5), we have

$$\frac{\left(-\frac{1}{2n} + \frac{1}{2n+1}\right)}{d_{2n+1}} > 0 \ \text{ implying } \ \frac{b_n}{b_{n+1}} > \frac{1}{4} \ \ \forall\, n = 1, 2, \dots$$

Thus,

$$b_{n+1} < 4b_n < \dots < 4^n b_1,$$

where $b_1 < 0$, in view of (4.5). This implies that

$$b_n \to -\infty \text{ as } n \to \infty. \tag{4.6}$$

Hence, $\underline{G}_s(1) = -\infty$. □

We give next the example of a function over the unit square, which is R-integrable, but not Lebesgue integrable, and satisfies the conditions of Theorems 3.1 and 3.4.

Proposition 4.2. *Let* $c_n = 1 - \frac{1}{2^n}$, $n = 0, 1, 2, \dots$ *Define* $f : [0,1]\times[0,1] \to \mathbb{R}$ *by*

$$f(x,y) = \begin{cases} \frac{(-1)^{k+1}2^k}{k} & \text{for } x \in [c_{k-1}, c_k), \quad k = 1, 2, \dots \\ 0 & \text{for } x = 1. \end{cases}$$

Then

(i) $\int_0^1 \int_0^1 f\,dx\,dy = \int_0^1 \int_0^1 f\,dy\,dx = \log 2.$

(ii) There exists a charge F *in* $[0,1]\times[0,1]$, *such that if* $[a,b]\times[c,d]$ *is any subinterval of* $[0,1]\times[0,1]$, *then*

$$F([a,b]\times[c,d]) = \int_c^d \int_a^b f\,dx\,dy.$$

(iii) $f \in R([0,1]\times[0,1])$, *and* F *is the indefinite R-integral of* f, *i.e.,*

$$\int_0^1 \int_0^1 f\,dx\,dy = \int_{[0,1]\times[0,1]} f.$$

(iv) f satisfies conditions of part (ii) of Theorem 3.1.

(v) f does not satisfy conditions of part (i) of Theorem 3.1.

(vi) f satisfies conditions of part (i) of Theorem 3.4.

(vii) f does not satisfy conditions of part (ii) of Theorem 3.4.

PROOF. Let g, G be as in the above Proposition 4.1.
(i) For a fixed $y \in [0,1]$,

$$f(x,y) = g(x)\ \forall\ x \in [0,1]. \tag{4.7}$$

Hence, by result (ii) of Proposition 4.1,

$$\int_0^1 \int_0^1 f\,dx\,dy = \int_0^1 \log 2 dy = \log 2.$$

And

$$\int_0^1 \int_0^1 f\,dy\,dx = \int_0^1 g(x)\,dx = \log 2.$$

(ii) By Theorem 2.3(ii), there exists a continuous function $v : \mathbb{R} \to \mathbb{R}$, such that for any $[a, b] \subset \mathbb{R}$, $G([a, b]) = v(b) - v(a)$. Define a vector field $u : [0, 1] \times [0, 1] \to \mathbb{R}^2$ by $u(x, y) = (v(x), 0)$. As v is a continuous function, u is also a continuous vector field. Let $E = [0, 1] \times [0, 1]$. Define a function $F : \mathcal{BV}(\mathbb{R}^2) \to \mathbb{R}$ by

$$F(A) = \int_{\partial_*(A\cap E)} u \cdot \nu_{(A\cap E)}\, d\mathcal{H}^1.$$

Then by Example 2.1.4 of [8], F is a charge in E. For any cell $A = [a, b] \times [c, d] \subset [0, 1] \times [0, 1]$, we have,

$$\begin{aligned}\partial_*(A \cap E) =& \partial_* A \text{ (as } A \subset E) = \partial A \\ =& \{[a, b] \times \{c\}\} \cup \{\{b\} \times [c, d]\} \cup \{[a, b] \times \{d\}\} \cup \{\{a\} \times [c, d]\}.\end{aligned}$$

Now, using Definition 2.1, we get

$$\nu_E(x, y) = \begin{cases} (0, -1) & \text{if } (x, y) \in (a, b) \times \{c\}, \\ (1, 0) & \text{if } (x, y) \in \{b\} \times (c, d), \\ (0, 1) & \text{if } (x, y) \in (a, b) \times \{d\}, \\ (-1, 0) & \text{if } (x, y) \in \{a\} \times (c, d). \end{cases}$$

Hence,

$$\begin{aligned} F(A) =& \int_{[a,b]\times\{c\}} u \cdot \nu_{(A\cap E)} d\mathcal{H}^1 + \int_{\{b\}\times[c,d]} u \cdot \nu_{(A\cap E)} d\mathcal{H}^1 \\ &+ \int_{[a,b]\times\{d\}} u \cdot \nu_{(A\cap E)} d\mathcal{H}^1 + \int_{\{a\}\times[c,d]} u \cdot \nu_{(A\cap E)} d\mathcal{H}^1 \\ =& 0 + v(b)(d - c) + 0 + v(a)(-1)(d - c) \\ =& (d - c)(v(b) - v(a)) = (d - c)G([a, b]) \\ =& \int_c^d G([a, b])\, dy = \int_c^d \int_a^b f\, dx\, dy, \end{aligned} \tag{4.8}$$

in view of (4.7).

(iii) Let $E_n = [0, b_n] \times [0, 1]$, where $b_n \in (0, 1)$, and $b_n \to 1$, $n = 1, 2, \ldots$ Then, for each n, as $b_n < 1$, $f|_{E_n}$ is Lebesgue integrable (and hence also R-integrable). Thus, by Fubini's Theorem,

$$\int_{E_n} f = \int_0^1 \int_0^{b_n} f\, dx\, dy. \tag{4.9}$$

By (4.8) and (4.9), we get $F(E_n) = \int_{E_n} f$. Also, $\{E_n\} \to E$. Hence, by Theorem 6.3.2 of [8], $f \in R(E)$, and F is the indefinite R-integral of f. Further, by (4.8),

$$F(E) = \int_0^1 \int_0^1 f\,dx\,dy = \log 2.$$

Thus, the iterated integrals and the double integral have the same value.

(iv) Clearly, the R-integral $\int_0^1 f\,dy$ exists for each $x \in [0,1]$. We verify the other condition. We denote by $\bar{f}$ the zero extension of the given function f; i.e.,

$$\bar{f}(x,y) = \begin{cases} f(x,y) & \text{for } (x,y) \in [0,1] \times [0,1], \\ 0 & \text{for } (x,y) \in \mathbb{R}^2 \setminus \{[0,1] \times [0,1]\}. \end{cases}$$

Clearly, $\bar{f}$ is R-integrable in $\mathbb{R}^2$, with F as the indefinite integral. Now choose any $x_0 \in (0,1) \setminus \cup_{n=0}^{\infty}\{c_n\}$. Clearly, f is continuous at the point (x_0, y), if $y \in (0,1)$. Also, over a small interval around (x_0, y), f is Lebesgue integrable. Hence by Theorem 2.5(iii),

$$F'_s(x_0,y) = \underline{F}_s(x_0,y) = \overline{F}_s(x_0,y) = f(x_0,y) \neq \pm\infty.$$

Now, we show that $\underline{F}_s(x_0,1) > -\infty$ and $\overline{F}_s(x_0,1) < \infty$. Now, either $\bar{f}(x_0,1) > 0$, or $\bar{f}(x_0,1) < 0$. If $\bar{f}(x_0,1) > 0$, then there exists a nondegenerate interval, say $A := [x_0 - \alpha, x_0 + \alpha] \times [1-\alpha, 1+\alpha]$ such that $\bar{f}$ is Lebesgue integrable over A, and

$$\bar{f}(x,y) > 0 \quad \text{over} \quad [x_0 - \alpha, x_0 + \alpha] \times [1-\alpha, 1]. \tag{4.10}$$

It is obvious that $\bar{f}$ is upper semicontinuous at the point $(x_0, 1)$. Hence, by Theorem 2.5(i), $\overline{F}_s(x_0,1) \le \bar{f}(x_0,1) < \infty$. If $B \subset A$ is any cell, then $F(B) \ge 0$, in view of (4.10). Hence it follows that $\underline{F}_s(x_0,1) \ge 0 > -\infty$. If $\bar{f}(x_0,1) < 0$, then also, similarly as above, we can show that $\underline{F}_s(x_0,1) > -\infty$ and $\overline{F}_s(x_0,1) < \infty$. Similarly, it can be easily seen that

$$\{\underline{F}_s(x_0,0), \overline{F}_s(x_0,0)\} \cap \{-\infty, \infty\} = \emptyset.$$

Let $C := \{0\} \cup \{1\} \cup \{\cup_{n=0}^{\infty}\{c_n\}\}$. Then, over $\{[0,1] \setminus C\} \times [0,1]$, we have $\underline{F}_s(x,y) > -\infty$ and $\overline{F}_s(x,y) < \infty$. (In fact, we can show that the above holds true over $\{[0,1] \setminus \{1\}\} \times [0,1]$). Thus, f satisfies conditions of part (ii) of Theorem 3.1.

(v) Choose any $y_0 \in (0,1)$. We will show that $\overline{F}_s(1,y_0) = \infty$. Let $A_n = [c_{2n}, 1] \times [y_0, y_0 + \frac{1}{2^{2n}}]\ \forall\, n = 1, 2, \ldots$ Then A_n tends to the point $(1, y_0)$. Thus, there exists a positive integer n_0 such that for all $n \geq n_0$, $A_n \subset E$. Hence, for all such n, by part (ii) above, we have,

$$F(A_n) = \int_{y_0}^{y_0+\frac{1}{2^{2n}}} \int_{c_{2n}}^{1} f\,dx\,dy = \frac{1}{2^{2n}} G([c_{2n}, 1]),$$

i.e.,

$$\begin{aligned}\frac{F(A_n)}{|A_n|} &= \frac{G([c_{2n}, 1])}{2^{2n}|[c_{2n}, 1]|\,|[y_0, y_0 + \frac{1}{2^{2n}}]|} \\ &= \frac{G([c_{2n}, 1])}{|[c_{2n}, 1]|} \to \infty \quad \text{(by (4.4))}.\end{aligned}$$

Thus, $\overline{F}_s(1, y_0) = \infty$. Hence, f does not satisfy conditions of part (i) of Theorem 3.1. In fact, if we take

$$B_n := [c_{2n-1}, 1] \times [y_0, y_0 + \frac{1}{2^{2n-1}}],\ \forall\, n = 1, 2, \ldots,$$

then using (4.6), we can show that

$$\frac{F(B_n)}{|B_n|} \to -\infty; \text{ i.e., } \underline{F}_s(1, y_0) = -\infty.$$

(vi) Since, $G([0, c_n]) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \to \log 2$, given $\epsilon > 0$, there exists a positive integer n_0 such that $|G([0, c_n])| < \log 2 + \epsilon$, $\forall\, n \geq n_0$. Hence, for each $y \in [0, 1]$,

$$\left| \int_0^{c_{n_0+p}} f(x, y)\,dx \right| < \log 2 + \epsilon,\ \ \forall\, p = 1, 2, \ldots$$

Thus the condition of part (c) of Theorem 3.4(i) is clearly satisfied. It is easily seen that the condition of part (b) of Theorem 3.4(i) is also satisfied. Also, the R-integral $\int_0^1 f(x, y)\,dx$ exists for each $y \in [0, 1]$. Hence, f satisfies conditions of part (i) of Theorem 3.4.

(vii) If possible, suppose f satisfies conditions of part (ii) of Theorem 3.4. Then there exists a sequence $(\alpha_n)_{n=1}^{n=\infty}$ in $(0, 1)$ such that $\alpha_n \to 1$, and there exist $\kappa_1, \kappa_2 \in R([0, 1])$, such that

$$\kappa_1(x) \leq \int_0^{\alpha_n} f(x, y)dy \leq \kappa_2(x)\ \ \forall\, n = 1, 2, \ldots, \text{ for almost all } x \in [0, 1].$$

Now,

$$\int_0^{\alpha_1} f(x,y)dy = \alpha_1 g(x) \ \ \forall\, x \in [0,1].$$

Thus, for all $k = 0,1,2,\dots,$ $\kappa_1(x) \le \alpha_1 \left(\frac{2^{2k+1}}{2k+1}\right)$ a.e. on $[c_{2k}, c_{2k+1})$, and for all $k = 1,2,\dots,$ $\kappa_1(x) \le \frac{-2^{2k}}{2k}$ a.e. on $[c_{2k-1}, c_{2k})$. As $\displaystyle\int_0^1 \kappa_1(x)\,dx = \lim_{n\to\infty} \int_0^{c_n} \kappa_1(x)\,dx$, we have,

$$\int_0^1 \kappa_1(x)\,dx \le \alpha_1 - \frac{1}{2} + \frac{1}{3}\alpha_1 - \frac{1}{4} + \frac{1}{5}\alpha_1 + \dots, \tag{4.11}$$

where $\alpha_1 \in (0,1)$. As κ_1 is integrable, $\int_0^1 \kappa_1(x)\,dx > -\infty$. But (4.11) implies that $\int_0^1 \kappa_1(x)\,dx = -\infty$. Thus we get a contradiction. Hence, f does not satisfy conditions of part (ii) of Theorem 3.4. □

Remark 4.1. From the above example, it is clear that there are situations when Theorem 3.1 can be applied, but Theorem 3.4 cannot be applied, and vice-versa.

We show in the next example (taken from [2], Exercise 16, page 185), that Theorem 3.1 can be applied in certain situations to prove that a function is not R-integrable.

Proposition 4.3. *Define* $f : [0,1]\times[0,1] \to \mathbb{R}$ *by*

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then f is not R-integrable over $[0,1]\times[0,1]$.

PROOF. It can be verified that,

$$\int_0^1 \int_0^1 f\,dy\,dx = \frac{\pi}{4}, \quad \text{and} \quad \int_0^1 \int_0^1 f\,dx\,dy = -\frac{\pi}{4}. \tag{4.12}$$

Suppose, if possible, that f is R-integrable over $[0,1]\times[0,1]$. Let F be the indefinite R-integral of f. Then it can be shown (as in the proof of the result (iv) of Proposition 4.2) that over $[0,1] \times \{[0,1] \setminus \{0\}\}$, $\overline{F}_s(x,y) < \infty$ and $\underline{F}_s(x,y) > -\infty$. Thus, f satisfies conditions of part (i) of Theorem 3.1. Hence,

$$\int_{[0,1]\times[0,1]} f = \int_0^1 f\,dx\,dy \tag{4.13}$$

Similarly, over $\{[0,1]\setminus\{0\}\}\times[0,1]$, $\overline{F}_s(x,y)<\infty$ and $\underline{F}_s(x,y)>-\infty$. Thus, f satisfies conditions of part (ii) of Theorem 3.1. Hence,

$$\int_{[0,1]\times[0,1]} f = \int_0^1 f\,dy\,dx \tag{4.14}$$

Thus, by (4.13) and (4.14), we get, $\int_0^1 f\,dx\,dy = \int_0^1 f\,dy\,dx$, which contradicts (4.12). Hence, f is not R-integrable over $[0,1]\times[0,1]$. □

For the example given in Proposition 4.2, both the iterated integrals exist. The various sets of conditions given in the last section ensure the existence and equality with double integral of only one iterated integral. We present the following example, given in [8](Example 5.1.14) and [7](Section 11.1), in which only one iterated integral exists, and conditions of Theorem 3.1(i) are satisfied. This example also shows that Fubini's Theorem does not hold for the R-integral.

Proposition 4.4. *Define a function $\phi\in C^\infty(\mathbb{R})$ by the formula*

$$\phi(s) := \begin{cases} \exp\left(\frac{1}{s^2-1}\right) & \text{if } |s|<1,\\ 0 & \text{if } |s|\ge 1.\end{cases}$$

If $K:=[a,b]$ is a one-dimensional cell, let

$$g_K(t) := \begin{cases} c^{-1}\int_{-1}^{u(t)}\phi(s)ds & \text{if } t>a,\\ 0 & \text{if } t\le a,\end{cases}$$

where

$$c := \int_{-1}^1 \phi(s)ds \quad \text{and} \quad u(t) := (2t-b-a)/(b-a).$$

For $n=0,1,2,\dots,$ let

$$A_n := [2^{-n-1},2^{-n}] \quad \text{and} \quad K_n := \left[\left(\frac{4}{3}\right)2^{-n-1},\left(\frac{5}{3}\right)2^{-n-1}\right].$$

Denote g_{K_n} by g_n, and define a function f on $\mathbb{R}^2$ as follows. Given $(x,y)\in\mathbb{R}^2$, let

$$f(x,y) := g_n(y)y^2\sin(8^n x) + [1-g_n(y)]y^2\sin(8^{n+1}x),$$

whenever $y\in A_n,\ n=0,1,\dots,$ and let

$$f(x,y) := \begin{cases} 0 & \text{if } y\le 0,\\ y^2\sin x & \text{if } y>1.\end{cases}$$

Let h be the divergence of the vector field $v:=(f,0)$. Then, over $D:=[0,2\pi]\times[0,1]$, the following hold:

(i) The R-integral $\int_0^1 h(x,y)dy$ *does not exist a.e. over* $[0,2\pi]$.

(ii) h is R-integrable over D.

(iii) $\int_D h = 0 = \int_0^1 \int_0^{2\pi} h(x,y)\,dx\,dy$.

(iv) h satisfies conditions of part (i) of Theorem 3.1 over D.

PROOF. Parts (i) and (ii) are proved in [8].
(iii) As has been shown in [8], the R-integrability of h follows from the Gauss-Green Theorem. Hence, $\int_D h = \int_{\partial_* D} v \cdot \nu_D \; d\mathcal{H}^{m-1}$. As in part (ii) of Proposition 4.2, we can compute ν_D, and show that $\int_{\partial_* D} v \cdot \nu_D \; d\mathcal{H}^{m-1} = 0$. Over D,

$$h(x,y) = \frac{\partial f}{\partial x} = 8^n y^2 g_n(y) \cos(8^n x) + 8^{n+1} y^2 [1 - g_n(y)] \cos(8^{n+1} x).$$

It can be easily verified that for each $y \in [0,1]$, $\int_0^{2\pi} h(x,y)\,dx = 0$. Hence, the result follows.

(iv) It can be shown as in part (iv) of Proposition 4.2. $\square$

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