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TWO-DIMENSIONAL MEAN INEQUALITIES IN CERTAIN BANACH FUNCTION SPACES

Abstract

Weight characterization is obtained for the L^p-X^q boundedness of the two-dimensional Hardy operator $(H_2f)(x_1, x_2) = \int_0^{x_1} \int_0^{x_2} f(t_1, t_2)\, dt_1\, dt_2$. By using a limiting procedure as well as by a direct method, the corresponding boundedness of the two-dimensional geometric mean operator $(G_2f)(x_1, x_2) = \exp\left(\frac{1}{x_1x_2} \int_0^{x_1} \int_0^{x_2} \ln f(t_1, t_2)\, dt_1\, dt_2\right)$ is obtained.

1 Introduction.

Let $\Omega \subset \mathbb{R}^n$. A real normed linear space $X = \{f : \|f\|_X < \infty\}$ of measurable functions on Ω is called a Banach function space (BFS), if in addition to the usual norm axioms, $\|f\|_X$ satisfies the following.
(1) $\|f\|_X = \|\,|f|\,\|_X$ for all $f \in X$.
(2) $0 \le f \le g$ a.e. $\Rightarrow \|f\|_X \le \|g\|_X$.
(3) $0 < f_n \uparrow f$ a.e. $\Rightarrow \|f_n\|_X \uparrow \|f\|_X$.
(4) $\operatorname{mes} E < \infty \Rightarrow \|\chi_E\|_X < \infty$.
(5) $\operatorname{mes} E < \infty \Rightarrow \int_E f \le C_E\|f\|_X$ for some constant C_E depending upon E.

Given a BFS X, its associate space X' is defined by

$$X' = \left\{g : \int_\Omega |fg| < \infty \text{ for all } f \in X\right\}$$

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and is endowed with the associate norm

$$\|g\|_{X'} = \sup\left\{ \int_\Omega |fg| : f \in X; \|f\|_X \le 1 \right\}.$$

Examples of BFS are Lebesgue L^p-spaces, Lorentz spaces, Orlicz spaces etc. It is known that the second associate space of X; i.e., $(X')' = X''$ coincides with X itself and consequently, the norm of the function in X can be written in terms of functions in X', i.e.,

$$\|f\|_X = \sup\left\{ \int_\Omega |fg| : g \in X'; \|g\|_{X'} \le 1 \right\}.$$

The idea of BFS was introduced by Luxemburg [8]. A good treatment of the theory of such spaces can be found, e.g., in [1].

Throughout the paper, we shall take $\Omega = (0,\infty)\times(0,\infty)$. We are concerned here with the space X^p, $-\infty < p < \infty$, $p \ne 0$ which is the space of all measurable functions f on Ω for which $\|f\|_{X^p} := \| \, |f|^p \, \|_X^{\frac{1}{p}} < \infty$, X being the underlying BFS. For $X = L^1$, the space X^p coincides with the L^p-space. It is known, see e.g., [9], [11] that for $1 \le p < \infty$, X^p is a BFS,

In this paper, we are concerned with the two-dimensional Hardy operator

$$(H_2 f)(x_1, x_2) = \int_0^{x_1} \int_0^{x_2} f(t_1, t_2)\, dt_1\, dt_2 \tag{1.1}$$

and the two-dimensional geometric mean operator

$$(G_2 f)(x_1, x_2) = \exp\left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} \ln f(t_1, t_2)\, dt_1\, dt_2 \right). \tag{1.2}$$

In fact, we obtain necessary and sufficient conditions for the L^p-X^q boundedness of these operators; i.e., we characterize the weighted inequalities

$$\left\|(H_2 f)^q u\right\|_X^{\frac{1}{q}} \le C\left(\int_0^\infty \int_0^\infty f^p(x_1, x_2) v(x_2, x_2)\, dx_1\, dx_2 \right)^{\frac{1}{p}} \tag{1.3}$$

and

$$\left\|(G_2 f)^q u\right\|_X^{\frac{1}{q}} \le C\left(\int_0^\infty \int_0^\infty f^p(x_1, x_2) v(x_1, x_2)\, dx_1\, dx_2 \right)^{\frac{1}{p}}. \tag{1.4}$$

When $X = L^1$, $1 < p \le q < \infty$, the inequality (1.3) has been characterized by Sawyer [13] giving three conditions. It was shown by Wedestig [16] (see also [14]) that if we take $v(x_2, x_2) = v(x_1)v(x_2)$, then only one condition is required for the corresponding inequality to hold. We extend this result of Wedestig for the L^p-X^q boundedness (See Theorem 1). We also discuss the corresponding boundedness of the conjugate Hardy operator

$$(H_2^* f)(x_1, x_2) = \int_{x_1}^{\infty} \int_{x_2}^{\infty} f(t_1, t_2)\, dt_1\, dt_2,$$

which is new even for L^p-L^q case.

As regards the inequality (1.4), we study it in two different ways. The first is to use the fact

$$\lim_{\alpha \to 0} \Big(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} f^{\alpha}(t_1, t_2)\, dt_1\, dt_2 \Big)^{\frac{1}{\alpha}} = \exp \left(\frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} \ln f(t_1, t_2)\, dt_1\, dt_2 \right)$$

in (1.3). Another is a direct way without using the limiting procedure. Moreover, in the later case, we study a more general inequality than (1.4) where the functions f are defined on $(0, b_1) \times (0, b_2)$, $0 < b_1,\ b_2 \le \infty$. Also, in this case, the weight on the R.H.S. of the corresponding inequality need not be of the product type (see Theorem 4). The corresponding L^p-L^q case has been proved for the case $p = q = 1$ by Heinig, Kerman and Krebec [2] and Jain and Hassija [5] while for the case $0 < p \le q < \infty$ by Wedestig [14], [16].

Let us mention that all the results cited or proved in this paper are known in the one dimensional situation. The L^p-L^q boundedness of the one dimensional Hardy operator $(Hf)(x) = \int_0^x f(t)dt$ and geometric mean operator $(Gf)(x) = \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)$ have been largely settled for all parameters, see [6], [7], [10], [12] and the references therein. While the corresponding L^p-X^q boundedness has recently been studied in [3], [4].

Throughout, all functions will be Lebesgue measurable. By a weight function, we shall mean a function which is measurable, positive and finite *a.e.* on the appropriate domain. We shall be using two-dimensional version of the Minkowski's integral inequality from [14], [15] stated below.

Proposition A. *Let* $r > 1$, $-\infty \le a_1 < b_1 \le \infty$, $-\infty \le a_2 < b_2 \le \infty$ *and* Φ, Ψ *be positive measurable functions on* $[a_1, b_1] \times [a_2, b_2]$. *Then*

$$\begin{aligned} &\int_{a_1}^{b_1} \int_{a_2}^{b_2} \Phi(x_1, x_2) \left(\int_{a_1}^{x_1} \int_{a_2}^{x_2} \Psi(y_1, y_2)\, dy_1\, dy_2 \right)^{r} dx_1\, dx_2 \\ &\le \int_{a_1}^{b_1} \int_{a_2}^{b_2} \Psi(y_1, y_2) \left(\int_{y_1}^{b_1} \int_{y_2}^{b_2} \Phi(x_1, x_2)\, dx_1\, dx_2 \right)^{1/r} dy_1\, dy_2. \end{aligned} \tag{1.5}$$

2 The Operators H_2 and H_2^*.

In this section, we give necessary and sufficient conditions for the L^p-X^q boundedness of the two-dimensional Hardy operator H_2 defined in (1.1) and its conjugate

$$(H_2^* f)(x_1,x_2) = \int_{x_1}^{\infty}\int_{x_2}^{\infty} f(t_1,t_2)\,dt_1\,dt_2.$$

We begin with the following precise result concerning H_2 .

Theorem 1. *Let $1 < p \le q < \infty$, $s_1, s_2 \in (1,p)$, u be a weight function on $\mathbb{R}_+^2$ and v_1, v_2 be weight functions on $\mathbb{R}_+$. Let $V_i(t_i) = \displaystyle\int_0^{t_i} v_i^{1-p'}(x_i)dx_i$, $i = 1,2$ and assume that $V_i(t_i) < \infty$, $0 < t_i < \infty$. Then the inequality*

$$\left\|(H_2 f)^q u\right\|_X^{\frac{1}{q}} \le C\left\{\int_0^{\infty}\int_0^{\infty} f^p(x_1,x_2)v_1(x_1)v_2(x_2)\,dx_1\,dx_2\right\}^{\frac{1}{p}} \tag{2.1}$$

holds for all measurable functions $f \ge 0$ if and only if $\sup\limits_{t_1,t_2>0} A(s_1,s_2) < \infty$, where

$$\begin{aligned}A(s_1,s_2) :=\, & V_1^{\frac{s_1-1}{p}}(t_1)V_2^{\frac{s_2-1}{p}}(t_2)\\ & \times \left\|\chi_{[t_1,\infty)}(x_1)\chi_{[t_2,\infty)}(x_2)u(x_1,x_2)V_1^{\frac{q(p-s_1)}{p}}(x_1)V_2^{\frac{q(p-s_2)}{p}}(x_2)\right\|_X^{\frac{1}{q}}\end{aligned}$$

and the best constant C in (2.1) has the estimates

$$\begin{aligned}&\sup_{1<s_1,s_2<p}\left[\frac{\left(\frac{p}{p-s_1}\right)^p}{\left(\frac{p}{p-s_1}\right)^p+\left(\frac{1}{s_1-1}\right)}\right]^{\frac{1}{p}}\left[\frac{\left(\frac{p}{p-s_2}\right)^p}{\left(\frac{p}{p-s_2}\right)^p+\left(\frac{1}{s_2-1}\right)}\right]^{\frac{1}{p}} A(s_1,s_2)\\ &\le C \le \inf_{1<s_1,s_2<p}\left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}}\left(\frac{p-1}{p-s_2}\right)^{\frac{1}{p'}} A(s_1,s_2)\,.\end{aligned}$$

Proof. The key step here is to use the following expression for the norm on X

$$\begin{aligned}\left\|(H_2 f)^q u\right\|_X^{\frac{1}{q}} = \sup_{h>0}\Bigg\{&\int_0^{\infty}\int_0^{\infty}\left(\int_0^{x_1}\int_0^{x_2} f(s,t)dsdt\right)^q\\ &\times u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'} \le 1\Bigg\}^{\frac{1}{q}}.\end{aligned}$$

Now, if we take $f^p(x_1,x_2)v_1(x_1)v_2(x_2) = g(x_1,x_2)$, then (2.1) becomes equivalent to

$$\sup_{h>0}\bigg\{\int_0^\infty\int_0^\infty\bigg(\int_0^{x_1}\int_0^{x_2} g^{\frac{1}{p}}(t_1,t_2)v_1^{-\frac{1}{p}}(t_1)v_2^{-\frac{1}{p}}(t_2)\,dt_1\,dt_2\bigg)^q u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\bigg\}^{\frac{1}{q}} \le C\bigg\{\int_0^\infty\int_0^\infty g(x_1,x_2)\,dx_1\,dx_2\bigg\}^{\frac{1}{p}} \tag{2.2}$$

We prove the necessity first. For fixed $t_1, t_2 > 0$, consider the test function

$$\begin{aligned} g(x_1,x_2) =&\left(\frac{p}{p-s_1}\right)^p\left(\frac{p}{p-s_2}\right)^p V_1^{-s_1}(t_1)v_1^{1-p'}(x_1)V_2^{-s_2}(t_2)v_2^{1-p'}(x_2)\\ &\times \chi_{(0,t_1)}(x_1)\chi_{(0,t_2)}(x_2) + \left(\frac{p}{p-s_1}\right)^p V_1^{-s_1}(t_1)v_1^{1-p'}(x_1)\\ &\times V_2^{-s_2}(x_2)v_2^{1-p'}(x_2)\chi_{(0,t_1)}(x_1)\chi_{(t_2,\infty)}(x_2)\\ &+\left(\frac{p}{p-s_2}\right)^p V_1^{-s_1}(x_1)v_1^{1-p'}(x_1)V_2^{-s_2}(t_2)v_2^{1-p'}(x_2)\\ &\times \chi_{(t_1,\infty)}(x_1)\chi_{(0,t_2)}(x_2) + V_1^{-s_1}(x_1)v_1^{1-p'}(x_1)\\ &\times V_2^{-s_2}(x_2)v_2^{1-p'}(x_2)\chi_{(t_1,\infty)}(x_1)\chi_{(t_2,\infty)}(x_2)\,, \end{aligned}$$

using which it is easy to check as in [16] that the R.H.S. of (2.2) is not greater than

$$\left(\left(\frac{p}{p-s_1}\right)^p + \frac{1}{s_1-1}\right)^{\frac{1}{p}}\left(\left(\frac{p}{p-s_2}\right)^p + \frac{1}{s_2-1}\right)^{\frac{1}{p}} V_1^{(1-s_1)/p}(t_1)V_2^{(1-s_2)/p}(t_2),$$

since $V_i^{1-s_i}(\infty) = 0$ if $V_i(\infty) = \infty$ and positive if $0 < V_i(\infty) < \infty$, $i = 1, 2$.

On the other hand, the L.H.S. can be estimated as follows.

$$\begin{aligned} &\sup_{h>0}\bigg\{\int_0^\infty\int_0^\infty\bigg(\int_0^{x_1}\int_0^{x_2} g^{\frac{1}{p}}(y_1,y_2)v_1^{-\frac{1}{p}}(y_1)v_2^{-\frac{1}{p}}(y_2)\,dy_2\bigg)^q\\ &\qquad\times u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\bigg\}^{\frac{1}{q}}\\ &\ge \sup_{h>0}\bigg\{\int_{t_1}^\infty\int_{t_2}^\infty\bigg[\bigg(\int_0^{t_1}\left(\frac{p}{p-s_1}\right)V_1^{-\frac{s_1}{p}}(t_1)v_1^{1-p'}(y_1)\,dy_1\bigg)\\ &\times\bigg(\int_0^{t_2}\left(\frac{p}{p-s_2}\right)V_2^{-\frac{s_2}{p}}(t_2)v_2^{1-p'}(y_2)\,dy_2\bigg) \end{aligned}$$

$$+\left(\int_0^{t_1}\left(\frac{p}{p-s_1}\right)V_1^{-\frac{s_1}{p}}(t_1)v_1^{1-p'}(y_1)\,dy_1\right)\left(\int_{t_2}^{x_2}V_2^{-\frac{s_2}{p}}(y_2)v_2^{1-p'}(y_2)\,dy_2\right)$$
$$+\left(\int_{t_1}^{x_1}V_1^{-\frac{s_1}{p}}(y_1)v_1^{1-p'}(y_1)\,dy_1\right)\left(\int_0^{t_2}\left(\frac{p}{p-s_2}\right)V_2^{-\frac{s_2}{p}}(t_2)v_2^{1-p'}(y_2)\,dy_2\right)$$
$$+\left(\int_{t_1}^{x_1}V_1^{-\frac{s_1}{p}}(y_1)v_1^{1-p'}(y_1)\,dy_1\right)\left(\int_{t_2}^{x_2}V_2^{-\frac{s_2}{p}}(y_2)v_2^{1-p'}(y_2)\,dy_2\right)\Bigg]^q$$
$$\times\; u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\Bigg\}^{\frac{1}{q}}$$
$$=\left(\frac{p}{p-s_1}\right)\left(\frac{p}{p-s_2}\right)\sup_{h>0}\Bigg\{\int_{t_1}^{\infty}\int_{t_2}^{\infty}u(x_1,x_2)V_1^{\frac{q(p-s_1)}{p}}(x_1)$$
$$\times\; V_2^{\frac{q(p-s_2)}{p}}(x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le\Bigg\}^{\frac{1}{q}}$$
$$=\left(\frac{p}{p-s_1}\right)\left(\frac{p}{p-s_2}\right)\left\|\chi_{[t_1,\infty)}\chi_{[t_2,\infty)}uV_1^{\frac{q(p-s_1)}{p}}V_2^{\frac{q(p-s_2)}{p}}\right\|_X^{\frac{1}{q}}.$$

Consequently, the inequality (2.2) takes the form

$$\left[\frac{\left(\frac{p}{p-s_1}\right)^p}{\left(\frac{p}{p-s_1}\right)^p+\frac{1}{s_1-1}}\right]^{\frac{1}{p}}\left[\frac{\left(\frac{p}{p-s_2}\right)^p}{\left(\frac{p}{p-s_2}\right)^p+\frac{1}{s_2-1}}\right]^{\frac{1}{p}}$$
$$\times\; V_1^{\frac{(s_1-1)}{p}}(t_1)V_2^{\frac{(s_2-1)}{p}}(t_2)\left\|\chi_{[t_1,\infty)}\chi_{[t_2,\infty)}V_1^{\frac{q(p-s_1)}{p}}(x_1)V_2^{\frac{q(p-s_1)}{p}}(x_2)u\right\|_X^{\frac{1}{q}}\le C;$$

i.e.,

$$\left[\frac{\left(\frac{p}{p-s_1}\right)^p}{\left(\frac{p}{p-s_1}\right)^p+\frac{1}{s_1-1}}\right]^{\frac{1}{p}}\left[\frac{\left(\frac{p}{p-s_2}\right)^p}{\left(\frac{p}{p-s_2}\right)^p+\frac{1}{s_2-1}}\right]^{\frac{1}{p}}A(s_1,s_2)\le C \qquad (2.3)$$

and the necessity follows.

Towards the sufficiency, first note that $\frac{d}{dt_1}V_1(t_1)=v_1^{1-p'}(t_1)$, $\frac{d}{dt_2}V_2(t_2)=v_2^{1-p'}(t_2)$. Now, by applying Hölder's inequality and Minkowski's inequality (1.5), the L.H.S. of (2.2) becomes

$$\sup_{h>0}\Bigg\{\int_0^{\infty}\int_0^{\infty}\Bigg(\int_0^{x_1}\int_0^{x_2}g^{\frac{1}{p}}(t_1,t_2)V_1^{\frac{s_1-1}{p}}(t_1)V_2^{\frac{s_2-1}{p}}(t_2)V_1^{\frac{-(s_1-1)}{p}}(t_1)v_1^{\frac{-1}{p}}(t_1)$$

$$
\begin{aligned}
&\times V_2^{\frac{-(s_2-1)}{p}}(t_2)v_2^{\frac{-1}{p}}(t_2)\,dt_1\,dt_2\Big)^q u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\Big\}^{\frac{1}{q}}\\
\le&\sup_{h>0}\Big\{\int_0^\infty\!\!\int_0^\infty\Big(\int_0^{x_1}\!\!\int_0^{x_2} g(t_1,t_2)V_1^{(s_1-1)}(t_1)V_2^{(s_2-1)}(t_2)\,dt_1\,dt_2\Big)^{\frac{q}{p}}\\
&\times\Big(\int_0^{x_1}V_1^{\frac{-(s_1-1)p'}{p}}(t_1)v_1^{\frac{-p'}{p}}(t_1)\,dt_1\Big)^{\frac{q}{p'}}\Big(\int_0^{x_2}V_2^{\frac{-(s_2-1)p'}{p}}(t_2)v_2^{\frac{-p'}{p}}(t_2)dt_2\Big)^{\frac{q}{p'}}\\
&\times u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\Big\}^{\frac{1}{q}}\\
=&\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}}\sup_{h>0}\Big\{\int_0^\infty\int_0^\infty\Big(\int_0^{x_1}\int_0^{x_2} g(t_1,t_2)\\
&\times V_1^{(s_1-1)}(t_1)V_2^{(s_2-1)}(t_2)\,dt_1\,dt_2\Big)^{\frac{q}{p}}V_1^{\frac{q(p-s_1)}{p}}(x_1)V_2^{\frac{q(p-s_2)}{p}}(x_2)\\
&\times u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\Big\}^{\frac{1}{q}}\\
\le&\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}}\sup_{h>0}\Big\{\int_0^\infty\int_0^\infty g(t_1,t_2)V_1^{s_1-1}(t_1)V_2^{s_2-1}(t_2)\\
&\times\Big(\int_{t_1}^\infty\int_{t_2}^\infty V_1^{\frac{q(p-s_1)}{p}}(x_1)V_2^{\frac{q(p-s_2)}{p}}(x_2)\\
&\times u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2\Big)^{\frac{p}{q}}dt_1\,dt_2 : \|h\|_{X'}\le 1\Big\}^{\frac{1}{p}}\\
\le&\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}}\Big\{\int_0^\infty\int_0^\infty g(t_1,t_2)V_1^{s_1-1}(t_1)V_2^{s_2-1}(t_2)\\
&\times\Big\|\chi_{[t_1,\infty)}\chi_{[t_2,\infty)}V_1^{\frac{q(p-s_1)}{p}}V_2^{\frac{q(p-s_2)}{p}}u\Big\|_X^{\frac{p}{q}}dt_1\,dt_2\Big\}^{\frac{1}{p}}\\
\le&\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}}A(s_1,s_2)\Big\{\int_0^\infty\int_0^\infty g(t_1,t_2)\,dt_1\,dt_2\Big\}^{\frac{1}{p}},
\end{aligned}
\tag{2.4}
$$

and, the sufficiency follows. The estimate for the best constant in (2.1) follows from (2.3) and (2.4). □

Remark 1. Theorem 1 extends a result of Wedestig [14], [15] who proved the L^p-L^q boundedness of H_2 which can be obtained by taking $X = L^1$.

We next consider the operator H_2^* (the conjugate of H_2) and characterize its L^p-X^q boundedness. The corresponding result for the L^p-L^q case is also not known. However, it is standard. Indeed, one can use either duality arguments or variable substitution method on the L^p-L^q boundedness of H_2. In the present situation, none of the methods is applicable as the dual of X^p is not known and also the expression of the X^p-norm does not support variable substitution. Therefore, we treat this case directly. However, the proof employs similar techniques as those in Theorem 1. We prove it below.

Theorem 2. *Let $1 < p \le q < \infty$, $s_1, s_2 \in (1,p)$, u be a weight function on $\mathbb{R}^2_+$ and v_1, v_2 be weight functions on $\mathbb{R}_+$. let $\widetilde{V}_i(t_i) = \int_{t_i}^\infty v_i^{1-p'}(x_i)dx_i$, $i = 1,2$ and assume that $\widetilde{V}_i(t_i) < \infty$, $0 < t_i < \infty$. Then the inequality*

$$\left\|(H_2^* f)^q u\right\|_X^{\frac{1}{q}} \le C\left\{\int_0^\infty \int_0^\infty f^p(x_1,x_2)v_1(x_1)v_2(x_2)\,dx_1\,dx_2\right\}^{\frac{1}{p}} \tag{2.5}$$

holds for all measurable functions $f \ge 0$ if and only if $\sup\limits_{t_1,t_2>0} A^*(s_1,s_2) < \infty$, where

$$\begin{aligned} A^*(s_1,s_2) &:= \widetilde{V}_1^{\frac{(s_1-1)}{p}}(t_1)\widetilde{V}_2^{\frac{(s_2-1)}{p}}(t_2) \\ &\quad \times \left\|\chi_{(0,t_1]}(x_1)\chi_{(0,t_2]}(x_2)u(x_1,x_2)\widetilde{V}_1^{\frac{q(p-s_1)}{p}}(x_1)\widetilde{V}_2^{\frac{q(p-s_2)}{p}}(x_2)\right\|_X^{\frac{1}{q}} \end{aligned} \tag{2.6}$$

and, the best possible constant C in (2.5) has the estimates

$$\begin{aligned} &\sup_{1<s_1,s_2<p}\left[\frac{\left(\frac{p}{p-s_1}\right)^p}{\left(\frac{p}{p-s_1}\right)^p + \left(\frac{1}{s_1-1}\right)}\right]^{\frac{1}{p}}\left[\frac{\left(\frac{p}{p-s_2}\right)^p}{\left(\frac{p}{p-s_2}\right)^p + \left(\frac{1}{s_2-1}\right)}\right]^{\frac{1}{p}} A^*(s_1,s_2) \\ &\le C \le \inf_{1<s_1,s_2<p}\left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}}\left(\frac{p-1}{p-s_2}\right)^{\frac{1}{p'}} A^*(s_1,s_2) \end{aligned}$$

PROOF. Take $f^p(x_1,x_2)v_1(x_1)v_2(x_2) = g(x_1,x_2)$ and we find that the inequality (2.5) becomes equivalent to

$$\begin{aligned} &\sup_{h>0}\left\{\int_0^\infty\int_0^\infty\left(\int_{x_1}^\infty\int_{x_2}^\infty g^{\frac{1}{p}}(t_1,t_2)v_1^{-\frac{1}{p}}(t_1)v_2^{-\frac{1}{p}}(t_2)\,dt_1\,dt_2\right)^q\right. \\ &\qquad\left. u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2\right\}^{\frac{1}{q}} \\ &\le C\left\{\int_0^\infty\int_0^\infty g(x_1,x_2)\,dx_1\,dx_2\right\}^{\frac{1}{p}} \end{aligned} \tag{2.7}$$

Assume first that (2.6) holds. Then Hölder's inequality, Minkowski's inequality (1.5) and the fact that $\frac{d}{dt_2}\widetilde{V}_2(t_2) = -v_2^{1-p'}(t_2) = -v_2^{\frac{-p'}{p}}(t_2)$, give

$$\sup_{h>0}\left\{\int_0^\infty\int_0^\infty\left(\int_{x_1}^\infty\int_{x_2}^\infty g^{\frac{1}{p}}(t_1,t_2)v_1^{\frac{-1}{p}}(t_1)v_2^{\frac{-1}{p}}(t_2)\,dt_1\,dt_2\right)^q\right.$$
$$\left.\times\ u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\right\}^{\frac{1}{q}}$$
$$=\sup_{h>0}\left\{\int_0^\infty\int_0^\infty\left(\int_{x_1}^\infty\int_{x_2}^\infty g^{\frac{1}{p}}(t_1,t_2)\widetilde{V}_1^{\frac{s_1-1}{p}}(t_1)\widetilde{V}_2^{\frac{s_2-1}{p}}(t_2)\right.\right.$$
$$\left.\times\ \widetilde{V}_1^{-\frac{s_1-1}{p}}(t_1)v_1^{\frac{-1}{p}}(t_1)\widetilde{V}_2^{-\frac{s_2-1}{p}}v_2^{\frac{-1}{p}}(t_2)\,dt_1\,dt_2\right)^q$$
$$\left.\times\ u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\right\}^{\frac{1}{q}}$$
$$\le\sup_{h>0}\left\{\int_0^\infty\int_0^\infty\left(\int_{x_1}^\infty\int_{x_2}^\infty g(t_1,t_2)\widetilde{V}_1^{(s_1-1)}(t_1)\widetilde{V}_2^{(s_2-1)}(t_2)\,dt_1\,dt_2\right)^{\frac{q}{p}}\right.$$
$$\times\left(\int_{x_1}^\infty\widetilde{V}_1^{\frac{-(s_1-1)p'}{p}}(t_1)v_1^{\frac{-p'}{p}}(t_1)dt_1\right)^{\frac{q}{p'}}\left(\int_{x_2}^\infty\widetilde{V}_2^{\frac{-(s_2-1)p'}{p}}(t_2)v_2^{\frac{-p'}{p}}(t_2)dt_2\right)^{\frac{q}{p'}}$$
$$\left.\times\ u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\right\}^{\frac{1}{q}}$$
$$=\left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}}\left(\frac{p-1}{p-s_2}\right)^{\frac{1}{p'}}\sup_{h>0}\left\{\int_0^\infty\int_0^\infty\left(\int_{x_1}^\infty\int_{x_2}^\infty g(t_1,t_2)\right.\right.$$
$$\left.\times\ \widetilde{V}_1^{(s_1-1)}(t_1)\widetilde{V}_2^{(s_2-1)}(t_2)\,dt_1\,dt_2\right)^{\frac{q}{p}}\widetilde{V}_1^{\frac{q(p-s_1)}{p}}(x_1)\widetilde{V}_2^{\frac{q(p-s_2)}{p}}(x_2)$$
$$\left.\times\ u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\right\}^{\frac{1}{q}}$$
$$\le\left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}}\left(\frac{p-1}{p-s_2}\right)^{\frac{1}{p'}}\sup_{h>0}\left\{\int_0^\infty\int_0^\infty g(t_1,t_2)\right.$$
$$\times\ \widetilde{V}_1^{(s_1-1)}(t_1)\widetilde{V}_2^{(s_2-1)}(t_2)\left(\int_0^{t_1}\int_0^{t_2}\widetilde{V}_1^{\frac{q(p-s_1)}{p}}(x_1)\widetilde{V}_2^{\frac{q(p-s_2)}{p}}(x_2)\right.$$
$$\left.\left.\times\ u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2\right)^{\frac{p}{q}}dt_1\,dt_2 : \|h\|_{X'}\le 1\right\}^{\frac{1}{p}}$$
$$\le\left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}}\left(\frac{p-1}{p-s_2}\right)^{\frac{1}{p'}}A^*(s_1,s_2)\left\{\int_0^\infty\int_0^\infty g(t_1,t_2)\,dt_1\,dt_2\right\}^{\frac{1}{p}}$$

and the sufficiency follows.

The necessity can be obtained by putting for fixed $t_1, t_2 > 0$, the following test function in (2.7).

$$\begin{aligned}
g(x_1,x_2) =& \left(\frac{p}{p-s_1}\right)^p \left(\frac{p}{p-s_2}\right)^p \widetilde{V}_1^{-s_1}(t_1) v_1^{1-p'}(x_1) \widetilde{V}_2^{-s_2}(t_2) \\
& \times\ v_2^{1-p'}(x_2)\chi_{(t_1,\infty)}(x_1)\chi_{(t_2,\infty)}(x_2) + \left(\frac{p}{p-s_1}\right)^p \widetilde{V}_1^{-s_1}(t_1) \\
& v_1^{1-p'}(x_1)\widetilde{V}_2^{-s_2}(x_2) v_2^{1-p'}(x_2)\chi_{(t_1,\infty)}(x_1)\chi_{(0,t_2)}(x_2) \\
& + \left(\frac{p}{p-s_2}\right)^p \widetilde{V}_1^{-s_1}(x_1) v_1^{1-p'}(x_1) \widetilde{V}_2^{-s_2}(t_2) v_2^{1-p'}(x_2) \\
& \times\ \chi_{(0,t_1)}(x_1)\chi_{(t_2,\infty)}(x_2) + \widetilde{V}_1^{-s_1}(x_1) v_1^{1-p'}(x_1) \widetilde{V}_2^{-s_2}(x_2) v_2^{1-p'}(x_2) \\
& \times \chi_{(0,t_1)}(x_1)\chi_{(0,t_2)}(x_2)\,.
\end{aligned}$$

Indeed, with the above test function, the RHS of (2.7) becomes

$$\begin{aligned}
& \Bigg\{ \left(\frac{p}{p-s_1}\right)^p \left(\frac{p}{p-s_2}\right)^p \widetilde{V}_1^{(1-s_1)}(t_1)\widetilde{V}_2^{(1-s_2)}(t_2) + \left(\frac{p}{p-s_1}\right)^p \left(\frac{1}{s_2-1}\right) \\
& \times \widetilde{V}_1^{(1-s_1)}(t_1)\left(\widetilde{V}_2^{(1-s_2)}(t_2) - \widetilde{V}_2^{(1-s_2)}(0)\right) + \left(\frac{p}{p-s_2}\right)^p \left(\frac{1}{s_1-1}\right)\widetilde{V}_2^{(1-s_2)}(t_2) \\
& \times \left(\widetilde{V}_1^{(1-s_1)}(t_1) - V_1^{(1-s_1)}(0)\right) + \left(\frac{1}{s_1-1}\right)\left(\frac{1}{s_2-1}\right) \\
& \times \left(\widetilde{V}_1^{(1-s_1)}(t_1) - \widetilde{V}_1^{(1-s_1)}(0)\right)\left(\widetilde{V}_2^{(1-s_2)}(t_2) - \widetilde{V}_2^{(1-s_2)}(0)\right)\Bigg\}^{\frac{1}{p}} \\
\le & \left(\left(\frac{p}{p-s_1}\right)^p + \frac{1}{s_1-1}\right)^{\frac{1}{p}} \left(\left(\frac{p}{p-s_2}\right)^p + \frac{1}{s_2-1}\right)^{\frac{1}{p}} \\
& \quad \times \widetilde{V}_1^{(1-s_1)/p}(t_1)\widetilde{V}_2^{(1-s_2)/p}(t_2)\,,
\end{aligned}$$

since $\widetilde{V}_i^{1-s_i}(0) = 0$ if $\widetilde{V}_i(0) = \infty$ and positive if $0 < \widetilde{V}_i(0) < \infty$, $i = 1, 2$.

On the other hand, the L.H.S. can be estimated as follows:

$$\begin{aligned}
& \sup_{h>0}\Bigg\{ \int_0^\infty \int_0^\infty \left(\int_{x_1}^\infty \int_{x_2}^\infty g^{\frac{1}{p}}(y_1,y_2) v_1^{-\frac{1}{p}}(y_1) v_2^{-\frac{1}{p}}(y_2)\, dy_1\, dy_2\right)^q \\
& \qquad \times u(x_1,x_2)h(x_1,x_2)\, dx_1\, dx_2 : \|h\|_{X'} \le 1\Bigg\}^{\frac{1}{q}} \\
\ge & \sup_{h>0}\Bigg\{ \int_0^{t_1} \int_0^{t_2} \Bigg[\Bigg(\int_{t_1}^\infty \left(\frac{p}{p-s_1}\right)\widetilde{V}_1^{-\frac{s_1}{p}}(t_1) v_1^{1-p'}(y_1)\, dy_1\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\int_{t_2}^{\infty}\left(\frac{p}{p-s_2}\right)\widetilde{V}_2^{-\frac{s_2}{p}}(t_2)v_2^{1-p'}(y_2)dy_2\right)\\
&+\left(\int_{t_1}^{\infty}\left(\frac{p}{p-s_1}\right)\widetilde{V}_1^{-\frac{s_1}{p}}(t_1)v_1^{1-p'}(y_1)\,dy_1\right)\left(\int_{x_2}^{t_2}\widetilde{V}_2^{-\frac{s_2}{p}}(y_2)v_2^{1-p'}(y_2)dy_2\right)\\
&+\left(\int_{x_1}^{t_1}\widetilde{V}_1^{-\frac{s_1}{p}}(y_1)v_1^{1-p'}(y_1)\,dy_1\right)\left(\int_{t_2}^{\infty}\left(\frac{p}{p-s_2}\right)\widetilde{V}_2^{-\frac{s_2}{p}}(t_2)v_2^{1-p'}(y_2)dy_2\right)\\
&+\left(\int_{x_1}^{t_1}\widetilde{V}_1^{-\frac{s_1}{p}}(y_1)v_1^{1-p'}(y_1)\,dy_1\right)\left(\int_{x_2}^{t_2}\widetilde{V}_2^{-\frac{s_2}{p}}(y_2)v_2^{1-p'}(y_2)dy_2\right)\Bigg]^q\\
&\times\left. u(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2:\|h\|_{X'}\le 1\right\}^{\frac{1}{q}}\\
=&\left(\frac{p}{p-s_1}\right)\left(\frac{p}{p-s_2}\right)\sup_{h>0}\left\{\int_0^{t_1}\int_0^{t_2}u(x_1,x_2)\widetilde{V}_1^{\frac{q(p-s_1)}{p}}(x_1)\right.\\
&\times\left.\widetilde{V}_2^{\frac{q(p-s_2)}{p}}(x_2)h(x_1,x_2)\,dx_1\,dx_2:\|h\|_{X'}\le\right\}^{\frac{1}{q}}\\
=&\left(\frac{p}{p-s_1}\right)\left(\frac{p}{p-s_2}\right)\left\|\chi_{(0,t_1)}(x_1)\chi_{(0,t_2)}(x_2)u(x_1,x_2)\widetilde{V}_1^{\frac{q(p-s_1)}{p}}(x_1)\widetilde{V}_2^{\frac{q(p-s_2)}{p}}(x_2)\right\|_X^{\frac{1}{q}}.
\end{aligned}
$$

Consequently, the inequality (2.7) takes the form

$$
\begin{aligned}
&\left(\frac{p}{p-s_1}\right)\left(\frac{p}{p-s_2}\right)\left\|\chi_{(0,t_1)}(x_1)\chi_{(0,t_2)}(x_2)u(x_1,x_2)\widetilde{V}_1^{\frac{q(p-s_1)}{p}}(x_1)\widetilde{V}_2^{\frac{q(p-s_2)}{p}}(x_2)\right\|_X^{\frac{1}{q}}\\
\le& C\left[\left(\frac{p}{p-s_1}\right)^p+\frac{1}{s_1-1}\right]^{\frac{1}{p}}\left[\left(\frac{p}{p-s_2}\right)^p+\frac{1}{s_2-1}\right]^{\frac{1}{p}}\\
&\times\widetilde{V}_1^{\frac{(1-s_1)}{p}}(t_1)\widetilde{V}_2^{\frac{(1-s_2)}{p}}(t_2)
\end{aligned}
$$

or,

$$
\begin{aligned}
&\left[\frac{\left(\frac{p}{p-s_1}\right)^p}{\left(\frac{p}{p-s_1}\right)^p+\frac{1}{s_1-1}}\right]^{\frac{1}{p}}\left[\frac{\left(\frac{p}{p-s_2}\right)^p}{\left(\frac{p}{p-s_2}\right)^p+\frac{1}{s_2-1}}\right]^{\frac{1}{p}}\\
&\times\widetilde{V}_1^{\frac{(s_1-1)}{p}}(t_1)\widetilde{V}_2^{\frac{(s_2-1)}{p}}(t_2)\left\|\chi_{[(0,t_1)}\chi_{(0,t_2)}\widetilde{V}_1^{\frac{q(p-s_1)}{p}}(t_1)\widetilde{V}_2^{\frac{q(p-s_1)}{p}}(t_2)u\right\|_X^{\frac{1}{q}}\le C;
\end{aligned}
$$

i.e.,

$$\left[\frac{\left(\frac{p}{p-s_1}\right)^p}{\left(\frac{p}{p-s_1}\right)^p+\frac{1}{s_1-1}}\right]^{\frac{1}{p}}\left[\frac{\left(\frac{p}{p-s_2}\right)^p}{\left(\frac{p}{p-s_2}\right)^p+\frac{1}{s_2-1}}\right]^{\frac{1}{p}} A^*(s_1,s_2)\le C$$

and the necessity follows. □

Remark 2. The assertion of Theorem 2 is new even for the case $X = L^1$, which gives the L^p-L^q boundedness of H_2.

3 The Operator G_2 As a Limiting Case of H_2.

In this section, we shall characterize the boundedness of the operator G_2 defined in (1.2). In fact, the idea is to use the boundedness of H_2 obtained in Theorem 1 and apply limiting arguments. The result generalizes a result of [14], [15] who proves it for $X = L^1$. Such technique has been used in the one dimensional situation to derive the boundedness of the geometric mean operator G. The corresponding L^p-L^q boundedness is obtained in [12] while the L^p-X^q boundedness in [3].

Theorem 3. *Let $0 < p \le q < \infty$, $s_1, s_2 \in (1,p)$, and u, v be weight functions defined on $\mathbb{R}^2_+$. Let $\theta_i(x_i) = x_i^{\frac{-s_i}{p}}$, $i = 1,2$ and*

$$w(x_1,x_2)=\left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\!\!\int_0^{x_2}\log\frac{1}{v(t_1,t_2)}\,dt_1\,dt_2\right)\right]^{\frac{q}{p}}u(x_1,x_2)\,. \tag{3.1}$$

Then the inequality

$$\left\|(G_2f)^q u\right\|_X^{\frac{1}{q}}\le C\left\{\int_0^\infty\int_0^\infty f^p(x_1,x_2)v(x_1,x_2)\,dx_1\,dx_2\right\}^{\frac{1}{p}} \tag{3.2}$$

holds for all positive and measurable functions f on $(0,\infty)\times(0,\infty)$ if and only if $\sup_{\substack{y_1\in(0,\infty)\\ y_2\in(0,\infty)}} B(s_1,s_2)<\infty$, where

$$B(s_1,s_2) := y_1^{\frac{s_1-1}{p}}\,y_2^{\frac{s_2-1}{p}}\left\|\theta_1(x_1)\theta_2(x_2)w(x_1,x_2)^{\frac{1}{q}}\chi_{[y_1,\infty)}(x_1)\chi_{[y_2,\infty)}(x_2)\right\|_{X^q}$$

and the best constant C in (3.2) satisfies

$$\begin{aligned}&\sup_{s_1,s_2>1}\left(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1}\right)^{\frac{1}{p}}B(s_1,s_2)r\\ &\le C\le\inf_{s_1,s_2>1}e^{\frac{s_1+s_2-2}{p}}B(s_1,s_2)\,.\end{aligned} \tag{3.3}$$

Proof. Writing $fv^{\frac{-1}{p}}$ for f, we find that the inequality (3.2) becomes equivalent to

$$\left\|(G_2 f)^q w\right\|_X^{\frac{1}{q}} \leq C\left\{\int_0^\infty \int_0^\infty f^p(x_1,x_2)\,dx_1\,dx_2\right\}^{\frac{1}{p}}, \tag{3.4}$$

with w as given by (3.1). Here, we have used the facts that $G_2(gh) = G_2(g)G_2(h)$ and $G_2(g^y) = [G_2(g)]^y$ almost everywhere on $(0,\infty)\times(0,\infty)$ for all measurable functions g and h for which $G_2(g)$ and $G_2(h)$ are defined almost everywhere on $(0,\infty)\times(0,\infty)$ and $y \in \mathbb{R}$.

Let $0 < \alpha < p$. Now, writing f^α, $wx_1^{\frac{-q}{\alpha}} x_2^{\frac{-q}{\alpha}}, 1, 1, \dfrac{p}{\alpha}, \dfrac{q}{\alpha}$ for, respectively, f, u, v_1, v_2, p, q in Theorem 1 we find that the inequality

$$\begin{aligned} &\sup_{h>0}\left\{\int_0^\infty \int_0^\infty \left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2} f^\alpha(t_1,t_2)\,dt_1\,dt_2\right)^{\frac{q}{\alpha}}\right. \\ &\qquad \left.\times\, w(x_1,x_2)h(x_1,x_2)dx_1, dx_2 : \|h\|_{X'} \leq 1\right\}^{\frac{1}{q}} \\ &\leq C\left\{\int_0^\infty \int_0^\infty f^p(x_1,x_2)\,dx_1\,dx_2\right\}^{\frac{1}{p}} \end{aligned} \tag{3.5}$$

holds for all $C > 0$ and for all measurable functions $f > 0$ if and only if

$$\sup_{t_1,t_2>0} \widetilde{A} := \sup_{t_1,t_2>0} t_1^{\frac{s_1-1}{p}} t_2^{\frac{s_2-1}{p}} \left\|\theta_1^q \theta_2^q w \chi_{[t_1,\infty)}\chi_{[t_2,\infty)}\right\|_X^{\frac{1}{q}} < \infty$$

and the constant C in (3.5) has the estimate

$$\begin{aligned} &\sup_{1<s_1,s_2<\frac{p}{\alpha}} \left(\frac{p}{p-\alpha s_1}\right)^{\frac{1}{\alpha}}\left[\left(\frac{p}{p-\alpha s_1}\right)^{\frac{p}{\alpha}} + \frac{1}{s_1-1}\right]^{\frac{-1}{p}} \\ &\qquad \times \left(\frac{p}{p-\alpha s_2}\right)^{\frac{1}{\alpha}}\left[\left(\frac{p}{p-\alpha s_2}\right)^{\frac{p}{\alpha}} + \frac{1}{s_2-1}\right]^{\frac{-1}{p}} \widetilde{A}^{\frac{1}{\alpha}} \\ &\leq C \leq \inf_{1<s_2,s_2<\frac{p}{\alpha}} \widetilde{A}^{\frac{1}{\alpha}} \left(\frac{p-\alpha}{p-\alpha s_1}\right)^{\frac{p-\alpha}{\alpha p}} \left(\frac{p-\alpha}{p-\alpha s_2}\right)^{\frac{p-\alpha}{\alpha p}}. \end{aligned} \tag{3.6}$$

Note that

$$\widetilde{A}^{\frac{1}{\alpha}} = B\,. \tag{3.7}$$

Now, taking the limit as $\alpha \to 0+$, we find that the inequality (3.5) becomes (3.2) which, in views of (3.7), holds if and only if $B < \infty$. Also, when $\alpha \to 0+$, the estimate (3.6) becomes (3.3). The lower bound in (3.3) can be obtained if we use the test function

$$
\begin{aligned}
g(x_1,x_2) = {} & t_1^{\frac{-1}{p}} t_2^{\frac{-1}{p}} \chi_{(0,t_1)}(x_1)\chi_{(0,t_2)}(x_2) + t_1^{\frac{-1}{p}} \chi_{(0,t_1)}(x_1) \frac{e^{\frac{-s_2}{p}} t_2^{\frac{s_2-1}{p}}}{x_2^{\frac{s_2}{p}}} \\
& \times \chi_{(t_2,\infty)}(x_2) + \frac{e^{\frac{-s_1}{p}} t_1^{\frac{s_1-1}{p}}}{x_1^{\frac{s_1}{p}}} \chi_{(t_1,\infty)}(x_1) t_2^{\frac{-1}{p}} \chi_{(0,t_2)}(x_2) \\
& + e^{\frac{-(s_1+s_2)}{p}} \frac{t_1^{\frac{s_1-1}{p}} t_2^{\frac{s_2-1}{p}}}{x_1^{\frac{s_1}{p}} x_2^{\frac{s_2}{p}}} \chi_{(t_1,\infty)}(x_1)\chi_{(t_2,\infty)}(x_2)
\end{aligned}
$$

in inequality (3.4) and follow similar arguments as in [16, theorem 3.1]. $\square$

4 The Operator G_2 Revisited.

In this section, we give another characterization for the L^p-X^q boundedness of G_2 with a different approach. In fact, we do not use here the limiting arguments as done in Theorem 3. Also, in this case, the weight on R.H.S. of the inequality need not be of product type. Furthermore, the functions, here, will be defined on $[0,b_1]\times[0,b_2]$, $0<b_i\le\infty$, $i=1,2$ so as to cover finite domains as well. Precisely, we prove the following.

Theorem 4. *Let $0<p\le q<\infty$, $0<b_1$, $b_2\le\infty$, $s_1,s_2>1$ and u,v be weight functions defined on $\mathbb{R}^2_+$. Then the inequality*

$$
\left\|(G_2f)^q u\chi_{(0,b_1)}\chi_{(0,b_2)}\right\|_X^{\frac{1}{q}} \le C\left\{\int_0^{b_1}\int_0^{b_2} f^p(x_1,x_2)v(x_1,x_2)\,dx_1\,dx_2\right\}^{\frac{1}{q}} \quad (4.1)
$$

holds for all measurable functions $f>0$ on $[0,b_1]\times[0,b_2]$ if and only if $\sup_{\substack{y_1\in(0,b_1)\\ y_2\in(0,b_2)}} \widetilde{B}(s_1,s_2)<\infty$, where

$$
\widetilde{B}(s_1,s_2) := y_1^{\frac{s_1-1}{p}} y_2^{\frac{s_2-1}{p}} \left\|\theta_1(x_1)\theta_2(x_2)w(x_1,x_2)^{\frac{1}{q}}\chi_{[y_1,b_1)}(x_1)\chi_{[y_2,b_2)}(x_2)\right\|_{X^q},
$$

where θ_i are as used in Theorem 3 and w is given by (3.1). Moreover, the best constant C in (4.1) has the estimate

$$
\begin{aligned}
& \sup_{s_1,s_2>1}\left(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1}\right)^{\frac{1}{p}}\widetilde{B}(s_1,s_2) \\
\le {} & C \le \inf_{s_1,s_2>1} e^{\frac{s_1+s_2-2}{p}}\widetilde{B}(s_1,s_2)
\end{aligned}
$$

Proof. Taking $g(x_1,x_2) = f^p(x_1,x_2)v(x_1,x_2)$, the inequality in (4.1) becomes

$$\begin{aligned}\sup_{h>0}\bigg\{\int_0^{b_1}\int_0^{b_2}\bigg[\exp\bigg(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log g(y_1,y_2)\,dy_1\,dy_2\bigg)\bigg]^{\frac{q}{p}}\\ \times\ w(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\bigg\}^{\frac{1}{q}}\\ \le C\bigg\{\int_0^{b_1}\int_0^{b_1} g(x_1,x_2)\,dx_1\,dx_2\bigg\}^{\frac{1}{p}},\end{aligned} \tag{4.2}$$

where w is as given in (3.1). For fixed t_1 and t_2, $0 < t_1 < b_1$, $0 < t_2 < b_2$, we choose the test function

$$\begin{aligned}g(x_1,x_2) =& t_1^{-1}t_2^{-1}\chi_{(0,t_1)}(x_1)\chi_{(0,t_2)}(x_2) + t^{-1}\chi_{(0,t_1)}(x_1)\frac{e^{-s_2}t_2^{s_2-1}}{x_2^{s_2}}\\ &\times\ \chi_{(t_2,\infty)}(x_2) + \frac{e^{-s_1}t_1^{s_1-1}}{x_1^{s_1}}\chi_{(t_1,\infty)}(x_1)t_2^{-1}\chi_{(0,t_2)}(x_2)\\ &+\ \frac{e^{-(s_2+s_1)}t_1^{s_1-1}t_2^{s_2-1}}{x_1^{s_1}x_2^{s_2}}\chi_{(t_1,\infty)}(x_1)\chi_{(t_2,\infty)}(x_2).\end{aligned}$$

The necessity can now be obtained if we use the above test function in (4.2) and follow the arguments similar to [15, Theorem 4.1].

In order to prove the sufficiency, take $y_1 = x_1t_1$ and $y_2 = x_2t_2$ so that (4.2) becomes

$$\begin{aligned}\sup_{h>0}\bigg\{\int_0^{b_1}\int_0^{b_2}\bigg[\exp\bigg(\int_0^{1}\int_0^{1}\log g(x_1t_1,x_2t_2)\,dt_1\,dt_2\bigg)\bigg]^{\frac{q}{p}}\\ \times\ w(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'}\le 1\bigg\}^{\frac{1}{q}}\\ \le C\bigg\{\int_0^{b_1}\int_0^{b_2} g(x_1,x_2)\,dx_1\,dx_2\bigg\}^{\frac{1}{p}}.\end{aligned} \tag{4.3}$$

By using the fact

$$\bigg(\exp\int_0^1\int_0^1\log t_1^{(s_1-1)}t_2^{(s_2-1)}\,dt_1\,dt_2\bigg)^{\frac{q}{p}} = e^{\frac{-(s_1+s_2-2)q}{p}}$$

and by Jensen's inequality, the L.H.S. of (4.3) becomes

$$e^{\frac{(s_1+s_2-2)}{p}}\sup_{h>0}\bigg\{\int_0^{b_1}\int_0^{b_2}\bigg[\exp\bigg(\int_0^1\int_0^1\log(t_1^{(s_1-1)}t_2^{(s_2-1)}g(x_1t_1,x_2t_2))\,dt_1\,dt_2\bigg)\bigg]^{\frac{q}{p}}$$

$$
\begin{aligned}
&\qquad \times\, w(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'} \le 1\Big\}^{\frac{1}{q}} \\
&\le e^{\frac{(s_1+s_2-2)}{p}} \sup_{h>0}\Big\{\int_0^{b_1}\int_0^{b_2}\Big(\int_0^1\int_0^1 t_1^{(s_1-1)}t_2^{(s_2-1)} \\
&\qquad \times\, g(x_1t_1,x_2t_2)\,dt_1\,dt_2\Big)^{\frac{q}{p}} w(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'} \le 1\Big\}^{\frac{1}{q}} \\
&= e^{\frac{(s_1+s_2-2)}{p}} \sup_{h>0}\Big\{\int_0^{b_1}\int_0^{b_2}\Big[\int_0^{x_1}\int_0^{x_2} y_1^{s_1-1}y_2^{s_2-1}g(y_1,y_2)\,dy_1\,dy_2\Big]^{\frac{q}{p}} \\
&\qquad \times\, \frac{w(x_1,x_2)}{x_1^{\frac{s_1q}{p}}\,x_2^{\frac{s_2q}{p}}}h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'} \le 1\Big\}^{\frac{1}{q}} \\
&\le e^{\frac{(s_1+s_2-2)}{p}} \sup_{h>0}\Big\{\int_0^{b_1}\int_0^{b_2} y_1^{s_1-1}y_2^{s_2-1}g(y_1,y_2)\Big(\int_{y_1}^{b_1}\int_{y_2}^{b_2} x_1^{\frac{-s_1q}{p}}\,x_2^{\frac{-s_2q}{p}} \\
&\qquad \times\, w(x_1,x_2)h(x_1,x_2)\,dx_1\,dx_2 : \|h\|_{X'} \le 1\Big)^{\frac{p}{q}}\,dy_1\,dy_2\Big\}^{\frac{1}{p}} \\
&\le\; e^{\frac{(s_1+s_2-2)}{p}}\widetilde{B}(s_1,s_2)\Big\{\int_0^{b_1}\int_0^{b_2} g(y_1,y_2)\,dy_1\,dy_2\Big\}^{\frac{1}{p}}
\end{aligned}
$$

and we are done. □

Theorem 4 extends a result of [14], [16] who proved it for $X = L^1$.

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