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ORDERS OF GROWTH OF REAL FUNCTIONS

Abstract

In this paper we define the notion of order of a function, which measures its growth rate with respect to a given function. We introduce the notions of continuity and linearity at infinity with which we characterize order-comparability and equivalence. Using the theory we have developed, we apply orders of functions to give a simple and natural criterion for the uniqueness of fractional and continuous iterates of a function.

1 Orders of Growth.

1.1 Introduction.

Consider functions between x and e^x, say $x, x^2, x^{\log x}, x^{(\log x)^{\log\log x}}, e^{x^{\frac{1}{\log\log x}}}$, $e^{\sqrt{x}}, 2^x, e^x$ in increasing order of largeness. Now, repeatedly take logs[1] of these and observe the asymptotic behavior at each level. Note that if [2]$f \sim g$, then

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[1]Throughout the paper, we denote the n^{th}-iterate of $\log x$ by $\log_n x$; that is $\log_n x = \log(\log_{n-1} x)$ and $\log_1 x = \log x$. Iterates of exp are denoted by e_n^x; $e_n^x = \exp(e_{n-1}^x)$ and $e_1^x = e^x$.

[2]We have the usual definitions for $f \sim g$, $f = o(g)$, $f \prec g$, $f \succ g$, namely: $f(x)/g(x)$ tends to $1, 0, 0, \infty$ respectively, as $x \to \infty$. By $f \asymp g$, we mean $\exists a, A > 0$ such that $a < f(x)/g(x) < A$ on a neighborhood of infinity.

$\log f \sim \log g$.

f	x	x^2	$x^{\log x}$	$x^{(\log x)^{\log\log x}}$	$e^{x^{\frac{1}{\log\log x}}}$	$e^{\sqrt{x}}$	2^x	e^x
$\log f$	$\log x$	$2\log x$	$(\log x)^2$	$(\log x)^{\log_2 x+1}$	$x^{\frac{1}{\log_2 x}}$	$\sqrt{x}$	$x\log 2$	x
$\log_2 f$	$\log_2 x$	$\sim \log_2 x$	$2\log_2 x$	$\sim (\log_2 x)^2$	$\frac{\log x}{\log_2 x}$	$\frac{1}{2}\log x$	$\sim \log x$	$\log x$
$\log_3 f$	$\log_3 x$	$\log_3 x$	$\sim \log_3 x$	$2(\log_3 x)$	$\sim \log_2 x$	$\sim \log_2 x$	$\log_2 x$	$\log_2 x$
$\log_4 f$	$\log_4 x$	$\log_4 x$	$\log_4 x$	$\sim \log_4 x$	$\log_3 x$	$\log_3 x$	$\log_3 x$	$\log_3 x$

We notice that for the first four functions, $\log_n f(x) \sim \log_n x$, while for the last four we have $\log_n f(x) \sim \log_{n-1} x$ whenever $n \geq 4$. In this sense, the former are not distinguishable from the function $f(x) = x$, while the latter are indistinguishable from e^x.

This observation motivates the following definition.

Definition 1.1. A function f *has order* $k \in \mathbb{Z}$ if for some $n \in \mathbb{Z}$ (and hence all larger n) $\log_n f(x) \sim \log_{n-k} x$ as $x \to \infty$. We write $O(f) = k$.

This notion is not new and quite recently, it was used by Rosenlicht in [7]. Our initial aim is to generalize this notion to non-integral values.

It is easy to prove the following basic properties given that $O(f)$ and $O(g)$ exist:
(1) $O(f+g) = O(f\cdot g) = \max\{O(f), O(g)\}$;
(2) $O(f(g)) = O(f) + O(g)$;
(3) $f \sim g \Longrightarrow O(f) = O(g)$;
(4) *if* $f < h < g$ *and* $O(f) = O(g)$, *then* $O(h)$ *exists and equals* $O(f)$;
(5) $O(e_k^x) = k$ *and* $O(\log_k x) = -k$, *for all* $k \in \mathbb{Z}$.
From these it follows that many standard functions tending to infinity have orders. Indeed, Hardy showed that for all so-called L-functions f tending to infinity (those obtained from a finite number of applications of the operations $+, -, \times, \div, \exp$ and $\log$ on the constant functions and x) there exist $r, s \in \mathbb{N}_0$ and $\mu > 0$ such that for all $\delta > 0$,

$$(\log_s x)^{\mu-\delta} < \log_r f(x) < (\log_s x)^{\mu+\delta},$$

for x sufficiently large (see [2]). Hence $\log_{r+1} f(x) \sim \mu \log_{s+1} x$, so that

$$\log_{r+2} f(x) \sim \log_{s+2} x.$$

Thus $O(f)$ exists and equals $r - s$.

Which functions, given that they tend to infinity, do not have orders? It is clear that if $f(x)$ tends to infinity sufficiently rapidly or slowly (i.e. faster than e_k^x or slower than $\log_k x$ respectively, for any k), then f does not have an order. In fact, one could then say that f has order ∞ or $-\infty$ respectively.

Other examples of functions not having an order are erratic functions which behave, say, like x and e^x along different sequences. More interesting examples come from observing that $O(f^k) = kO(f)$ (where f^k is f iterated k times), which follows from (2) above. For example, suppose g is such that $g(g(x)) = e^x$. Then $O(g^2) = 1$, so that if $O(g)$ existed, it should equal $\frac{1}{2}$! Furthermore, suppose that $x \prec g(x) \prec e^x$. Then, using $\log_n g(x) = g(\log_n x)$, we have

$$\log_n x \prec \log_n g(x) \prec \log_{n-1} x \text{ for every } n.$$

More generally, if g is such that $g^n(x) = e_m^x$, then we should have $O(g) = \frac{m}{n}$. This suggests we should generalize the notion of order to fractional and even irrational values. Further, it indicates that there is, in some sense, a 'continuum' of functions between x and e^x, with orders ranging from 0 to 1.

We end this introduction with a brief summary of the rest of the paper. In §1.2, we generalize the notion of order to real values and, in §1.3, to a more general setting, basing it on composition of functions (Definition 1.3). In §1.5, we make a more in-depth study of functions of order 0, leading to the notion of degree. The main result of this section (Theorem 1.13) shows how the growth of iterates of a function is determined by its degree.

In Section 2, we develop the theory further to obtain a characterization of order-comparability and order-equivalence (i.e. when different functions give rise to the same orders), in Theorem 2.4 and Corollary 2.5. This is done through the (new) notions of continuity and linearity at infinity.

In Section 3, we apply the notion of order to fractional and continuous iteration. In general, a strictly increasing and continuous function tending to infinity has infinitely many choices for fractional iterates. We give a simple and natural criterion based on orders which gives unique fractional (and continuous) iterates of a particular growth rate. Furthermore, it applies in great generality. To make the application, we consider the Abel functional equation, to which the idea of orders is closely related. Indeed, we can view the notion of order one functions as an approximation to the Abel equation, with the solution of the Abel equation being obtained through a limiting process. We give sufficient conditions for the existence of such a limit and its derivative.

Notation. Unless stated otherwise, all functions are considered to be defined on a neighborhood of infinity. By 'f is continuous/increasing/etc.' we mean 'f is continuous/increasing/etc. on some interval $[A, \infty)$'. Also we write $f < g$ to mean $\exists x_0$ such that for $x \geq x_0$, $f(x) < g(x)$.

For a given f, we write f^n for the n^{th}-iterate. For the special functions exp and log we use the notation e_n^x and $\log_n x$ for the n^{th}-iterates respectively.

We define the function spaces

$$\mathrm{SIC}_\infty = \{f : f \text{ is continuous, strictly increasing and } f(x) \to \infty \text{ as } x \to \infty\},$$
$$D_\infty^+ = \{f \in \mathrm{SIC}_\infty : f \text{ is continuously differentiable and } f' > 0\}.$$

Note that SIC_∞ and D_∞^+ are groups under composition, if we identify functions which are equal on a neighborhood of infinity.

1.2 Generalizing Orders.

There are of course various ways of making generalizations. One way would be to generalize $\log_n x$ from integral n to real n in a suitable way, and hence to define $O(f) = \alpha$ if $\log_n f(x) \sim \log_{n-\alpha} x$ as $x \to \infty$ for some n. Apart from the problem of deciding how to define $\log_\lambda x$ for non-integral λ, such an approach is unsatisfactory in that it *lacks* the following desired 'completeness' property: *given a function f and $\lambda \in \mathbb{R}$, suppose that for all $\varepsilon > 0$, there exist functions g_ε and h_ε such that*

$$O(g_\varepsilon) = \lambda - \varepsilon,\ O(h_\varepsilon) = \lambda + \varepsilon \textit{ and } g_\varepsilon < f < h_\varepsilon,$$

then $O(f)$ exists and equals λ. This will be made clear later.

A more profitable way of generalizing orders (whereby the above property is obtained) is by finding an increasing function, say F, which has the property that

$$F(e^x) = F(x) + 1 + o(1) \text{ as } x \to \infty. \tag{1.1}$$

Then $F(x) = F(\log_k x) + k + o(1)$ for each $k \in \mathbb{Z}$. Such an F tends to infinity extremely slowly — slower than any iterate of log, since it follows from (1.1) that $F(e_n^1) \sim n$ as $n \to \infty$. Assume also that $F(x+y) = F(x) + o(1)$ whenever $y = o(x)$ (in fact, after (1.1) it suffices to assume this for $y = o(1)$).

Now suppose that $O(f) = k$, so that $\log_n f(x) \sim \log_{n-k} x$ for some n. Then

$$F(f(x)) = F(\log_n f(x)) + n + o(1) = F(\log_{n-k} x) + n + o(1) = F(x) + k + o(1),$$

i.e. $O(f) = k \implies \lim_{x\to\infty}\{F(f(x)) - F(x)\} = k$. This gives an immediate generalization to non-integral orders: *f has order $\lambda \in \mathbb{R}$ if*

$$\lim_{x\to\infty}\{F(f(x)) - F(x)\} = \lambda.$$

It will be seen however that there are many such functions F (satisfying (1.1)), each giving rise to (possibly different) orders. As such, we shall write

$$O_F(f) = \lambda,$$

emphasizing the dependence on F.

One example of such an F is the following function Ξ defined below. First we define $\chi(t)$ for $t \geq 0$ by the equations

$$\chi(t) = 1 \text{ for } 0 \leq t \leq 1, \text{ and } \chi(t) = t\chi(\log t) \text{ for } t \geq 1.$$

Then χ is continuous and increasing. Now define $\Xi(x)$ for $x \geq 0$ by

$$\Xi(x) = \int_0^x \frac{1}{\chi(t)}\, dt.$$

Then

$$\Xi(e^x) = \Xi(1) + \int_1^{e^x} \frac{1}{\chi(t)}\, dt = 1 + \int_0^x \frac{e^t}{\chi(e^t)}\, dt = 1 + \int_0^x \frac{1}{\chi(t)}\, dt = \Xi(x) + 1.$$

Thus Ξ satisfies (1.1) with no error term. Also for all $x, y \geq 0$,

$$\frac{y}{\chi(x+y)} \leq \int_0^y \frac{dt}{\chi(t+x)} = \Xi(x+y) - \Xi(x) \leq \frac{y}{\chi(x)}. \tag{1.2}$$

In particular, if $y = o(x)$, then $\Xi(x+y) = \Xi(x) + o(1)$ (since $\chi(x) \geq x$).

The basic properties of orders referred to earlier all still hold in this more general setting. It is natural to ask whether we have also generalized the integer orders. This is indeed the case but, as we shall see later in §2.2, every such generalization leads to the same (new) integer orders. For example, there exists f such that $O_\Xi(f) = 0$ but $O(f) \neq 0$ in the original definition. Indeed, we can characterize functions of order 0 (in the original sense) using Ξ as follows.

Theorem 1.2. *We have $\log_n f(x) \sim \log_n x$ for some n if and only if there exists an integer k such that*

$$\Xi(f(x)) = \Xi(x) + o\Big(\frac{1}{\log_k x}\Big) \text{ as } x \to \infty.$$

PROOF. Suppose $\log_n f(x) \sim \log_n x$ for some n. Then for all x sufficiently large, $|\log_{n+1} f(x) - \log_{n+1} x| \leq 1$. Hence for such x,

$$\begin{aligned}\Xi(f(x)) - \Xi(x) &= \Xi(\log_{n+1} f(x)) - \Xi(\log_{n+1} x)\\ &\leq \Xi(\log_{n+1} x + 1) - \Xi(\log_{n+1} x) \leq \frac{1}{\chi(\log_{n+1} x)},\end{aligned}$$

using (1.2). In the same way, for a lower bound, we have

$$\Xi(f(x))-\Xi(x)\geq \Xi(\log_{n+1}x-1)-\Xi(\log_{n+1}x)\geq -\frac{1}{\chi(\log_{n+1}x-1)}$$

Combining these gives $\Xi(f(x))-\Xi(x)=o(\frac{1}{\log_{n+1}x})$, since $\chi(x)\succ x$.

Conversely, suppose $\Xi(f(x))-\Xi(x)=o(\frac{1}{\log_k x})$ for some k. Then for all x large enough, $\Xi(f(x))-\Xi(x)\leq \frac{1}{\chi(\log_{k+1}x+1)}$, since $\chi(x)=o(e^x)$. But then, by (1.2),

$$\Xi(\log_{k+1}f(x))-\Xi(\log_{k+1}x)\leq \frac{1}{\chi(\log_{k+1}x+1)}\leq \Xi(\log_{k+1}x+1)-\Xi(\log_{k+1}x),$$

which gives $\log_{k+1}f(x)\leq \log_{k+1}x+1$ for sufficiently large x. Similarly for a lower bound. □

For example, the function $h(x)$ defined by

$$h(x)=\Xi^{-1}\Big(\Xi(x)+\frac{1}{\Xi(x)}\Big), \tag{1.3}$$

has $O_\Xi(h)=0$ but $\log_k h(x)\succ \log_k x$ for all $k\in\mathbb{N}$ since $\Xi(x)=o(\log_n x)$ for all n.

Note that if we had used $\log_\lambda x$ (with $\lambda\in\mathbb{R}$) to generalize orders to non-integral values, then we should expect that

$$\liminf_{x\to\infty}\{\Xi(x)-\Xi(\log_\varepsilon x)\}>0$$

for every fixed $\varepsilon>0$. But then for the above function h,

$$\lim_{x\to\infty}\{\Xi(\log_n h(x))-\Xi(x)+n\}=\lim_{x\to\infty}\{\Xi(h(x))-\Xi(x)\}=0, \text{ while}$$
$$\liminf_{x\to\infty}\{\Xi(\log_{n-\varepsilon}x)-\Xi(x)+n\}>0.$$

It follows that $\log_n x\prec \log_n h(x)<\log_{n-\varepsilon}x$ for all $n\in\mathbb{N}$ and all $\varepsilon>0$. Hence for such a generalization, we do not have the completeness property referred to earlier.

1.3 Orders of Functions.

In the preceding discussion, we considered functions satisfying (1.1) and we defined orders with respect to these. However, there is no reason why we should restrict ourselves to such functions. More generally therefore, we define orders with respect to any increasing function tending to infinity.

Definition 1.3. Let f be increasing and such that $f(x) \to \infty$ as $x \to \infty$. We say a function g has *order λ with respect to f* if

$$\lim_{x\to\infty} \{f(g(x)) - f(x)\} = \lambda.$$

We denote this by $O_f(g) = \lambda$.

In this general setting, there are no rules for sums and products of functions, but only for compositions. Thus if $O_f(g) = \lambda$ and $O_f(h) = \mu$, then $O_f(g \circ h) = \lambda + \mu$, as can be easily verified. In particular, this implies that $O_f(g^k) = k\lambda$ for every $k \in \mathbb{N}$.

The notion of order gives a simple, but non-trivial, estimate for the growth of iterates of a function. For if $O_f(g) = 1$, then for every fixed x for which $g^n(x) \to \infty$ as $n \to \infty$, $f(g^n(x)) \sim n$. To see this, let $\varepsilon > 0$. Then for all suitably large x, $g(x) > x$, f is increasing and $1 - \varepsilon < f(g(x)) - f(x) < 1 + \varepsilon$. Hence $1 - \varepsilon < f(g^{r+1}(x)) - f(g^r(x)) < 1 + \varepsilon$, and summing from $r = 0$ to $n - 1$ gives $(1 - \varepsilon)n < f(g^n(x)) - f(x) < (1 + \varepsilon)n$.

Examples 1.4. (a) Let $f(x) = \log x$. Then $O_f(g) = \lambda \iff \log g(x) = \log x + \lambda + o(1)$; i.e. $g(x) \sim e^\lambda x$ as $x \to \infty$.

(b) Let $f(x) = \log\log x/\log 2$. Then $O_f(g) = \lambda \iff \log\log g(x) = \log\log x + \lambda \log 2 + o(1)$; i.e., $g(x) = x^{2^\lambda + o(1)}$.

Thus log distinguishes between multiples of x and log log distinguishes between powers of x. We have already seen how Ξ distinguishes between iterates of exp.

Throughout the rest of this article, when we refer to orders w.r.t. f, we shall implicitly assume that f is increasing and tends to infinity, and we shall not repeat the phrase 'let f be increasing etc.'.

Proposition 1.5. *Suppose $O_f(g) < O_f(h)$ for some functions g, h. Then $g < h$; that is, $g(x) < h(x)$ for all x sufficiently large.*

PROOF. Immediate from the definition. □

Next we prove the completeness property referred to earlier.

Theorem 1.6. *Let g be a function and $\lambda \in \mathbb{R}$. Suppose that for every $\varepsilon > 0$, there exist functions h_ε and k_ε such that*

$$O_f(h_\varepsilon) = \lambda - \varepsilon,\ O_f(k_\varepsilon) = \lambda + \varepsilon \text{ and } h_\varepsilon < g < k_\varepsilon.$$

Then $O_f(g)$ exists and equals λ.

PROOF. Since f is increasing and $h_\varepsilon < g < k_\varepsilon$, it follows that for all sufficiently large x,

$$f(h_\varepsilon(x)) - f(x) \le f(g(x)) - f(x) \le f(k_\varepsilon(x)) - f(x).$$

Letting $x \to \infty$ yields

$$\lambda - \varepsilon \le \liminf_{x\to\infty}\{f(g(x)) - f(x)\} \le \limsup_{x\to\infty}\{f(g(x)) - f(x)\} \le \lambda + \varepsilon.$$

Since ε is arbitrary, this implies $\lim_{x\to\infty}\{f(g(x))-f(x)\}=\lambda$; i.e., $O_f(g)=\lambda$. $\square$

Next we investigate the relationship between functions with respect to which a given function has the same positive order. We start with two elementary results.

Proposition 1.7. *Let $h \in \mathrm{SIC}_\infty$. If $O_f(h) = O_g(h) = 1$, then $f \sim g$.*

PROOF. There exists x_0 such that for $x \ge x_0$, $h(x)$ is strictly increasing, continuous, and $h(x) > x$. Hence, every $y \ge x_0$ can be written uniquely as $y = h^n(x)$ for some $n \ge 0$ and $x \in [x_0, h(x_0))$. In particular, $h^n(x_0) \le y < h^{n+1}(x_0)$. But $f(h^n(x_0)) \sim n \sim g(h^n(x_0))$ as $n \to \infty$. Thus

$$\underbrace{\frac{f(h^n(x_0))}{g(h^{n+1}(x_0))}}_{\to 1} < \frac{f(y)}{g(y)} < \underbrace{\frac{f(h^{n+1}(x_0))}{g(h^n(x_0))}}_{\to 1},$$

and the result follows. $\square$

Proposition 1.8. *Suppose that $O_f(h) = O_g(h) = 1$, for some function h. Further suppose that $O_f(H)$ exists for some other function H. Then either $O_g(H)$ exists and equals $O_f(H)$ or it does not exist.*

PROOF. Let $O_f(H) = \lambda$ and suppose $O_g(H)$ exists and equals $\mu > \lambda$. Then there exist integers m, n with $m \ge 1$ such that $m\lambda < n < m\mu$. But then

$$O_f(H^m) = m\lambda < n = O_f(h^n) = O_g(h^n) < m\mu = O_g(H^m).$$

By Proposition 1.5, the left and right hand inequalities imply that $H^m < h^n$ and $h^n < H^m$ respectively — a contradiction. A similar contradiction follows if we assume $\mu < \lambda$. Hence $\mu = \lambda$. $\square$

1.4 Order Comparability and Equivalence.

When do two functions f and g give rise to the same orders? i.e. when do we have

$$\lim_{x\to\infty}\{f(h(x)) - f(x)\} = \lambda \iff \lim_{x\to\infty}\{g(h(x)) - g(x)\} = \lambda,$$

for all functions h and $\lambda \in \mathbb{R}$? If this holds we shall write $O_f \equiv O_g$. For example, if $f = g + o(1)$, then (trivially) $O_f \equiv O_g$.

There is also a more subtle notion of whether we can *compare* orders w.r.t. f and g. Roughly speaking, by this we mean that if a given function has the same order, say λ, w.r.t. both f and g, then O_f and O_g will agree on all other functions of order λ. For example, suppose F_1 and F_2 are two functions satisfying (1.1), is it true that $O_{F_1}(h) = 1 \iff O_{F_2}(h) = 1$?

For reasons which will become clear later, it is more convenient to consider functions which are invertible, so for the following definition we shall assume that f and g are *strictly* increasing and *continuous*.

Definition 1.9. Let $f, g \in \mathrm{SIC}_\infty$.

(a) We say f and g are *order-comparable* if, whenever $O_f(h) = O_g(h) = \lambda$ for a given function h and some λ, then for any other function H, we have $O_f(H) = \lambda$ if and only if $O_g(H) = \lambda$.[3] In this case we may define the set of values on which O_f and O_g agree. We denote this set by $O_{f,g}$; i.e.,

$$O_{f,g} = \{\lambda \in \mathbb{R} : O_f(h) = \lambda \iff O_g(h) = \lambda\}.$$

(b) f and g are *order-equivalent* if they give rise to exactly the same orders; i.e $O_{f,g} = \mathbb{R}$.

Note that if f and g are order-comparable, then they always agree on order zero since the function $h(x) = x$ is always of order zero; i.e., $0 \in O_{f,g}$ whenever $O_{f,g}$ exists.

Note also that $O_{f,g} = \mathbb{R}$ implies $O_f \equiv O_g$, but not vice versa, since the former requires f and g to be invertible.

The relations of order-comparability and order-equivalence are clearly equivalence relations.

1.5 Degree of a Function; Growth Rates of Iterates.[4]

The equality $O_f(g) = \lambda$ gives information about the growth rate of g. However, for fixed g, there is a difference between the cases where $\lambda = 0$ and $\lambda \neq 0$.

[3] As we saw in Proposition 1.8, if $O_g(H)$ exists, it cannot take any other value.

[4] This section may be omitted on a first reading.

For $\lambda \neq 0$, f is determined up to asymptotic equivalence (by Proposition 1.7), but for $\lambda = 0$, there are many other functions F for which $O_F(g) = 0$. This implies that if $\lambda = 0$ we actually obtain less information. To illustrate this, consider $g(x) = ex + 1$. Then

$$\text{(i)} \quad O_{\log}(g) = 1 \text{ and (ii) } O_{\log\log}(g) = 0.$$

(i) is clearly more informative, saying $g(x) \sim ex$, whereas (ii) says only that $g(x) = x^{1+o(1)}$.

In this section we make a further distinction between functions of order 0, by defining the *degree* of a function. In this sense we can think of this as a second-order theory.

Let f be increasing and suppose $O_f(g) = 0$, where $g(x) > x$. Then

$$f(g(x)) = f(x) + \frac{1}{h(x)},$$

where $h(x) \to \infty$. Clearly the growth rate of g depends on how quickly h tends to infinity. The faster (slower) h tends to infinity, the slower (faster) g does. The motivation for the definition of degree comes from considering the growth rate of iterates. Consider the following functions and their iterates:

0. Let $g(x) = x + a$ with $a > 0$. Then $g^n(x) = an + x$, which, as a function of n, has order 0 (in the original sense, or indeed w.r.t. Ξ) for any fixed x.

1. Let $g(x) = bx$ with $b > 1$. Then $g^n(x) = b^n x$, which has order 1 for any fixed $x > 0$.

2. Let $g(x) = x^c$ with $c > 1$. Then $g^n(x) = x^{c^n}$, which has order 2 for any fixed $x > 1$.

These examples show that the growth rate of $g^n(x)$ doesn't depend significantly on particular values of x. In each case $O_\Xi(g) = 0$, so consider the differences $\Xi(g) - \Xi$. From (1.2), we have

$$\Xi(x+a) - \Xi(x) \sim \frac{a}{\chi(x)}, \qquad \text{as } x \to \infty.$$

Hence

$$\Xi(bx) - \Xi(x) = \Xi(\log x + \log b) - \Xi(\log x) \sim \frac{\log b}{\chi(\log x)},$$

and

$$\Xi(x^c) - \Xi(x) = \Xi(c\log x) - \Xi(\log x) \sim \frac{\log c}{\chi(\log\log x)}.$$

Now the differences between the cases become apparent. In case (i), $(i = 0, 1, 2)$ the difference $\Xi(g) - \Xi$ tends to zero with order $-i$. Hence in each case, the order of $g^n(\cdot)$ is -1 times the order of $\dfrac{1}{\Xi(g) - \Xi}$.

Definition 1.10. Let $f \in \mathrm{SIC}_\infty$ and let $g(x) > x$ for all x sufficiently large. Then g has *degree* λ *with respect to* f, if

$$f(g(x)) = f(x) + \frac{1}{h(x)} \text{ and } O_f(h) = -\lambda.$$

We denote this by $\partial_f(g) = \lambda$. More compactly, $\partial_f(g) = -O_f(\frac{1}{f(g)-f})$.

Examples 1.11. (a) Let $f(x) = \log x$. Using the fact that $O_{\log}(h) = \lambda \iff h(x) \sim e^\lambda x$, we have

$$\partial_{\log}(g) = \lambda \iff g(x) = x + e^\lambda + o(1).$$

(b) Let $f(x) = \log\log x$. Using $O_{\log\log}(h) = \lambda \iff h(x) = x^{e^\lambda + o(1)}$, we have

$$\partial_{\log\log}(g) = \lambda \iff g(x) = x + x^{1-e^{-\lambda}+o(1)}.$$

(c) Let $f = \Xi$. Then $\partial_\Xi(x+1) = 0$, $\partial_\Xi(2x) = 1$, $\partial_\Xi(x^3) = 2$, $\partial_\Xi(x^{\log x}) = 3$. Without too much difficulty one can show that if $\partial_\Xi(f)$ and $\partial_\Xi(g)$ exist, then

$$\begin{aligned}\partial_\Xi(f(g)) &= \max\{\partial_\Xi(f), \partial_\Xi(g)\},\\ \partial_\Xi(f+g) &= \max\{\partial_\Xi(f), \partial_\Xi(g), 1\}, \text{ and}\\ \partial_\Xi(f \cdot g) &= \max\{\partial_\Xi(f), \partial_\Xi(g), 2\}.\end{aligned}$$

In the last two statements one must take into account that $f(x)+g(x) > 2x$ and $f(x)g(x) > x^2$ for all large x. Note also that after Theorem 1.2, all functions of finite degree have order 0 in the original sense, while the function defined by (1.3) has infinite degree.

Proposition 1.12. *Suppose $\partial_f(g) < \partial_f(h)$ for some functions g, h. Then $g < h$; that is, $g(x) < h(x)$ for all x sufficiently large.*

Proof. We have

$$f(g(x)) = f(x) + \frac{1}{k(x)} \text{ and } f(h(x)) = f(x) + \frac{1}{l(x)},$$

with $O_f(k) > O_f(l)$. Hence $k > l$ (Proposition 1.5), and so $g < h$. $\square$

As the earlier discussion indicated, the notion of degree is closely related to the order of the n^{th} iterate. This is the main result of this section.

Theorem 1.13. *Let $f \in \mathrm{SIC}_\infty$ such that $\partial_f(x+1) = 0$. Let $\partial_f(g) = \lambda$ and suppose $g^n(\alpha) \to \infty$ as $n \to \infty$ for some constant α. Then $O_f(g^n(\alpha)) = \lambda$ (as $n \to \infty$). More compactly, $O_f(g^n) \equiv \partial_f(g)$.*

PROOF. Since $\partial_f(x+1) = 0$, we can write

$$f(x+1) = f(x) + \frac{1}{l(x)},$$

where $O_f(l) = 0$. For $\beta \in \mathbb{R}$, define the functions p_β and q_β by

$$p_\beta(x) = f^{-1}(f(x) - \beta) \text{ and } q_\beta(x) = p_\beta^{-1}(p_\beta(x) + 1).$$

(We have used the fact that p_β is invertible, with $p_\beta^{-1}(x) = f^{-1}(f(x) + \beta)$.) Observe that $O_f(p_\beta) = -\beta$. Then

$$\begin{aligned} f(q_\beta(x)) &= f(p_\beta^{-1}(p_\beta(x) + 1)) = f(p_\beta(x) + 1) + \beta \\ &= f(p_\beta(x)) + \frac{1}{l(p_\beta(x))} + \beta = f(x) + \frac{1}{l(p_\beta(x))}. \end{aligned}$$

Hence $\partial_f(q_\beta) = -O_f(l \circ p_\beta) = \beta$.

Let $\varepsilon > 0$. Then there exists x_0 such that for all $x \geq x_0$,

$$q_{\lambda-\varepsilon}(x) < g(x) < q_{\lambda+\varepsilon}(x),$$

by Proposition 1.12. Hence for each $n \geq 1$,

$$q_{\lambda-\varepsilon}^n(x) < g^n(x) < q_{\lambda+\varepsilon}^n(x),$$

using the fact that q_β is strictly increasing. Since $g^n(\alpha) \to \infty$ with n, we can find m such that $g^m(\alpha) > x_0$. Let $x_1 = g^m(\alpha)$. Hence for $n \geq m$,

$$q_{\lambda-\varepsilon}^{n-m}(x_1) < g^n(\alpha) < q_{\lambda+\varepsilon}^{n-m}(x_1). \tag{1.4}$$

But $q_\beta^n(x) = p_\beta^{-1}(n + p_\beta(x))$. Thus, as a function of n,

$$O_f(q_\beta^{n-m}(x_1)) = O_f(p_\beta^{-1}) + O_f(n - m + p_\beta(x_1)) = \beta,$$

since $f(x+A) = f(x) + o(1)$ for every A follows from the assumptions. Hence from (1.4)

$$\lambda - \varepsilon \leq \liminf_{n\to\infty}\{f(g^n(\alpha)) - f(n)\} \leq \limsup_{n\to\infty}\{f(g^n(\alpha)) - f(n)\} \leq \lambda + \varepsilon.$$

This holds for all $\varepsilon > 0$, and so it follows that $O_f(g^n(\alpha)) = \lambda$. $\square$

Remark 1.14. The condition $\partial_f(x+1)=0$ is satisfied by functions f which tend to infinity smoothly and slowly, functions such as $\log\log x$ or $\Xi(x)$.

To illustrate Theorem 1.13, take $f=\log\log$. Then $g(x)=x+x^{1-\beta+o(1)}$ implies $g^n(\alpha)=n^{\frac{1}{\beta}+o(1)}$ for $\beta>0$ and every α sufficiently large.

2 Continuity and Linearity at Infinity.

In this section we shall introduce two notions of behavior at infinity with which we can characterize when functions are order-comparable and order-equivalent (Theorem 2.4 and Corollary 2.5). These notions will also be useful in §3.

2.1 Continuity and Linearity at Infinity.

Definition 2.1. (a) A function f is *continuous at infinity* if, for every $\varepsilon(x)\to 0$ as $x\to\infty$, we have

$$f(x+\varepsilon(x))=f(x)+o(1) \text{ as } x\to\infty.$$

More compactly, $f(x+o(1))=f(x)+o(1)$.

(b) A function f is *linear at infinity* if, for every $\lambda\in\mathbb{R}$, we have

$$f(x+\lambda)=f(x)+\lambda+o(1) \text{ as } x\to\infty.$$

For SIC_∞-functions, (b) is a stronger notion. Indeed, we have the following.

Lemma 2.2. *Let $f\in\mathrm{SIC}_\infty$ and suppose that f is linear at infinity. Then f is continuous at infinity. Furthermore, f^{-1} is also linear (and hence continuous) at infinity.*

PROOF. Let $\varepsilon(x)\to 0$. Then, for any $\varepsilon>0$, we have $|\varepsilon(x)|<\varepsilon$ for $x\geq x_0$ say. Hence, for such x,

$$|f(x+\varepsilon(x))-f(x)|<f(x+\varepsilon)-f(x-\varepsilon)\to 2\varepsilon \text{ as } x\to\infty.$$

But ε is arbitrary, so f is continuous at infinity.

Now suppose that f^{-1} is not linear at infinity. Then $\exists\lambda\in\mathbb{R}$ such that $f^{-1}(x+\lambda)-f^{-1}(x)\not\to\lambda$. Hence $\exists x_n\nearrow\infty$ and $\delta>0$ such that

$$|f^{-1}(x_n+\lambda)-f^{-1}(x_n)-\lambda|\geq\delta.$$

This means that

$$x_n+\lambda\geq f(f^{-1}(x_n)+\lambda+\delta) \text{ or } x_n+\lambda\leq f(f^{-1}(x_n)+\lambda-\delta).$$

But
$$f(f^{-1}(x_n)+\lambda\pm\delta)=x_n+\lambda\pm\delta+o(1),$$
and in both cases we obtain a contradiction, since $\delta>0$. □

Lemma 2.3. *(a) $f'\asymp 1\Rightarrow f$ and f^{-1} are continuous at infinity;*

(b) $f'\sim 1\Rightarrow f$ and f^{-1} are linear at infinity.

PROOF. (a) Let $\varepsilon(x)\to 0$. Then for all y sufficiently large, $|f'(y)|\le A$ for some A. Hence, for all large x,
$$|f(x+\varepsilon(x))-f(x)|=\left|\int_x^{x+\varepsilon(x)} f'(y)\,dy\right|\le A|\varepsilon(x)|\to 0,$$
as $x\to\infty$, showing f is continuous at infinity. On the other hand, $(f^{-1})'=1/f'(f^{-1})\asymp 1$. So the same argument applies to f^{-1}.

For (b), we know that for any $\varepsilon>0$, $|f'(x)-1|<\varepsilon$ for all x sufficiently large. Let $\lambda>0$ without loss of generality. Hence for such x,
$$|f(x+\lambda)-f(x)-\lambda|=\left|\int_0^{\lambda} f'(x+t)-1\,dt\right|\le\int_0^{\lambda}|f'(x+t)-1|\,dt\le\lambda\varepsilon.$$
Since ε is arbitrary, it follows that f is linear at infinity. The result for f^{-1} follows from Lemma 2.2. □

2.2 Applications to Orders.

In this section we apply the concepts of §2.1 to give necessary and sufficient conditions for two functions to be order-comparable and order-equivalent.

Theorem 2.4. *Let $f,g\in\mathrm{SIC}_\infty$. Then*

(a) f and g are order-comparable if and only if $f\circ g^{-1}$ and $g\circ f^{-1}$ are continuous at infinity, in which case either $O_{f,g}=\alpha\mathbb{Z}$ for some $\alpha\ge 0$, or $O_{f,g}=\mathbb{R}$.

(b) f and g are order-equivalent (i.e. $O_{f,g}=\mathbb{R}$) if and only if $g\circ f^{-1}$ is linear at infinity.

PROOF. (a) Suppose $f\circ g^{-1}$ and $g\circ f^{-1}$ are continuous at infinity. Let $\theta=g\circ f^{-1}$. Suppose for some function h and some $\lambda\in\mathbb{R}$, we have $O_f(h)=$

$O_g(h) = \lambda$. Then

$$\begin{aligned} g(h(x)) &= \theta(f(h(x))) = \theta(f(x) + \lambda + o(1)) \text{ (since } O_f(h) = \lambda) \\ &= \theta(f(x) + \lambda) + o(1) \\ &\quad \text{(since } \theta \text{ is continuous at infinity,)} \end{aligned}$$

but the left side equals

$$g(x) + \lambda + o(1) = \theta(f(x)) + \lambda + o(1).$$

Since f is invertible, it follows that

$$\theta(x + \lambda) = \theta(x) + \lambda + o(1). \tag{2.1}$$

Now, if for any other function H, we have $O_f(H) = \lambda$, then

$$\begin{aligned} g(H(x)) &= \theta(f(H(x))) = \theta(f(x) + \lambda + o(1)) = \theta(f(x)) + \lambda + o(1) \quad \text{(by (2.1))} \\ &= g(x) + \lambda + o(1), \end{aligned}$$

so that $O_g(H) = \lambda$ also. Similarly, $O_g(H) = \lambda \Longrightarrow O_f(H) = \lambda$ by using the fact that θ^{-1} is continuous at infinity. Hence f and g are order-comparable.

Conversely, suppose that f and g are order-comparable. As before let $\theta = g \circ f^{-1}$. Let $\varepsilon(x)$ be any function tending to 0 as $x \to \infty$, and put $h(x) = f^{-1}(f(x) + \varepsilon(f(x)))$. Then

$$f(h(x)) = f(x) + \varepsilon(f(x)) = f(x) + o(1),$$

so $O_f(h) = 0$. By supposition, we have $O_g(h) = 0$ also. Thus

$$\begin{aligned} g(h(x)) &= \theta(f(h(x))) = \theta(f(x) + \varepsilon(f(x))), \text{ and} \\ g(h(x)) &= g(x) + o(1) = \theta(f(x)) + o(1). \end{aligned}$$

Hence, writing y for $f(x)$, we have $\theta(y + \varepsilon(y)) = \theta(y) + o(1)$ as $y \to \infty$, and θ is continuous at infinity. Similarly for $f \circ g^{-1}$. This proves the first part of (a).

Observe from above that $\lambda \in O_{f,g}$ if and only if

$$\begin{aligned} \theta(x + \lambda) &= \theta(x) + \lambda + o(1) \quad \text{and} \\ \theta^{-1}(x + \lambda) &= \theta^{-1}(x) + \lambda + o(1), \end{aligned}$$

where $\theta = g \circ f^{-1}$. Suppose $\lambda, \mu \in O_{f,g}$. Then

$$\theta(x + (\lambda + \mu)) = \theta(x + \lambda) + \mu + o(1) = \theta(x) + (\lambda + \mu) + o(1),$$

(and similarly for θ^{-1}) so that $\lambda+\mu \in O_{f,g}$. Also $-\lambda \in O_{f,g}$ as

$$\theta(x-\lambda)=\theta(x-\lambda+\lambda)-\lambda+o(1)=\theta(x)-\lambda+o(1)$$

(and for θ^{-1}). In particular, $\lambda \in O_{f,g}$ implies $\lambda\mathbb{Z} \subset O_{f,g}$.

Now suppose $O_{f,g} \neq \mathbb{R}$ and $O_{f,g} \neq \{0\}$. Let

$$\alpha = \inf\{\lambda \in O_{f,g} : \lambda > 0\}.$$

Then $\alpha > 0$, since $\alpha = 0$ implies there exists $\lambda_n \in O_{f,g}$ with $\lambda_n \searrow 0$. From above, this implies $O_{f,g} \supset \cup_{n=1}^{\infty} \lambda_n\mathbb{Z}$, which is dense in $\mathbb{R}$. Hence, given $\lambda \in \mathbb{R}$, there exist sequences $\mu_n, \nu_n \in O_{f,g}$ such that $\mu_n \nearrow \lambda$ and $\nu_n \searrow \lambda$ as $n \to \infty$. Then

$$\begin{aligned} &\theta(x+\lambda) \leq \theta(x+\nu_n) = \theta(x)+\nu_n+o(1), \text{ (as } x\to\infty) \\ \text{and}\quad &\theta(x+\lambda) \geq \theta(x+\mu_n) = \theta(x)+\mu_n+o(1). \end{aligned}$$

Hence

$$\mu_n \leq \liminf_{x\to\infty}\{\theta(x+\lambda)-\theta(x)\} \leq \limsup_{x\to\infty}\{\theta(x+\lambda)-\theta(x)\} \leq \nu_n$$

for each n. This forces $\lim_{x\to\infty}\{\theta(x+\lambda)-\theta(x)\} = \lambda$. Similarly for θ^{-1}. Hence $\lambda \in O_{f,g}$, and so $O_{f,g} = \mathbb{R}$ — a contradiction. Thus $\alpha > 0$.

By the same argument, we actually have $\alpha = \min_{\lambda \in O_{f,g}, \lambda>0} \lambda$, for if there exist $\lambda_n \in O_{f,g}$ such that $\lambda_n \searrow \alpha$, then $O_{f,g} \supset \cup_{n=1}^{\infty}(\lambda_n - \lambda_{n+1})\mathbb{Z}$, which is dense in $\mathbb{R}$, and again we obtain the contradiction $O_{f,g} = \mathbb{R}$. Thus $\alpha \in O_{f,g}$.

Now we claim $O_{f,g} = \alpha\mathbb{Z}$. For we already have $O_{f,g} \supset \alpha\mathbb{Z}$, and if $O_{f,g} \neq \alpha\mathbb{Z}$, then $\exists \lambda \in O_{f,g} \setminus \alpha\mathbb{Z}$. Hence $k\alpha < \lambda < (k+1)\alpha$ for some $k \in \mathbb{Z}$. But then $\lambda - k\alpha \in O_{f,g}$ and $0 < \lambda - k\alpha < \alpha$, which contradicts the minimality of α. This proves (a).

(b) From (a), $O_{f,g} = \mathbb{R}$ if and only if θ and θ^{-1} are continuous at infinity and

$$\forall \lambda \in \mathbb{R}, \quad \begin{cases} \theta(x+\lambda) = \theta(x)+\lambda+o(1) \\ \theta^{-1}(x+\lambda) = \theta^{-1}(x)+\lambda+o(1); \end{cases}$$

i.e., θ and θ^{-1} are continuous and linear at infinity. By Lemma 2.2, this is equivalent to saying that θ is linear at infinity. $\square$

Corollary 2.5. *Let $f, g \in D_{\infty}^{+}$. Then:*

(a) $f' \asymp g'$ implies f and g are order-comparable.

(b) $f' \sim g'$ implies f and g are order-equivalent.

PROOF. (a) Let $\theta = g \circ f^{-1}$. Then

$$\theta' = \frac{g'(f^{-1})}{f'(f^{-1})} \asymp 1.$$

Hence, by Lemma 2.3, θ and θ^{-1} are continuous at infinity. Theorem 2.4(a) implies the result.

For (b), we have $\theta' \to 1$, so that, after Lemma 2.3, θ is linear at infinity. Theorem 2.4(b) implies the result. □

Remark 2.6. (i) The case $\alpha = 0$ (so that $O_{f,g} = \{0\}$) is possible. For example take $g(x) = \beta f(x)$ for some positive constant $\beta \neq 1$. Less trivially, let $g = \theta \circ f$, where θ is given by

$$\theta(x) = x + \sqrt{x} \sin \sqrt{x}.$$

Then $\theta(x) \sim x$ and $\theta'(x) \asymp 1$ (so that $f \sim g$ and, if f' exists, $f' \asymp g'$), but

$$\theta(x+\lambda) - \theta(x) - \lambda = \frac{\lambda}{2} \cos \sqrt{x} + O(x^{-1/2}) \not\to 0,$$

as $x \to \infty$, for any $\lambda \neq 0$. Thus $O_{f,g} = \{0\}$.

(ii) It is also possible to have $O_{f,g} = \mathbb{R}$ but $f' \not\sim g'$. Let $f \in D^+_\infty$ and let $g = \theta \circ f$ where θ is given by

$$\theta(x) = x + \frac{1}{2} \int_0^x \sin(t^2)\, dt,$$

Then

$$\begin{aligned} \theta(x+\lambda) - \theta(x) - \lambda &= \frac{1}{4} \int_{x^2}^{(x+\lambda)^2} \frac{\sin u}{\sqrt{u}}\, du = \frac{1}{4} \int_0^{2\lambda x + \lambda^2} \frac{\sin(x^2+v)}{\sqrt{x^2+v}}\, dv \\ &= \frac{1}{4x} \int_0^{2\lambda x+\lambda^2} \sin(x^2+v)\, dv + O\Big(\frac{1}{x}\Big) = O\Big(\frac{1}{x}\Big), \end{aligned}$$

so θ is linear at infinity. But $\theta'(x) = 1 + \frac{1}{2}\sin(x^2) \not\to 1$ and hence $f' \not\sim g'$.

Indeed, it is even possible to choose f and g such that $O_{f,g} = \mathbb{R}$ but $f' \not\asymp g'$, by choosing $\theta(x) = x - \frac{1}{x} + \int_0^x \sin(t^2) dt$. This function is again linear at infinity and $\theta' > 0$, but $\theta' \not\asymp 1$. These examples show that order-comparability cannot be usefully characterized in terms of derivatives.

(iii) Theorem 2.4(a) shows that the generalization to integer orders (referred to in §1.2) is unique. For example, suppose F_1 and F_2 are two order-comparable functions satisfying (1.1), then $1 \in O_{F_1,F_2}$. Hence $\mathbb{Z} \subset O_{F_1,F_2}$.

2.3 Equivalent Degrees.

If f and g are order-comparable, it is meaningful to ask whether they give rise to the same degrees; that is, when is $\partial_f \equiv \partial_g$? Clearly, we would need to have $O_f \equiv O_g$. With some extra conditions the degrees are also equivalent.

Theorem 2.7. *Let $f, g \in D_\infty^+$ such that $f' \sim g'$. Further suppose that $f(x + o(x)) = f(x) + o(1)$ and similarly for g. Then $\partial_f \equiv \partial_g$; i.e., $\partial_f(h) = \lambda \Longleftrightarrow \partial_g(h) = \lambda$ for all h and λ.*

PROOF. Suppose $\partial_f(h) = \lambda$ for some function h and some $\lambda \in \mathbb{R}$. Then

$$f(h(x)) = f(x) + \frac{1}{k(x)},$$

for some k with $O_f(k) = -\lambda$. By Corollary 2.5(b), $O_{f,g} = \mathbb{R}$. Thus $O_g(k) = -\lambda$ also.

Now $g = \theta \circ f$, where $\theta'(x) \to 1$. Hence

$$g(h(x)) - g(x) = \theta(f(h(x)) - \theta(f(x)) = \theta\Big(f(x) + \frac{1}{k(x)}\Big) - \theta(f(x)) = \frac{\theta'(c_x)}{k(x)},$$

for some $c_x \in (f(x), f(x)+1/k(x))$. But $c_x \to \infty$, so $\theta'(c_x) \to 1$. In particular, $\frac{1}{g(h(x))-g(x)} \sim k(x)$, and by the assumption, it follows that $\partial_g(h) = \lambda$.

By symmetry, we also have $\partial_g(h) = \lambda \Longrightarrow \partial_f(h) = \lambda$. □

Slightly different sets of assumptions also give equivalent degrees. For example, the same method of proof shows that the conditions $f' \asymp g'$, $O_{f,g} = \mathbb{R}$ and $f(\lambda x) = f(x) + o(1)$ for all $\lambda > 0$ (and similarly for g), are also sufficient to prove $\partial_f \equiv \partial_g$.

3 Fractional Iteration and the Abel Functional Equation.

In this section, we apply the notion of order to obtain a simple and natural criterion for the uniqueness of fractional iterates of a function.

Let $f \in \mathrm{SIC}_\infty$ and suppose we know that $O_F(f) = 1$ for some F. Consider the problem of finding a function g satisfying $g \circ g = f$. In general, there are infinitely many such g. As we pointed out in §1, if $O_F(g)$ exists, it must equal $\frac{1}{2}$. We shall show in Theorem 3.9 that, under some mild conditions on f and F, there is only one such g for which $O_F(g) = \frac{1}{2}$. This gives a natural criterion for the unique 'half-iterate' of f which is best behaved at infinity with respect to F. The same argument applies to other fractional iterates and continuous iterates.

To prove these results we must first consider the Abel functional equation to which fractional iteration is closely linked.

3.1 The Abel Functional Equation.

Given a function f such that $f(x) > x$, the equation

$$F(f(x)) = F(x) + 1 \tag{3.1}$$

(to solve for F) is called the *Abel functional equation.* For example, the function Ξ from §1.2 satisfies (3.1) with $f = \exp$. This equation has been studied by many authors[5]. For an exposition on the Abel functional equation, see for example [3], [4]. Szekeres ([9], [10], [11], [12]) studied it to develop a theory of 'regular growth', based on completely monotonic functions.

Equation (3.1) is closely related to the problem of finding a function F for which $O_F(f) = 1$; i.e.,

$$F(f(x)) = F(x) + 1 + o(1) \text{ as } x \to \infty. \tag{3.2}$$

In §1.2, we required such an F with $f = \exp$.

As Abel had discovered, the solutions of (3.1) can be used to define the *fractional iterates* of f. For if F^{-1} exists, we may define the 'λ^{th}-iterate' of f by

$$f^\lambda(x) = F^{-1}(F(x) + \lambda).$$

Then f^n corresponds to the usual n^{th}-iterate of f (for $n \in \mathbb{N}$), f^{-1} corresponds to its inverse, and $f^{\lambda+\mu} = f^\lambda \circ f^\mu$ for all $\lambda, \mu \in \mathbb{R}$.

For $f \in \mathrm{SIC}_\infty$, equation (3.1) always has an infinite number of SIC_∞-solutions. For suppose f is strictly increasing, continuous, and $f(x) > x$ for $x \geq a$. Then define F strictly increasing and continuous on $[a, f(a)]$ arbitrarily such that $F(f(a)) = F(a) + 1$. Extend F using (3.1) to $[a, \infty)$. This gives an infinite number of solutions. We show below that any two such solutions are necessarily order-comparable. Furthermore, if F_1 solves (3.1), then so does $F_2 = F_1 + c$ for any constant c. Trivially, F_1 and F_2 are then order-equivalent. We show below that the converse is also true.

Theorem 3.1. *Let $f \in \mathrm{SIC}_\infty$ and let F_1, F_2 be SIC_∞-solutions of the Abel functional equation*

$$F(f(x)) = F(x) + 1.$$

Then (i) F_1 and F_2 are order-comparable; and (ii) F_1 and F_2 are order-equivalent if and only if they differ by a constant.

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[5]In many articles, the function f is defined in a right neighborhood of 0. Using the transformation $g(\cdot) = 1/f(1/\cdot)$, this is equivalent to our setup.

PROOF. (i) After Theorem 2.4(a), it suffices to show that $F_1 \circ F_2^{-1}$ is continuous at infinity (since by the same reasoning the same is true for $F_2 \circ F_1^{-1}$). Let $\theta = F_1 \circ F_2^{-1}$. Then $\theta \in \mathrm{SIC}_\infty$ and

$$\theta(x+1) = F_1 \circ F_2^{-1}(x+1) = F_1 \circ f \circ F_2^{-1}(x) = F_1 \circ F_2^{-1}(x) + 1 = \theta(x) + 1.$$

It follows that $\theta(x) - x$ is periodic and extends to a continuous function on $\mathbb{R}$. Hence $\theta(x) - x$ is *uniformly continuous* on $\mathbb{R}$. Let $\varepsilon(x) \to 0$ as $x \to \infty$. Thus $\theta(x+\varepsilon(x)) - (x+\varepsilon(x)) - (\theta(x) - x) \to 0$; i.e. $\theta(x+\varepsilon(x)) = \theta(x) + o(1)$, and θ is continuous at infinity.

(ii) Suppose that $O_{F_1,F_2} = \mathbb{R}$. Then θ is linear at infinity, by Theorem 2.4(b). But $\theta(x+1) = \theta(x) + 1$. Hence,

$$\begin{aligned}\theta(x) &= \theta(n+x) - n \text{ (for all } n \in \mathbb{N})\\ &= \theta(n) + x - n + o(1) \text{ (as } n \to \infty\text{, since } \theta \text{ is linear at infinity)}\\ &= \theta(0) + x + o(1).\end{aligned}$$

Letting $n \to \infty$ gives $\theta(x) = x + \theta(0)$, and so $F_1 - F_2$ is constant.

The converse is immediate. □

In particular, if F_1 and F_2 are differentiable solutions of (3.1) and $F_1' \sim F_2'$, then $F_1 = F_2$+constant.

3.2 Constructing Solutions.

In many cases it is possible, via an iterating process, to go from (3.2) to (3.1). The simplest case is when the $o(1)$-term is sufficiently small.

Theorem 3.2. *Let $f \in \mathrm{SIC}_\infty$ and suppose that $F \in \mathrm{SIC}_\infty$ is such that*

$$F(f(x)) = F(x) + 1 + \delta(F(x)),$$

where $\delta(x) \to 0$ in such a way that $|\delta(x)| \le \varepsilon(x)$ for some $\varepsilon(x) \searrow 0$ and $\int^\infty \varepsilon(t)dt < \infty$. Then the limit

$$\rho(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} \{F(f^n(x)) - n\} \tag{3.3}$$

exists for all x sufficiently large, the convergence being uniform in a neighborhood of infinity. Further, ρ is continuous and increasing, satisfies $\rho(f(x)) = \rho(x) + 1$, and $\rho(x) = F(x) + o(1)$ as $x \to \infty$.

Proof. The assumptions implicitly imply that $f(x) > x$ eventually. Hence for x sufficiently large, $f^n(x) \to \infty$ as $n \to \infty$. For such x, we have by iteration,

$$F(f^n(x)) - n = F(x) + \sum_{r=0}^{n-1} \delta(F(f^r(x))).$$

Now $|\delta(F(f^r(x)))| \leq \varepsilon(F(f^r(x))) \leq \varepsilon(r/2)$ for r sufficiently large, since $\varepsilon(\cdot)$ is decreasing and $F(f^r(x)) \sim r$. Also $\sum_{r=m}^{n} \varepsilon(r/2) \to 0$ as $m, n \to \infty$ by comparison to $\int_m^n \varepsilon(t)dt$. Hence $\sum_{r=0}^{\infty} \delta(F(f^r(x)))$ converges absolutely, and the convergence is uniform in a neighborhood of infinity. Thus

$$\lim_{n\to\infty} \{F(f^n(x)) - n\} = F(x) + \sum_{r=0}^{\infty} \delta(F(f^r(x))) \stackrel{\text{def}}{=} \rho(x).$$

Since each term in the above series is continuous, and the convergence is uniform, it follows that ρ is continuous. Further

$$\begin{aligned} \rho(f(x)) &= \lim_{n\to\infty} \{F(f^n(f(x))) - n\} = \lim_{n\to\infty} \{F(f(f^n(x))) - n\} \\ &= \lim_{n\to\infty} \{F(f^n(x)) - n\} + 1 = \rho(x) + 1 \qquad (\text{since } O_F(f) = 1). \end{aligned}$$

Also $\rho(x) - F(x) = \sum_{r=0}^{\infty} \delta(F(f^r(x)))$ which tends to 0 as $x \to \infty$ since $\delta(F(f^r(x))) \to 0$ for every r and the convergence is uniform in closed neighborhoods of ∞. This shows $\rho(x) = F(x) + o(1)$. □

Remark 3.3. Since $\rho(x) = F(x) + o(1)$, it follows that $O_\rho \equiv O_F$. We could further write $O_{F,\rho} = \mathbb{R}$ if ρ is strictly increasing (since, strictly speaking, order-comparability is only defined for SIC_∞-functions). But it can happen that ρ is not strictly increasing even if F is (see Remark 3.7 (i) for an example).

Without the strong bound on the error $\delta(x)$ in the assumptions of Theorem 3.2, the limit (3.3) may not necessarily exist. However, in many cases the limit

$$\sigma(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} \{F(f^n(x)) - F(f^n(x_0))\} \tag{3.4}$$

does exist for all sufficiently large x, x_0, and the limit function $\sigma(x)$, also satisfies the corresponding Abel equation (3.1):

$$\begin{aligned} \sigma(f(x)) &= \lim_{n\to\infty} \{F(f^n(f(x))) - F(f^n(x_0))\} = \lim_{n\to\infty} \{F(f(f^n(x))) - F(f^n(x_0))\} \\ &= \lim_{n\to\infty} \{F(f^n(x)) - F(f^n(x_0))\} + 1 = \sigma(x) + 1 \ (\text{since } O_F(f) = 1). \end{aligned}$$

Of course, if ρ exists, then so does σ, in which case $\sigma(x) = \rho(x) - \rho(x_0)$. We will give various conditions under which σ exists in §3.3.

Unlike the case for ρ, we shall see (in Remark 3.7) that it may happen that $O_F \not\equiv O_\sigma$. However, if we know that $\sigma \in \mathrm{SIC}_\infty$, then we do have $O_{F,\sigma} = \mathbb{R}$, which we prove below. Hence in this case, we may equally well use F or σ to define orders.

Theorem 3.4. *Let $f, F \in \mathrm{SIC}_\infty$ such that $O_F(f) = 1$. Suppose that $f^n(x) \to \infty$ as $n \to \infty$ for $x \geq x_0$ and that the limit function σ in (3.4) exists and $\sigma \in \mathrm{SIC}_\infty$. Then $O_{F,\sigma} = \mathbb{R}$.*

Proof. After Theorem 2.4(b), we just need to show that $h \stackrel{\text{def}}{=} F \circ \sigma^{-1}$ is linear at infinity. Now $\sigma^{-1}(x+1) = f(\sigma^{-1}(x))$ and hence

$$\begin{aligned} h(x+1) = F(\sigma^{-1}(x+1)) = F(f(\sigma^{-1}(x))) &= F(\sigma^{-1}(x)) + 1 + o(1) \\ &\qquad\qquad (\text{since } O_F(f) = 1) \\ &= h(x) + 1 + o(1), \ \text{ as } x \to \infty. \end{aligned}$$

By iteration, $h(x+k) = h(x) + k + o(1)$ for all $k \in \mathbb{Z}$.

Assume, without loss of generality, that $\sigma(x)$ is strictly increasing and continuous for $x \geq x_0$. Let $\lambda \geq 0$. Then for $n \in \mathbb{N}$, $\sigma^{-1}(n+\lambda) = f^n(\sigma^{-1}(\lambda))$ and, by definition of σ,

$$F(f^n(\sigma^{-1}(\lambda))) - F(f^n(x_0)) \to \lambda \text{ as } n \to \infty.$$

Hence

$$\begin{aligned} h(n+\lambda) &= F(\sigma^{-1}(n+\lambda)) = F(f^n(\sigma^{-1}(\lambda))) = F(f^n(x_0)) + \lambda + o(1) \\ &= h(n) + \lambda + o(1). \end{aligned}$$

Clearly, this extends to all values of λ.

Now we want to show that 'n' can be replaced by 'x'; i.e. $h(x+\lambda) - h(x) \to \lambda$ as $x \to \infty$ through real values.

Fix $k \in \mathbb{N}$. Let $n = [x]$. Then x lies in one of the intervals $[n + \frac{r-1}{k}, n + \frac{r}{k})$ $(r = 1, \ldots, k)$. By monotonicity of h we have (for all x sufficiently large)

$$h\Big(n + \frac{r-1}{k}\Big) - h(n) \leq h(x) - h(n) < h\Big(n + \frac{r}{k}\Big) - h(n).$$

Let $k(x) = h(x) - x$. Then $k(x+1) = k(x) + o(1)$ and the above inequalities imply that

$$h\Big(n+\frac{r-1}{k}\Big) - h(n) - \frac{r-1}{k} - \frac{1}{k} < k(x) - k(n) < h\Big(n+\frac{r}{k}\Big) - h(n) - \frac{r}{k} + \frac{1}{k}. \quad (3.5)$$

Let $x \to \infty$ (so that $n \to \infty$). Now for each $r \in \{1, \ldots, k\}$, $h(n + \frac{r}{k}) - h(n) - \frac{r}{k} \to 0$; i.e. $\forall \varepsilon > 0$, $\exists n_r$ such that

$$\left| h\Big(n + \frac{r}{k}\Big) - h(n) - \frac{r}{k} \right| < \varepsilon \text{ for } n \geq n_r.$$

Hence for $n \geq N = \max_{1 \leq r \leq k} n_r$, the above holds *uniformly* for all such r. Thus the LHS of (3.5) tends to $-\frac{1}{k}$ while the RHS tends to $\frac{1}{k}$ as $x \to \infty$. This shows that

$$\limsup_{x\to\infty} |k(x) - k([x])| \leq \frac{1}{k}.$$

However, this is true for all $k \in \mathbb{N}$, so it follows that the above limsup is zero; i.e. $k(x) = k([x]) + o(1)$. Let $\lambda \in [0, 1)$. Then

$$k(x + \lambda) - k(x) = k([x + \lambda]) - k([x]) + o(1).$$

But $[x + \lambda] = [x]$ or $[x] + 1$ and, in either case, the RHS tends to 0. By iteration, $k(x + \lambda) - k(x) \to 0$ for all λ, and hence

$$h(x + \lambda) = h(x) + \lambda + o(1),$$

as required. □

Corollary 3.5. *Let $f \in \mathrm{SIC}_\infty$ and suppose that $O_{F_1}(f) = O_{F_2}(f) = 1$ for some order-equivalent functions $F_1, F_2 \in \mathrm{SIC}_\infty$ for which the corresponding σ_1 and σ_2 exist and $\sigma_1, \sigma_2 \in \mathrm{SIC}_\infty$. Then $\sigma_1 = \sigma_2$+constant.*

PROOF. We have $O_{F_1,\sigma_1} = O_{F_2,\sigma_2} = \mathbb{R}$, from Theorem 3.4. Since $O_{F_1,F_2} = \mathbb{R}$ we also have $O_{\sigma_1,\sigma_2} = \mathbb{R}$. Theorem 3.1 yields $\sigma_1 = \sigma_2$+constant. □

3.3 Conditions for Existence of σ.

In this section, we consider sufficient conditions for which σ exists, and for which σ is invertible.

Theorem 3.6. *Let $f, F \in \mathrm{SIC}_\infty$ such that $O_F(f) = 1$ and $F \circ f - F$ is monotonic. Suppose that $f^n(x) \to \infty$ as $n \to \infty$ for $x \geq x_0$. Then the limit $\sigma(x)$, given by (3.4), exists and is an increasing function of x. Moreover, σ is continuous if $F \circ f - F$ is decreasing, and σ is strictly increasing if $F \circ f - F$ is increasing.*

PROOF. Let $\sigma_n(x) = F(f^n(x)) - F(f^n(x_0))$. Since $f^n(x_0) \to \infty$, we have $x_0 \leq x \leq f^m(x_0)$ for some m. Thus

$$0 \leq \sigma_n(x) \leq F(f^n(f^m(x_0))) - F(f^n(x_0)) = F(f^m(f^n(x_0))) - F(f^n(x_0)) \to m,$$

as $n \to \infty$. Hence $\sigma_n(x)$ is bounded as $n \to \infty$ in any case.

Let $g = F \circ f - F$, so that $g(x)$ is either increasing or decreasing, and tends to 1. Then $\sigma_{n+1}(x) - \sigma_n(x) = g(f^n(x)) - g(f^n(x_0))$, which is either ≥ 0 for all n sufficiently large, or ≤ 0. In either case, $\sigma_n(x)$ is monotonic and bounded, and converges to $\sigma(x)$. That σ is increasing follows immediately from (3.4).

Now, if g is decreasing, then for $y > x$ (both sufficiently large),

$$F(y)-F(x) \geq F(f(y))-F(f(x)) \geq \cdots \geq F(f^n(y))-F(f^n(x)) \to \sigma(y)-\sigma(x),$$

so that σ is continuous. On the other hand, if g is increasing, then the above inequalities are reversed and σ is strictly increasing. $\square$

In particular, if ρ exists and $F(f) - F$ is increasing, then $\rho \in \mathrm{SIC}_\infty$ (since ρ is continuous already) and so also $\sigma \in \mathrm{SIC}_\infty$.

Remark 3.7. (i) The assumptions of Theorem 3.6 are not sufficient to give $\sigma \in \mathrm{SIC}_\infty$. It may happen that $F(f) - F$ is decreasing but σ is not strictly increasing. For instance, take $f(x) = x + 1$ and F to be the SIC_∞-function

$$F(x) = x - \frac{1}{x} + \begin{cases} 1 - \{x\} & \text{if } \{x\} \leq \frac{1}{2} \\ \{x\} & \text{if } \{x\} > \frac{1}{2} \end{cases} \qquad (x \geq 1)$$

(Here, as usual, $[x]$ is the largest integer less than or equal to x and $\{x\} = x - [x]$.) We can write $F(x) = G(x) - \frac{1}{x}$ where $G(x+1) = G(x) + 1$. Thus $F(x+1) - F(x) = 1 + \frac{1}{x(x+1)}$, which decreases to 1, and

$$F(f^n(x)) - n = F(n+x) - n = F(x) + \frac{1}{x} - \frac{1}{n+x} \to G(x) \text{ as } n \to \infty.$$

Hence ρ exists and $\rho(x) = G(x)$, but ρ (and σ) is not strictly increasing.

(ii) A similar example (where $f(x) = x + 1$) can be constructed where $F(f) - F$ is increasing and σ is not continuous. We omit the details.

(iii) It is also possible that σ exists (without the condition of monotonicity of $F(f) - F$), but that σ is neither continuous nor strictly increasing. For example, let $f(x) = x + 1 - \frac{\{x\}(1-\{x\})}{[x]}$ and $F(x) = x$. Then $f, F \in \mathrm{SIC}_\infty$, $O_F(f) = 1$, and σ exists for all $x, x_0 \geq 1$. But $\sigma(x) = [x] - [x_0]$ which is neither continuous nor strictly increasing. (To see this, note that $f[k, k+1) = [k+1, k+2)$ for every $k \in \mathbb{N}$. Thus for $x \in [1, 2)$, $f^n(x) = n + 1 + \alpha_n$ for some $\alpha_n \in [0, 1)$. This leads to

$$\alpha_n = \alpha_{n-1} - \frac{\alpha_{n-1}(1 - \alpha_{n-1})}{n},$$

which in turn implies $\alpha_n \to 0$ as $n \to \infty$. Thus $f^n(x) = n + 1 + o(1)$.) Furthermore, for this example, $O_F \not\equiv O_\sigma$, since $[x+\lambda] - [x] \not\to \lambda$ for $\lambda \notin \mathbb{Z}$.

For the purposes of fractional iterates it is important to have σ invertible. In that case Theorem 3.4 also applies. One way to ensure invertibility is for σ to have a positive derivative. So consider the case when the functions f and F are differentiable. Under what conditions is σ differentiable? Let $\sigma_n(x) = F(f^n(x)) - F(f^n(x_0))$. Suppose that $f, F \in D^+_\infty$ and that σ exists. If σ'_n tends uniformly to a limit τ say, then σ is differentiable and $\sigma' = \tau$ (see for example, [6] p. 402). The following theorem gives conditions for the existence of $\lim_{n\to\infty} \sigma'_n$. Note that

$$\sigma'_n(x) = F'(f^n(x))(f^n)'(x) = F'(f^n(x)) \prod_{r=0}^{n-1} f'(f^r(x)).$$

Theorem 3.8. *Let $f, F \in D^+_\infty$ be such that $F(f(x)) = F(x) + 1 + \delta(F(x))$, where $\delta(x) \to 0$ and $\delta'(x) \to 0$ in such a way that $|\delta'(x)| < \varepsilon(x) \searrow 0$ and $\int^\infty \varepsilon(t)dt$ converges. Then*

$$\lim_{n\to\infty} F'(f^n(x))(f^n)'(x) = \tau(x)$$

exist for all x sufficiently large, the convergence being uniform on compact subsets of a neighborhood of infinity. The function τ is continuous, positive, and $f'(x)\tau(f(x)) = \tau(x)$. Further, $F'(x) \sim \tau(x)$ as $x \to \infty$.

Moreover, σ exists, $\sigma' = \tau$, and $\sigma \in D^+_\infty$.

Proof. Let $\tau_n(x) = (F(f^n(x)))' = F'(f^n(x))(f^n)'(x)$ (> 0). Then

$$\frac{\tau_{n+1}(x)}{\tau_n(x)} = \frac{F'(f^{n+1}(x))(f^{n+1})'(x)}{F'(f^n(x))(f^n)'(x)} = \frac{F'(f(f^n(x)))f'(f^n(x))}{F'(f^n(x))} = 1+\delta'(F(f^n(x))).$$

Hence

$$\tau_N(x) = \tau_0(x) \prod_{n=0}^{N-1} \Big(1 + \delta'(F(f^n(x)))\Big).$$

No term in the product equals zero and $|\delta'(F(f^n(x)))| < \varepsilon(n/2)$ for all n and all x in a fixed bounded interval (since $F(f^n(\cdot)) \sim n$). As $\sum_n \varepsilon(n/2)$ converges (by comparison with $\int^\infty \varepsilon(t/2)dt$), it follows that τ_N converges uniformly to some positive, continuous function τ. Furthermore,

$$\begin{aligned} f'(x)\tau(f(x)) &= f'(x) \lim_{n\to\infty} F'(f^n(f(x)))(f^n)'(f(x)) \\ &= \lim_{n\to\infty} F'(f^{n+1}(x))(f^{n+1})'(x) = \tau(x), \end{aligned}$$

so that $f'(x)\tau(f(x)) = \tau(x)$. By iteration, $\tau(x) = (f^n)'(x)\tau(f^n(x))$ for every $n \in \mathbb{N}$. Now consider F'/τ. We have

$$\frac{F'(f^n(x))}{\tau(f^n(x))} = \frac{F'(f^n(x))(f^n)'(x)}{\tau(x)} = \frac{\tau_n(x)}{\tau(x)} \to 1 \text{ as } n \to \infty.$$

This convergence is uniform in x on compact subsets of $[A, \infty)$ (for some A). Hence $F'(x)/\tau(x) \to 1$, as required.

To deduce the existence of σ, let $T_n(x) = \int_{x_0}^x \tau_n(t)dt$ and $T(x) = \int_{x_0}^x \tau(t)dt$, for x, x_0 sufficiently large. Since $\tau_n \to \tau$ uniformly, $T_n \to T$ uniformly on compact subsets of a neighborhood of infinity. But

$$T_n(x) = \int_{x_0}^x (F(f^n))'(t)\,dt = F(f^n(x)) - F(f^n(x_0)),$$

so σ exists and $\sigma(x) = T(x) = \int_{x_0}^x \tau(t)dt$. Since τ is continuous, we have $\sigma' = \tau$ and so $\sigma \in D_\infty^+$. $\square$

Theorem 3.8 gives a sufficient condition for $\sigma' > 0$ and hence for σ to be invertible. There are other sufficient conditions (for σ to be invertible). One such follows from a result of Lévy [5], which was rigorously proved by Szekeres [9]: if, with the same setup as in Theorem 3.8, we assume $\delta(x), \delta'(x) \to 0$ and $\delta'(\cdot)$ is of bounded variation, then $\sigma \in \mathrm{SIC}_\infty$ (and hence is invertible).

3.4 Application to Fractional and Continuous Iteration.

Given a function f, consider the functional equation,

$$g \circ g = f. \tag{3.6}$$

We say g is a $\frac{1}{2}^{\text{th}}$-*iterate* of f. More generally, for a positive rational $r = \frac{p}{q}$, an r^{th}-*iterate* of f is a function g for which

$$g^q = f^p. \tag{3.7}$$

(Recall that g^q denotes the q^{th}-iterate of g, not the q^{th}-power). We shall restrict ourselves to the case where f is in SIC_∞. As for the case of solutions to the Abel equation (3.1), there are always infinitely many solutions g; if F solves (3.1), then $g(x) = F^{-1}(F(x) + \frac{1}{2})$ solves (3.6).

Now suppose we know that for some F, we have $O_F(f) = 1$. If g solves (3.6) and *if* $O_F(g)$ exists, then it must equal $\frac{1}{2}$. It turns out that this extra condition (i.e. the existence of $O_F(g)$) distinguishes g uniquely from all the other solutions of (3.6). Similar considerations apply to (3.7).

We say f has a *continuous iteration* f_λ ($\lambda \in \mathbb{R}$) if $f_n = f^n$ for $n \in \mathbb{Z}$ and $f_\lambda \circ f_\mu = f_{\lambda+\mu}$ for all $\lambda, \mu \in \mathbb{R}$. As for the fractional iterates, there are always (infinitely many) solutions, given by $F^{-1}(F(x) + \lambda)$ where F solves (3.1).

Theorem 3.9. *Let $f, F \in \mathrm{SIC}_\infty$ be such that $O_F(f) = 1$. Suppose also that the corresponding σ exists (given by (3.4)) and $\sigma \in \mathrm{SIC}_\infty$. Then*

(a) for $p, q \in \mathbb{N}$, there exists a unique SIC_∞-function g satisfying (3.7) *for which $O_F(g)$ exists, namely, $\sigma^{-1}(\sigma(x) + \frac{p}{q})$;*

(b) there exists a unique continuous iteration f_λ of f satisfying $O_F(f_\lambda) = \lambda$, namely, $\sigma^{-1}(\sigma(x) + \lambda)$.

PROOF. (a) Since $O_F(g)$ exists, we must have $O_F(g) = \frac{p}{q} = r$, say. Now

$$g \circ f^p = g \circ g^q = g^{q+1} = g^q \circ g = f^p \circ g.$$

Hence, for all $n \in \mathbb{N}$, $g \circ f^{np} = f^{np} \circ g$. Now, for x, x_0 sufficiently large,

$$F(g(f^{np}(x))) - F(f^{np}(x_0)) = F(f^{np}(x)) - F(f^{np}(x_0)) + r + o(1) \to \sigma(x) + r,$$

as $n \to \infty$, since $O_F(g) = r$. On the other hand,

$$F(g(f^{np}(x))) - F(f^{np}(x_0)) = F(f^{np}(g(x))) - F(f^{np}(x_0)) \to \sigma(g(x)).$$

Hence $\sigma(g(x)) = \sigma(x) + r$ and so g is given uniquely by $g(x) = \sigma^{-1}(\sigma(x) + r)$. Furthermore, this g indeed satisfies $O_F(g) = r$, since $O_\sigma(g) = r$ and F and σ are order-equivalent by Theorem 3.4.

(b) In this case, $F(f_\lambda(x)) = F(x) + \lambda + o(1)$. Hence, as $n \to \infty$,

$$F(f_\lambda(f^n(x))) - F(f^n(x_0)) = F(f^n(x)) - F(f^n(x_0)) + \lambda + o(1) = \sigma(x) + \lambda + o(1),$$

and

$$F(f_\lambda(f^n(x))) - F(f^n(x_0)) = F(f^n(f_\lambda(x))) - F(f^n(x_0)) = \sigma(f_\lambda(x)) + o(1).$$

This yields $\sigma(f_\lambda(x)) = \sigma(x) + \lambda$. Again $O_F(f_\lambda) = \lambda$, since $O_\sigma(f_\lambda) = \lambda$ and $O_{F,\sigma} = \mathbb{R}$. □

Remark 3.10. The condition that $\sigma \in \mathrm{SIC}_\infty$ is crucial for both existence and uniqueness of the fractional iterates satisfying the order condition. For instance, if σ exists but $\sigma \notin \mathrm{SIC}_\infty$, then it may happen that there are many or no solutions g for which $g \circ g = f$ and $O_F(g) = \frac{1}{2}$. By following the proof of (a), this is equivalent to solving

$$\sigma(g(x)) = \sigma(x) + \frac{1}{2}.$$

For example, let f and F be as in Remark 3.7 (iii), so that $\sigma(x) = [x] - [x_0]$. Then there is no function g for which $[g(x)] = [x] + \frac{1}{2}$, as the LHS is an integer and the RHS is not.

As a second example, let $f(x) = x + 1$ and $F(x) = K(x) - \frac{1}{x}$, where

$$K(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2} \\ 3x - \frac{3}{2} & \text{if } \frac{1}{2} \le x < \frac{2}{3} \\ \frac{1}{2} & \text{if } \frac{2}{3} \le x < \frac{5}{6} \\ 3x - 2 & \text{if } \frac{5}{6} \le x < 1 \end{cases},$$

and $K(x+1) = K(x)+1$. Thus σ exists and $\sigma(x) = K(x) - K(x_0)$ is continuous but not strictly increasing. Then there are infinitely many functions g such that $g^2 = f$ and $\sigma(g) = \sigma + \frac{1}{2}$; namely

$$g(x) = \begin{cases} h(x) & \text{if } 0 \le x < \frac{1}{2} \\ x + \frac{1}{3} & \text{if } \frac{1}{2} \le x < \frac{2}{3} \\ h^{-1}(x) + 1 & \text{if } \frac{2}{3} \le x < \frac{5}{6} \\ x + \frac{2}{3} & \text{if } \frac{5}{6} \le x < 1 \end{cases},$$

and extended to $\mathbb{R}$ via $g(x + 1) = g(x) + 1$ (here h is any strictly increasing function from $[0, \frac{1}{2}]$ onto $[\frac{2}{3}, \frac{5}{6}]$).

Of course, this last example is somewhat artificial, since given $f(x) = x+1$, it is plainly absurd to choose F (for which $O_F(f) = 1$) to be such a badly behaved function.

Examples 3.11. (a) Let $f(x) = cx + w(x)$ where $c > 1$ and $w(x) = o(x)$. Taking $F(x) = \frac{\log x}{\log c}$, we have $O_F(f) = 1$ and

$$\frac{F'(f(x))f'(x)}{F'(x)} - 1 = \frac{xf'(x)}{f(x)} - 1 = \frac{xw'(x) - w(x)}{cx + w(x)}.$$

If we assume further that $w'(x) = o(1/(\log x)(\log\log x)^{1+\eta})$ for some $\eta > 0$, then (after Theorem 3.8) σ exists and $\sigma \in D_\infty^+$. Hence there exists a unique continuous iteration f_λ subject to the condition $f_\lambda(x) \sim c^\lambda x$ as $x \to \infty$.

(b) Let $f(x) = x^2 + 1$. Then, taking $F(x) = \frac{\log\log x}{\log 2}$, we have $O_F(f) = 1$ and

$$\frac{F'(f(x))f'(x)}{F'(x)} - 1 = \frac{2x^2 \log x}{(x^2 + 1)\log(x^2 + 1)} - 1 \sim -\frac{1}{x^2}.$$

Thus (after Theorem 3.8) σ exists and $\sigma \in D_\infty^+$. Hence there exists a unique continuous iteration f_λ subject to the condition $f_\lambda(x) = x^{2^\lambda + o(1)}$. In particular, there is a unique g such that

$$g(g(x)) = x^2 + 1 \quad \text{and} \quad g(x) = x^{\sqrt{2}+o(1)}.$$

Similarly for any other D_∞^+-function of the form $x^{\alpha+\varepsilon(x)}$ with $\alpha > 1$ and $\varepsilon(x) \to 0$ in such a way that $\varepsilon'(x) = o(1/x(\log x)^{1+\eta})$ for some $\eta > 0$. In this case take $F(x) = \frac{\log\log x}{\log \alpha}$. In particular, this includes all polynomial functions of degree greater than 1.

(c) Let f be a function of positive order α w.r.t. Ξ. If

$$\frac{f'(x)\chi(x)}{\chi(f(x))} - 1 = o\Big(\frac{1}{\Xi(x)^{1+\delta}}\Big)$$

for some $\delta > 0$, then

$$\sigma(x) = \frac{1}{\alpha} \lim_{n\to\infty} \{\Xi(f^n(x)) - \Xi(f^n(x_0))\}$$

exists and $\sigma \in D_\infty^+$ by Theorem 3.8. All such f have a unique continuous iteration for which $O_\Xi(f^\lambda) = \lambda\alpha$, given by

$$f^\lambda(x) = \sigma^{-1}(\sigma(x) + \lambda).$$

Examples (a) and (b) are known in various guises — see for example [8] where case (a) is shown to hold under the condition $w'(x) = o(1/x^\delta)$ (some $\delta > 0$), while case (b) is proven under the assumption $\varepsilon'(x) = o(1/x^{1+\delta})$ (some $\delta > 0$) (see also [13], where an alternative (convexity) condition is also given). Furthermore, these cases are usually treated separately. On the contrary, our approach applies in great generality, uniting these varying results.

3.5 Best-Behaved Abel Functions.

The unique fractional and continuous iterates obtained in Theorem 3.8, are not necessarily 'best-behaved' (or 'most regular') iterates in some absolute sense, but 'best-behaved' with respect to the function F. If one is interested in 'best-behaved' iterates, then the function F should be chosen as 'best-behaved' at infinity as possible, since the condition $O_F(f_{1/2}) = \frac{1}{2}$ (or $O_F(f_\lambda) = \lambda$) is just a regularity condition at infinity.

In example (a) (similarly (b)), the F is well-chosen in the above sense. For $\frac{\log x}{\log c}$ is the best-behaved function G satisfying $G(cx) = G(x)+1$; if such a G is differentiable, then $cG'(cx) = G'(x)$. But then $c^n G'(c^n x) = G'(x)$ for every n. If we assume that G' is continuous and positive, then this implies $yG'(y) \asymp 1$ as $y \to \infty$. The most regular choice for G is one for which $\lim_{y\to\infty} yG'(y)$ exists. This forces $G'(x) = a/x$ for some a.

For example (c), the continuous iteration of exponentially growing functions would be the natural choice of iterates *if* we could show that Ξ is the 'best-behaved' solution of the Abel equation for exp. In fact, Ξ can be seen not to be the best choice in this regard. For in this case, $\Xi' = 1/\chi$ is continuous, but χ not differentiable at each of the points e_n^1. In [10], Szekeres claimed to have obtained such a function based on the complete monotonicity of the solution of the related Abel equation for $e^x - 1$.

The point to be stressed here is that, once a function F (like log) has been accepted as having regular growth, it can be used to take orders with, and in this way the most regular fractional iterates can be found of a large class of functions.

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