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MINIMAL AND ω-MINIMAL SETS OF FUNCTIONS WITH CONNECTED G_δ GRAPHS

Abstract

Let $I = [0, 1]$, and let $\mathcal{J}$ be the class of functions $I \to I$ with connected G_δ graph. Recently it was shown that dynamical systems generated by maps in $\mathcal{J}$ have some nice properties. Thus, the Sharkovsky's theorem is true, and a map has zero topological entropy if and only if every periodic point has period 2^n, for an integer $n \geq 0$. In this paper we consider, for a map φ in $\mathcal{J}$, properties of ω-minimal sets; i.e., sets $M \subset I$ such that the ω-limit set $\omega_\varphi(x)$ is M, for every $x \in M$. If φ is continuous, then, as is well-known, M is minimal if and only if M is non-empty, closed, $\varphi(M) \subseteq M$, any point in M is uniformly recurrent, and no proper subset of M has this property. In this paper we prove that the same is true for $\varphi \in \mathcal{J}$ with zero topological entropy, but not for an arbitrary $\varphi \in \mathcal{J}$.

1 Introduction.

Denote by $\mathcal{J}$ the class of maps of the interval I with connected G_δ graph. We say that $f \in \text{Conn}$ if f has a connected graph, $f \in \mathcal{G}_\delta$ if the graph of f is a G_δ set, $f \in \mathcal{D}$ if f has Darboux property and $f \in \mathcal{B}_1$ if f is a Baire-1 function. It is well known that

$$\mathcal{DB}_1 := \mathcal{D} \cap \mathcal{B}_1 \subset \text{Conn} \cap \mathcal{G}_\delta =: \mathcal{J} \subset \mathcal{D}.$$

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In [4] it was proved that, for maps in $\mathcal{J}$, the Sharkovsky's theorem is true. In [2] we proved that the notion of topological entropy can be defined for an arbitrary discontinuous map of a compact metric space such that the main properties, known in the continuous case, remain true. Using this we prove the following.

Proposition 1.1. (Cf. [2].) *Let $f \in \mathcal{J}$. Then f has a positive topological entropy if and only if f has a periodic point whose period is not a power of* 2.

We also showed that maps in $\mathcal{J}$ fail to have other standard properties. E.g., the ω-limit set $\omega_f(x)$, with $f \in \mathcal{J}$, need not be strongly invariant. Actually, as we will see later, it even need not be invariant. Based on these results we are interested in properties of minimal sets for maps in $\mathcal{J}$. For continuous maps there are several equivalent definitions of this notion. Unfortunately, in $\mathcal{J}$ this is not the case. Thus for $f \in \mathcal{J}$ we define minimal and ω-minimal sets of function f.

Definition 1.1. A non-empty set M is minimal if it is closed, $f(M) = M$ and no proper subset of M has these properties.

Definition 1.2. A non-empty set M is ω-minimal if $\omega_f(x) = M$ for every $x \in M$.

For continuous functions the notion ω-minimal set is the same as the notion minimal set. Our main result is the following theorem; its proof is based on Lemmas 3.1, 3.5, 3.6 and 3.7 below, and can be found at the end of Section 3.

Theorem 1.1. *Let $f \in \mathcal{J}$ have zero topological entropy, and let $\emptyset \neq M \subset I$. Then the following two conditions are equivalent:*

1. *M is an ω-minimal set of f,*
2. *M is closed, $f(M) \subseteq M$, every point in M is recurrent, and no proper subset of M has these properties.*

Moreover, if M is ω-minimal, then $f(M) = M$ and any point in M is uniformly recurrent.

It should be noted that Theorem 1.1 is true for continuous maps. But it is nontrivial since $\mathcal{J}$ contains discontinuous maps with zero topological entropy. An easy example is a discontinuous map f with $f(x) > x$ for $x \in [0, 1)$, and $f(1) = 1$. A less trivial example can be obtained in the following way. Let f be a continuous function with zero topological entropy such that, for some $x \in I$, $\omega_f(x)$ is the Cantor set M, and J_0 is an open wandering interval whose endpoints lie in M. Then, for every $i \in \mathbb{N}$, there is an open interval J_i with end-points in M such that $f(J_{i+1}) = J_i$. Let g_i be a surjective map from J_i onto the closure of J_{i-1} of the form $\lambda_i \cdot \sin \frac{1}{\rho(x,J_i)} + \theta_i$, where $\rho(x, J_i)$ is the distance of x from $I \setminus J_i$, and λ_i, θ_i are suitable coefficients. Then the set of points of discontinuity is dense in M and the ω-limit set of any such point is the set M.

The proof of Theorem 1.1, like the proof of Proposition 1.1 or Sharkovsky's theorem for maps in $\mathcal{J}$, is based on the fact [4] (see also [3] or [5]) that, for any interval $K \subset I$ and any $f \in \mathcal{J}$,

$$f(K) \text{ is an interval}$$

and

$$f^n(\overline{K}) \subseteq \overline{K} \text{ or } f^n(\overline{K}) \supseteq \overline{K} \Rightarrow \; f^n(p) = p, \text{ for some } p \in \overline{K}.$$

We will use these properties in the sequel without specific references.

The paper is organized as follows. The next section contains some preliminary results and examples describing properties of ω-minimal sets. In particular, examples given in Lemmas 2.2–2.4 show that zero topological entropy of the map f is essential in Theorem 1.1, whose proof can be found in Section 3. Terminology and notions used in this paper are standard ones, if not stated otherwise; see also [1] or [2].

2 Preliminary Results Concerning Functions in $\mathcal{J}$.

Some properties of ω-minimal sets of continuous functions are shared by the maps in the class $\mathcal{J}$. These general properties are described in the following lemma. First we recall some notions.

Let $f \in \mathcal{J}$. A point $x \in I$ is *recurrent*, if $x \in \omega_f(x)$. We denote the set of recurrent points of f by $\operatorname{Rec} f$. The map f is *topologically transitive* on $A \subseteq I$ (hereafter simply $f|_A$ is transitive), if for every open, nonempty subsets $U, V \subset A$ there is a positive integer n such that $U \cap f^n(V) \neq \emptyset$. Note that in general $f(A) \nsubseteq A$.

Lemma 2.1. *If $f \in \mathcal{J}$ and M is an infinite ω-minimal set of f, then M has the following properties:*

1. *M is closed (thus $M = \overline{M}$),*
2. *M is a subset of the set of recurrent points of f ($M \subset \operatorname{Rec} f$),*
3. *$f|_M$ is transitive,*

4. *M is a nowhere dense set,*

5. *if $M' \neq \emptyset$ is a closed subset of M such that $f(M') \subseteq M'$, then $M' = M$.*

PROOF. 1. Since $M = \omega_f(x)$ for $x \in M$, M is closed.

2. Obvious by definition.

3. Let U, V be open disjoint sets such that there are an $x \in U \cap M$ and a $y \in V \cap M$. Since $x \in \omega_f(y)$, there is $m \in \mathbb{N}$ with $f^m(y) \in U \cap M$ and consequently, $f^m(V) \cap U \neq \emptyset$.

4. If M contains a subinterval J, then no periodic point lies in J and hence, J contains no recurrent point, which is not possible (cf. [2] - Lemmas 2.4 and 2.6).

5. If $M \neq M'$, $x \in M' \subseteq M$, M' is closed, and $f(M') \subseteq M'$, then $\omega_f(x) \subseteq M' \neq M$, a contradiction. □

It is well-know that, for a continuous map φ of a compact metric space, any ω-limit set, and hence, any minimal set M is strongly invariant so that $\varphi(M) = M$. Also, any ω-limit set contains a minimal set. Nothing of this is generally true for functions in $\mathcal{J}$, even when M is finite. By the next two lemmas, it may happen that $\varphi(M)$ is disjoint from M, or is a proper subset of M, and that an ω-limit set contains no ω-minimal set.

Lemma 2.2. *There is an $f \in \mathcal{J}$ with an infinite ω-minimal set M such that $f(M) \cap M = \emptyset$ and M contains an isolated point.*

PROOF. Let $M = Q \cup \{a\}$ where Q is the middle-third Cantor set and a does not belongs to the closure of the set Q. Thus a is isolated point of M. Let $f|_M$ is identically equal to $\alpha \notin M$ and $\omega_f(\alpha) = M$. Now it is sufficient to define f on the complement of the set $M \cup \{\alpha\}$ such that $f \in \mathcal{J}$. Then M is ω-minimal set with isolated point. Obviously, the map f has positive topological entropy. □

The next lemma shows that, for $f \in \mathcal{J}$, image of an ω-minimal set M can be a proper invariant subset of M.

Lemma 2.3. *There is a map $f \in \mathcal{J}$ possessing an ω-minimal set M such that $f(M)$ is a proper subset of M.*

PROOF. Let $f \in \mathcal{J}$ be a function with graph given by Figure 1. The ω-minimal set M of this function f is the middle-third Cantor set and every point of M is uniformly recurrent. If $f(1) = 2/3$, then there is a proper f-invariant subset $M\setminus\{0\}$ of M. Obviously, this function has positive topological entropy since any two intervals U, V complementary to M form a horseshoe; i.e., $U \cup V \subset f^m(U) \cap f^n(V)$, for positive integers m, n. $\square$

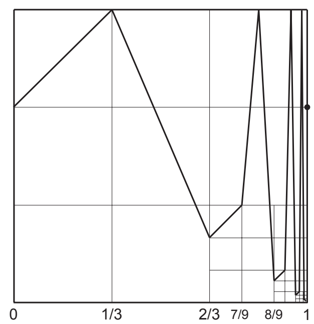

Figure 1: f has an ω-minimal set M such that $f(M) \subset M$.

Recall that a point $x \in [0, 1]$ is *uniformly recurrent* if, for each open set U containing x, there exists a positive integer N such that if $f^m(x) \in U$ with $m \geq 0$, then $f^{m+k}(x) \in U$ for some k with $0 < k \leq N$. In view of the above examples we are distinguishing the notions invariant and strongly invariant. We say that a set A is *f-invariant* if $f(A) \subseteq A$ and is *strongly f-invariant* if $f(A) = A$.

Lemma 2.2 shows, that Theorem 1.1 is not true for maps in $\mathcal{J}$ with positive topological entropy. In particular, in this case, condition (1) does not imply (2). We show that, in general, the converse also is not true.

Lemma 2.4. *There is an $f \in \mathcal{J}$ (with $h(f) > 0$) possessing a set $M \neq \emptyset$ which is not ω-minimal but satisfies condition (2) of Theorem 1.1.*

PROOF. Let $A \subset I$ be an infinite minimal set of a continuous map $g : I \to I$ defined by

$$g(x) = \begin{cases} 0 & \text{for } x = 1, \\ x + \frac{2}{3} & \text{for } 0 \leq x \leq \frac{1}{3}, \\ \frac{16}{9} - \frac{7}{3}x & \text{for } \frac{1}{3} \leq x \leq \frac{2}{3}, \\ \frac{g(3x-2)}{3} & \text{for } \frac{2}{3} \leq x \leq 1. \end{cases}$$

Then A is the middle-third Cantor set (see e.g. [1]) and $g|_A$ is a homeomorphism.

Let φ be the affine map $I \to [\frac{1}{3}, \frac{2}{3}]$, given by $x \mapsto \frac{1}{3}(1+x)$, and let $B = \varphi(A)$. Thus, the sets A and B have just two points, $a_1 = \frac{1}{3}$ and $b_1 = \frac{2}{3}$, in common. Let $g_B = \varphi \circ g \circ \varphi^{-1}$. Then $g|_A$ and $g_B|_B$ are conjugate. Let $M = A \cup B$.

Since $g|_A$ is a homeomorphism, there is a unique point $a_0 \in A$ such that $g(a_0) = a_1$. Similarly, there is unique $b_0 \in B$ with $g_B(b_0) = b_1$. Let $C = \{a_0, a_1, b_0, b_1\}$. Define f on M by $f(x) = g(x)$ if $x \in A \setminus C$, $f(x) = g_B(x)$ if $x \in B \setminus C$, $f(a_1), f(b_0) \in B \setminus C$, and $f(a_0), f(b_1) \in A \setminus C$.

Clearly, M is closed and $f(M) = M$. Moreover, any point in M is recurrent, since $\omega_f(x) = A = \omega_f(b_1)$ for $x \in A$, $x \neq a_1$, and $\omega_f(x) = B = \omega_f(a_1)$ if $x \in B$, $x \neq b_1$. Finally, $f(A) \not\subseteq A$, $f(B) \not\subseteq B$ and consequently, M contains no closed invariant proper subset. To finish the argument, extend f to

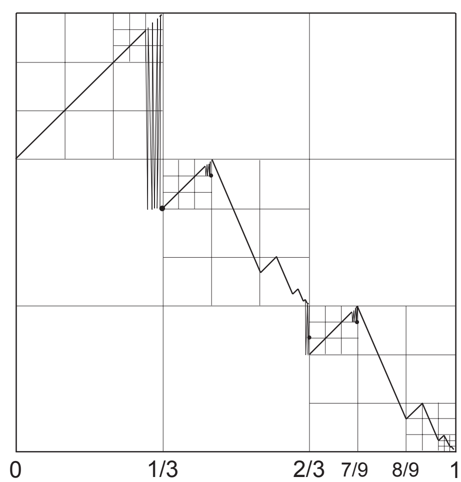

Figure 2: f has a minimal set which is not ω-minimal.

a map in $\mathcal{J}$. This is possible since $f|_M$ is continuous, except for the points in C. Therefore there is a sequence $\{I_n\}_{n=1}^{\infty}$ of intervals complementary to M, whose cluster points are just the points in C. Let f map continuously any I_n onto I, and let f connect linearly the endpoints of any complementary interval different from I_n. The function f has the graph given by Figure 2. □

We conclude this section by the following simple result. Its proof is obvious and we omit it.

Lemma 2.5. *Let $f \in \mathcal{J}$, and let $M \neq \emptyset$ be a finite invariant set. Then M contains a cycle. Hence, if M is an invariant ω-minimal set, then M is a cycle.*

3 Proof of the Main Theorem.

Before stating the next lemma we need some terminology. Given a map $f : I \to I$, an interval J is *weakly periodic* with period k if $f^k(J) \subset J$, and $f^j(J) \not\subset J$, whenever $1 \leq j < k$. A *simple system* for f is a system $\mathcal{S} = \{J_\alpha; \alpha \in \{0,1\}^n, n \in \mathbb{N}\}$ of minimal compact intervals in I such that the intervals $\{J_\alpha\}_{\alpha \in \{0,1\}^n}$ form a weakly f-periodic orbit of period 2^n and, for any α, $J_{\alpha 0} \cup J_{\alpha 1} \subset J_\alpha$. Here $\alpha 0$ is the concatenation of words α and 0. Moreover, the coding can be chosen such that $f(J_\alpha) = J_{\alpha+1}$ where 1 is added to α mod 2 from the left; e.g., $f(J_{01}) = J_{11}, f(J_{11}) = J_{00}$. By induction it is obvious that there are such minimal intervals J_α, $\alpha \in \{0,1\}^n$. Since intervals $J_{00}, J_{01}, J_{10}, J_{11}$ are compact and form a weakly periodic orbit, it follows that their union cannot be an interval. Hence the sets $J_{00} \cup J_{01}$ and $J_{10} \cup J_{11}$ are contained in disjoint compact intervals. By induction, if $\alpha, \beta \in \{0,1\}^n$ have different digits at the place $i < n$, then $J_\alpha \cap J_\beta = \emptyset$. It is clear that $J_{00} \cup J_{01} \cup J_{10} \cup J_{11}$ is the subset of two disjoint intervals I_1 and I_2 whereas $J_{00} \cup J_{01} \subset I_1$ and $J_{10} \cup J_{11} \subset I_2$. By induction the union of intervals J_α, $\alpha \in \{0,1\}^n$ with period 2^n is subset of 2^{n-1} intervals $I_1, I_2, \ldots, I_{2^{n-1}}$ whereas for $i = 1, 2, \ldots, 2^{n-1}$ we have $\bigcup_{\alpha \in \{0,1\}^n} J_\alpha \cap I_i \neq \emptyset$ and, for any $a \in \{0,1\}^{n-1}$, there is $1 \leq i \leq 2^{n-1}$ such that $J_{a0} \cup J_{a1} \subset I_i$. An invariant nonempty set $A \subset I$ is a *simple set* for f if there is a simple system $\mathcal{S}$ such that

$$A \subseteq \bigcap_{n=1}^{\infty} \bigcup \{J_\alpha; \alpha \in \{0,1\}^n\}. \tag{3.1}$$

A *maximal simple set* is a simple system A given by (3.1) such that $A \supseteq B$ for any simple set $B \subseteq \bigcap_{n=1}^{\infty} \bigcup \{J_\alpha; \alpha \in \{0,1\}^n\}$. Obviously, any maximal simple set is closed and any simple set is infinite. Also, the above notions slightly differ from those for continuous maps of the intervals since continuous image of a closed interval is closed, which is not true for maps in $\mathcal{J}$. However, the main properties that any simple set is infinite and any maximal set is compact, remain satisfied in $\mathcal{J}$, as we show later.

Lemma 3.1. *Let $A \subset I$ be a maximal simple set for a map $f : I \to I$. Then there is an unique ω-minimal set $M \subset A$ such that $f(M) = M$, and $\omega_f(a) = M$, for any $a \in A$.*

PROOF. Assume that A is given by (3.1). Then $A = \bigcup_{\alpha \in \{0,1\}^{\mathbb{N}}} I_\alpha$, where I_α, for $\alpha = a_1 a_2 \dots$, is the intersection of the nested system $\{I_{a_1 a_2 \dots a_n}\}_{n=1}^{\infty}$. Since the sets I_α are disjoint and there are uncountably many of them, there is an $\alpha_0 \in \{0,1\}^{\mathbb{N}}$ such that $I_{\alpha_0} = \{p\}$. Let $\omega_f(p) = M$. Let $a \in A$. For every $k \in \mathbb{N}$ there are $\alpha_k, \beta_k \in \{0,1\}^k$ such that $a \in J_{\alpha_k}$ and $p \in J_{\beta_k}$. Since J_{α_k} and J_{β_k} are f-periodic intervals of period 2^k belonging to the same orbit, there is an n_k with $0 \le n_k < 2^k$ such that $f^{n_k}(a) \in J_{\beta_k}$. Since $\operatorname{diam} J_{\beta_k}$ vanishes for $k \to \infty$, $\lim_{k\to\infty} f^{n_k}(a) = p$ and consequently, $p \in \omega_f(a)$. Hence, for any $a \in A$, $M \subseteq \omega_f(a) \subseteq A$ since A is closed and f-invariant and consequently, M is ω-minil. □

Lemma 3.2. (Cf. [2].) *Assume that $f \in \mathcal{J}$ has a horseshoe; i.e., compact intervals U, V with at most one point in common such that $U \cup V \subset f^m(U) \cap f^n(V)$, for positive integers m, n. Then $h(f) > 0$.*

Lemma 3.3. *Assume $f \in \mathcal{J}$ and $h(f) = 0$. If $\omega_f(x) =: \tilde{\omega}$ contains at least four distinct points for some $x \in I$, $a = \min \tilde{\omega}$ and $b = \max \tilde{\omega}$, then there is a fixed point $c \in (a, b)$.*

PROOF. Assume there is no fixed point in (a, b). Let $\{x_n\}_{n=0}^{\infty}$ be the trajectory of x. Then $f(y) < y$ for any $y \in (a, b)$, or $f(y) > y$ for any $y \in (a, b)$. Assume, e.g., the first case. Then $f(a) = a$. Indeed, otherwise we would have $f(a) = d < a$. For every $\delta > 0$ there is a $n(\delta) \in \mathbb{N}$ such that $x_n > a - \delta$ for every $n \ge n(\delta)$. Then a lies in the closure of set of fixed points since $a < b$. Consequently, there are fixed points p, q and $n_1, n_2 \in \mathbb{N}$ such that $d < x_{n_1} < p < x_{n_2} < q < a < x_{n_i+1}$, for $i = \{1, 2\}$. Then $f^2([x_{n_1}, p]) \supset f([p, a]) \supset [d, a]$, and $f^2([x_{n_2}, q]) \supset f([q, a]) \supset [d, a]$, contrary to Lemma 3.2. Thus, we may assume that

$$f(a) = a \text{ and } f(y) < y, \text{ for any } y \in (a, b). \tag{3.2}$$

To finish the argument we consider two cases.

CASE 1. Assume $a < x_{n+1} < x_n < b$, for some $n \in \mathbb{N}$. By (3.2), there are $m, k \in \mathbb{N}$ such that $n < m < k$ and $x_{m+1} < a < x_n < x_k$. Let $U = [a, x_{n+1}]$ and $V = [x_{n+1}, x_n]$. Then, contrary to Lemma 3.2,

$$U \cup V \subset f^{k-n-1}(U) \cap f^{k-n-1}(V).$$

CASE 2. Suppose that $x_{n+1} < a$ whenever $x_n \in (a, b)$. If $f(b) > a$, then, for any $x_m \in (a, b)$ there is a $p \in (x_m, b)$ such that $f(p) = a$. Take $k \in \mathbb{N}$ with $x_{m+k} > p$. Then, for $U = [a, x_m]$ and $V = [x_m, p]$, we would have $U \cup V \subset f^k(U) \cap f^k(V)$, a contradiction. Thus, $f(b) \le a$. Let c, d be distinct points in $(a, b) \cap \tilde{\omega}$. Then there are increasing sequences $\{m_k\}_{k=1}^{\infty}, \{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that the compact intervals U, V with endpoints c, x_{m_1}, resp. d, x_{n_1} are disjoint, and $x_{m_k} \in U$, $x_{n_k} \in V$, for any k. Let $s \geq \max\{m_1, n_1\}$ be such that x_s is greater than the endpoints of U and V. Since $f(x_{m_k}) < a$, for any k, and $a = \min \tilde{\omega}$, $\lim_{k\to\infty} f(x_{m_k}) = a$. Thus, $f(U) \supset [\min_k f(x_{m_k}), a)$. Consequently $f^{s-m_1}(U) = f^{s-m_1-1}(f(U)) \supset (a, x_s] \supset U \cup V$. Similarly $f^{s-n_1}(V) \supset U \cup V$, contrary to Lemma 3.2. □

Lemma 3.4. *Let $f \in \mathcal{J}$ with $h(f) = 0$, and let $\tilde{\omega} := \omega_f(x)$. Let $a < b$ be the minimal and maximal points of $\tilde{\omega}$, and $c \in (a,b)$ a fixed point of f. If $x \in (a,b)$, then c lies between x_n and x_{n+1}, for each $n \in \mathbb{N}$. Consequently, either $\omega_{f^2}(x) \subset [a,c]$ and $\omega_{f^2}(f(x)) \subset [c,b]$, or $\omega_{f^2}(x) \subset [c,b]$ and $\omega_{f^2}(f(x)) \subset [a,c]$.*

PROOF. Let $\{x_n\}_{n=0}^{\infty}$ be the trajectory of x such that, e.g., $x_n, x_{n+1} > c$ for some $n \in \mathbb{N}$. If $x_{n+1} > x_n$ let $m > n$ be the minimal integer such that $x_{m+1} < x_m$. Then there is a fixed point $d \in (x_{m-1}, x_m)$. Let $U = [c, x_{m-1}]$ and $V = [x_{m-1}, d]$. Then $f(U) \supset U \cup V$ and $f^k(V) \supset U$ for any k such that $x_{k+m-1} < c$, contrary to Lemma 3.2. So assume that, for any n, $x_n, x_{n+1} > c$ implies $x_{n+1} < x_n$. Then there is $m > 0$ such that $x_{m+1} < c < x_m < x_{m-1}$. Let $U = [c, x_m]$ and $V = [x_m, x_{m-1}]$. Then $f(V) \supset U$, and since $x_0 \in (a,b)$ and $x_0 \neq c$, there is a $k \geq 0$ such that $f^k(U)$ contains the interval with endpoints x_0, c. But then $f^{k+m-1}(U) \supset U \cup V$, and f would have a horseshoe. The second statement is obvious. □

Lemma 3.5. *Assume $f \in \mathcal{J}$, $h(f) = 0$ and $\tilde{\omega} = \omega_f(x)$ is infinite. Then there is a maximal simple set containing $\tilde{\omega}$.*

PROOF. By Lemmas 3.3 and 3.4, there are periodic portions $\tilde{\omega}_0$ and $\tilde{\omega}_1$ of $\tilde{\omega}$ separated by a fixed point p which form a decomposition of $\tilde{\omega}$. Assume that, e.g., $\tilde{\omega}_0$ is to the left of $\tilde{\omega}_1$. Let U_i be the convex hull of $\tilde{\omega}_i$, $V_i = \bigcup_{n=0}^{\infty} f^{2n}(U_i)$, and $J_i = \overline{V_i}$, $i = 0, 1$. Since $f^2(\tilde{\omega}_i) = \tilde{\omega}_i$, we have $f^2(V_i) = V_i$, and since f has the Darboux property, $f^2(J_i) \subseteq J_i,\ i = 0, 1$. We show that

$$J_0 \subseteq [0,p] \text{ and } J_1 \subseteq [p,1]. \tag{3.3}$$

Assume that, e.g., $U_0 = [a_0, b_0] \subset J_0 = [u, v]$ such that $u \leq a_0 < b_0 \leq p < v$. Then $f^{2n}(U_0) \nsubseteq [u,p]$, for some $n \in \mathbb{N}$ and, by the Darboux property, there is a $w \in (a_0, b_0)$ with $f^{2n}(w) > p$. Find integers $k < s$ such that $x_{2s} < w < x_{2k} (< p)$, and let $U = [w, x_{2k}]$ and $V = [x_{2k}, p]$. Then

$$f^{2(s-k)}(U) \cap f^{2(s-k)}(V) \supset [x_{2s}, p] \supset U \cup V,$$

contrary to Lemma 3.2.

By induction from (3.3), for every periodic portion $\tilde{\omega}_{\alpha_1 \ldots \alpha_k}$ of $\tilde{\omega}$ of period $2^k, k > 1$, there is a minimal compact weakly periodic interval $J_{\alpha_1 \alpha_2 \ldots \alpha_k}$ containing it. Then $\mathcal{S} = \{J_\alpha : \alpha \in \{0,1\}^n, n \in \mathbb{N}\}$ is a simple system of intervals for $\tilde{\omega}$. □

Lemma 3.6. *Let $f \in \mathcal{J}$, and $A \subseteq I$ such that $f(A) \subseteq A$. If for every open set V such that $V \cap A \neq \emptyset$ there is $k \in \mathbb{N}$ with $\bigcup_{j=0}^{k} f^{-j}(V) \supset A$, then every point in A is uniformly recurrent.*

PROOF. Let $x \in A$ and V be an open set containing x. Then there is $k \in \mathbb{N}$ such that $\bigcup_{j=0}^{k} f^{-j}(V) \supset A$. Let $f^n(x) \in V$. If $f^{n+1}(x) \notin V$, then there is $0 < j \leq k$ such that $f^{n+1}(x) \in f^{-j}(V)$ and hence $f^{n+1+j}(x) \in V$, $1+j \leq k+1$ and consequently x is uniformly recurrent. □

Lemma 3.7. *Let $f \in \mathcal{J}$ and $h(f) = 0$. If M is an infinite ω-minimal set then, for every open set V such that $V \cap M \neq \emptyset$, there is a $k \in \mathbb{N}$ with $\bigcup_{j=0}^{k} f^{-j}(V) \supset M$.*

PROOF. Let $x \in M$. By Lemma 3.5, M is a subset of a maximal simple set A satisfying (3.1). By Lemma 3.1, M is the unique ω-minimal set in A. Let V be open, and $x \in M \cap V$. Then there are $k \in \mathbb{N}$ and $\alpha_0 \in \{0,1\}^k$ such that $x \in J_{\alpha_0} \subset V$. But then

$$\bigcup_{j=0}^{2^k} f^{-j}(V) \supset M$$

since $\{J_\alpha\}_{\alpha \in \{0,1\}^k}$ is a weakly f-periodic orbit of period 2^k. □

Lemma 3.8. (Cf. [2].) *Let $f \in \mathcal{J}$, $a \in \operatorname{Rec} f$, and let $f^k(a) > a$, for some k. If there is a sequence of fixed points $p_n > a$, $p_n \to a$, then f has positive topological entropy.*

Lemma 3.9. *Assume $f \in \mathcal{J}$ with $h(f) = 0$, and let M be a finite ω-minimal set. Then $f(M) = M$.*

PROOF. If M contains a periodic point, the statement is true. Assume, that M contains no periodic points. Then, for every $x \in M$, there is $n_x \in \mathbb{N}$ such that $f^n(x) \notin M$ whenever $n \geq n_x$. Put $\min M = a, \max M = b$. If $\#M > 2$, then there is a point $c \in M$, $a < c < b$. Since $\omega_f(c) = M$, Lemma 3.4 implies existence of a fixed point $p \in (a,b)$ such that $M_1 = \omega_{f^2}(c) \subset M$ is a subset of $[a,p]$ or $[p,b]$. Hence, $\#M_1 < \#M$. Consequently, replacing f by a suitable iterate of f, we may assume that $\#M \leq 2$. Now we proceed with several stages.

STAGE 1. We prove the statement for $M = \{a\}$. Let $\{a_n\}_{n=0}^{\infty}$ be the trajectory of a. Without loss of generality suppose that $f(a) > a$. If $a_n > a$ for infinitely many n, then for infinitely many such n we have $f(a_n) < a_n$ and thus there is a sequence of fixed points $p_n > a$ such that $p_n \to a$ and, by Lemma 3.8, f would have positive topological entropy which is impossible.

If, on the other hand, $a_n < a$ whenever $n \geq m$, then, applying Lemma 2.9 in [2] to $g = f^m$ we obtain that g has positive topological entropy. Thus, $f(M) = M$.

STAGE 2. Assume that $M = \{a, b\}$ contains two distinct points and $a < b$. Since, for some $k, s \in \mathbb{N}$, $f^k(a) > a$ and $f^s(b) < b$, by Lemma 3.8 there are $u, v \in (a, b)$, $u < v$, such that $[a, u] \cup [v, b]$ contains no fixed point of f. Then $f(a) > a$. Indeed, assume $f(a) < a$. Then $f(x) < x$ for $x \in [a, u]$, and since $\{a, b\} \subset \omega_f(a)$ there is an increasing sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $a_{n_k} := f^{n_k}(a) < a$ and $a_{n_{k+1}} > a$, for infinitely many k. But then any left neighborhood of a would contain a fixed point, contrary to Lemma 3.8. Similarly, $f(b) < b$ and consequently, f has a fixed point $c \in (a, b)$.

Let $g = f^2$, and G_a, G_b disjoint neighborhoods of a and b, respectively. By Stage 1, we may assume that $\omega_g(a) \neq \{a\}$ and $\omega_g(b) \neq \{b\}$. Then there are two possible cases.

STAGE 2a. Let $\omega_g(a) = \{b\}$ and $\omega_g(b) = \{a\}$. Then $a_{2n} \in G_b$ and $b_{2n} \in G_a$, for large n in $\mathbb{N}$. Assume $c > b_{2n} > a$ for infinitely many n. Obviously there is an $m \in \mathbb{N}$ such that $a < b_{2m} < c$ and $g^n(b_{2m}) < b_{2m}$ for any n. Moreover, there is $n_0 \in \mathbb{N}$ with $g^{n_0}(a) > c$. Then there exists a point q between a and b_{2m} satisfying $g^{n_0}(q) = q$ and consequently there is a point $p \in (a, b_{2m})$ such that $g^{n_0}(p) = c$. It is easily seen that $U = [p, b_{2m}]$ and $V = [b_{2m}, c]$ form a horseshoe which is a contradiction. We have the same case if $c < a_{2n} < b$ for infinitely many n.

Next it suffices to assume that $a_{2n}, b_{2n} \notin [a, u] \cup [v, b]$. Consequently, $b_{2n} < a$ and $a_{2n} > b$, for large n, and $U = [a, c]$ and $V = [c, b]$ would form a horseshoe which is impossible.

STAGE 2b. By symmetry, assume $\omega_g(a) = \{a, b\}$. Then, by Lemma 3.4, $a_n \in (a, b)$ for no $n \in \mathbb{N}$. Moreover, there is an n such that both a_n, a_{n+1} belong to one of the neighborhoods G_a or G_b, say to G_a, since otherwise $\omega_g(a)$ would be a singleton which is impossible, as we shown above. Let $U = [a, c]$ and let V be the interval with endpoints a_n, a_{n+1}. Then $f^n(U) \cup f^{n+1}(U) \supset U \cup V$, and $f^k(V) \supset U$ if $k > 1$ is the minimal integer such that $a_{n+k} > b$. Thus, f would have a horseshoe, a contradiction. □

PROOF OF THEOREM 1.1. Suppose 1. By Lemmas 3.1, 3.5, 3.9, and 2.5, $f(M) = M$ and there is no proper invariant subset of M. By definition, M is closed and by Lemmas 3.6, 3.7 and 2.5, every $x \in M$ is uniformly recurrent.

To prove that 2. implies 1., take $x \in M$. Then $x \in \omega_f(x)$ and $f(x) \in M$ since $M \subset \operatorname{Rec} f$ and $f(M) \subseteq M$. Hence, $f(x) \in \omega_f(f(x)) = \omega_f(x)$ and, by induction, $f^n(x) \in \omega_f(x)$, for any $n \in \mathbb{N}$. Consequently, $\omega_f(x)$ either is a cycle and obviously an ω-minimal set, or is infinite and then ω-minimal by Lemmas

3.5 and 3.1. □

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