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CONSISTENT RECOVERY AND POLYGONAL APPROXIMATION OF FUNCTIONS†

Abstract

We consider real-valued functions defined on the unit interval. It is known that the class of first-return recoverable functions is the same as the class of polygonally approximable functions and that this class consists of the Baire one functions. Here we introduce the more restrictive classes of consistently first-return recoverable functions and consistently polygonally approximable functions. We show these classes are identical and consist of those functions which are continuous except at countably many points.

1 Introduction

Functions considered here are real-valued and defined on the interval $I = [0, 1]$. It has been shown in [4] that such a function f belongs to Baire class one if and only if f is what is called a first-return recoverable function; and it has been shown in [1] that f belongs to Baire class one if and only if f is what is called a polygonally approximable function. Here, we define what we mean for a function f to be *consistently* first-return recoverable and to be *consistently*

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polygonally approximable. We show that the two concepts are equivalent and characterize functions in either class as those functions having only countably many points of discontinuity. Before proceeding we need to review definitions and introduce the new concepts.

Underlying most of our subsequent definitions is the notion of what we call a *trajectory.* A *trajectory* is any sequence $\overline{x} = \{\overline{x}(n)\} = \{x_n\}$ of distinct points in I, whose range is dense in I. Any countable dense set $S \subset I$ is called a *support set* and, of course, any enumeration of S becomes a trajectory. For a given trajectory $\overline{x} = \{x_n\}$ and a finite union H of intervals, we let $r(\overline{x}, H)$ denote the first x_n that belongs to H.

For $x \in [0,1]$ and $\rho > 0$ we let $B_\rho(x) = \{y \in [0,1] : |y - x| < \rho\}$. As is standard, we denote the restriction of a function $f : I \to \mathbb{R}$ to a set $D \subseteq I$ by $f|_D$. As a final bit of notation, we find it convenient to let $L^{(u,v)}_{(s,t)} : I \to \mathbb{R}$ denote the linear function passing through the two points $(s,t), (u,v)$.

Definition 1. Let $x \in I$ and let $\overline{x} = \{x_n\}$ be a fixed trajectory. The *first return route to* x, $\mathcal{R}(\overline{x})_x = \{w_k(x)\}_{k=1}^{\infty}$, is defined recursively via

$$w_1(x) = x_0,$$

$$w_{k+1}(x) = \begin{cases} r\left(\overline{x}, B_{|x-w_k(x)|}(x)\right) & \text{if } x \neq w_k(x) \\ w_k(x) & \text{if } x = w_k(x). \end{cases}$$

We say that f is *first return recoverable with respect to* $\overline{x}$ at x provided that

$$\lim_{k\to\infty} f(w_k(x)) = f(x),$$

and if this happens for each $x \in I$, we say that f is *first return recoverable with respect to* $\overline{x}$. Finally, we say that f is *first-return recoverable* if it is first-return recoverable with respect to some trajectory.

Definition 2. Let D be a support set. We shall say a function f is *consistently first-return recoverable with respect to* D provided that f is first-return recoverable with respect to every ordering of D. A function is said to be *consistently recoverable* if there exists a support set D with respect to which f is consistently first-return recoverable.

Definition 3. Let $f : I \to \mathbb{R}$.

a) We say that a function $h : I \to \mathbb{R}$ is a *polygonal function for* f if there is a partition $\tau = \{0 = a_0 < a_1 < a_2 < \cdots < a_m = 1\}$ such that h agrees with f at each partition point and is linear on the intervening closed intervals. We call $a_0, a_1, \ldots, a_m$ the *nodes of* h and

$(a_0, h(a_0)), (a_1, h(a_1)), \ldots, (a_m, h(a_m))$ the *vertices of* h. The maximum distance between adjacent nodes is called the *mesh of* h and the maximum distance between adjacent vertices is called the *graph-mesh of* h. These are denoted $mesh(h)$ and graph-mesh(h), respectively.

b) We say that a sequence $\{h_n\}$ of polygonal functions for f *polygonally approximates* f *on* I provided $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in I$ and $lim_{n\to\infty} mesh(h_n) = 0$. In this case we say that f *is polygonally approximable*.

c) If graph-mesh(h_n) replaces $mesh(h_n)$ in b) then we obtain the notion of a *strongly polygonally approximable function*.

Now, there is a natural way in which a trajectory $\overline{x} = \{x_n\}$ can generate a sequence $H_{n,\overline{x}}$ of polygonal functions for f.

Definition 4. Let $\overline{x} = \{x_n\}$ be a trajectory. For each n let $H_{n,\overline{x}}$ be the polygonal function for f determined by the partition $\mathcal{P}_n$ consisting of the points $0, 1, x_1, x_2, \ldots, x_n$. If the trajectory $\overline{x}$ is understood, we abbreviate $H_{n,\overline{x}}$ by H_n. If the sequence $\{H_n\}$ polygonally approximates f, we say that f is *polygonally approximable with respect to* $\overline{x}$.

In this notation the proof of Theorem 4.1 in [1] shows that f is polygonally approximable if and only if there is a trajectory $\overline{x}$ such that f is polygonally approximable with respect to $\overline{x}$.

Definition 5. If f is polygonally approximable with respect to every ordering of a support set D we shall say that f is *consistently polygonally approximable with respect to* D. Finally, if there exists a support set D with respect to which f is consistently polygonally approximable, we say that f is *consistently polygonally approximable*.

2 Characterization of Consistently First-Return Recoverable and Consistently Polygonally Approximable Functions

If a function f is first-return recoverable with respect to a specific trajectory $\overline{x}$, it is easily seen that f is polygonally approximated by $\{H_{n,\overline{x}}\}$. However, it is also easy to think of function f and a trajectory $\overline{x}$ such that $\{H_{n,\overline{x}}\}$ polygonally approximates f, yet f is not first-return recoverable with respect to any rearrangement of $\overline{x}$. However, once the word "consistent" is added the situation changes dramatically:

Theorem 1. *Let D be a support set and $f : I \to \mathbb{R}$. The following are equivalent:*

(A) f is consistently polygonally appoximable with respect to D.

(B) For each $x \in I \setminus D$, $\lim_{t\to x} f|_D(t) = f(x)$.

(C) f is consistently first-return recoverable with respect to D.

(D) f is continuous at each $x \in I \setminus D$.

Proof. $(A) \Rightarrow (B)$: Assume that f is consistently polygonally appoximable with respect to D. Let $x \in I \setminus D$ and suppose that $\lim_{t\to x} f|_D(t) \neq f(x)$. Via various symmetries we may reduce the possibilities to be considered to the following two:

Case 1: $f|_D^-(x) \equiv \lim_{t\to x^-} f|_D(t) \leq f(x) < \lim_{t\to x^+} f|_D(t) \equiv f|_D^+(x)$.
Case 2: $f|_D^-(x) > f(x)$ and $f|_D^+(x) > f(x)$.

Consider Case 1. Let $5\epsilon = f|_D^+(x) - f(x)$, and set $a = f(x) + \epsilon$, $b = f(x) + 2\epsilon$, and $c = f|_D^+(x) - f(x)$. Choose a positive number δ so that for all $t \in D \cap (x - \delta, x)$, $f(t) < a$ and for all $t \in D \cap (x, x + \delta)$, $f(t) > c$. Let $\{x_n\}$ be an arbitrary enumeration of D. We shall define an enumeration $\overline{s} = \{s_n\}$ of D with respect to which f will not be polygonally approachable at x. In particular, we shall show that for infintely many n we have $H_{n,\overline{s}}(x) > b$. We shall accomplish this inductively. First, choose $s_1 \in D \cap (x - \delta, x)$ so close to x that $u \equiv \left(L_{(s_1, f(s_1))}^{(x,b)}\right)^{-1}(c) < x + \delta$. Choose $s_2 \in D \cap (x, u)$. There are finitely many terms x_n in the set $\{x_n : n \leq \overline{x}^{-1}(s_1) \text{ and } x_n < s_1\} \cup \{x_n : n \leq \overline{x}^{-1}(s_2) \text{ and } x_n > s_2\}$. List these finitely many points in any order as $s_3, s_4, \ldots, s_{n_1}$. Set $m_1 = 1$. For each $n \leq n_1$ we let H_n be the polygonal function for f determined by the points $\{0, 1, s_1, s_2, \ldots s_n\}$ we see that $H_2(x) = H_{m_1+1}(x) > b$.

Next, suppose that $j \in \mathbb{N}$, that m_j and $n_j \in \mathbb{N}$ have been defined with $m_j < n_j$, that $s_1, s_2, \ldots, s_{n_j}$ have all been selected, that $s_{m_j} < x < s_{m_j+1}$, and that $H_{m_j+1}(x) > b$. Set $m_{j+1} = n_j + 1$ and choose $s_{m_{j+1}} \in D \cap ((s_{m_j} + x)/2, x)$ so close to x that $u \equiv \left(L_{(s_{n_j+1}, f(s_{n_j+1}))}^{(x,b)}\right)^{-1}(c) < s_{m_j+1}$. Choose $s_{m_{j+1}+1} \in D \cap (x, (x + u)/2)$. There are finitely many terms x_n in the set $\{x_n : n \leq \overline{x}^{-1}(s_{m_{j+1}}) \text{ and } x_n < s_{m_{j+1}}\} \cup \{x_n : n \leq \overline{x}^{-1}(s_{m_{j+1}+1}) \text{ and } x_n > s_{m_{j+1}+1}\}$ which have not been appended to the list of s_n's in previous stages. List these finitely many points in any order as $s_{m_{j+1}+2}, s_{m_{j+1}+3}, \ldots, s_{n_{j+1}}$. For each $n \leq n_{j+1}$ we let H_n be the polygonal function for f determined by the points $\{0, 1, s_1, s_2, \ldots s_n\}$ and we see that $H_{m_{j+1}+1}(x) > b$.

In this manner, we have inductively defined a rearrangement $\{s_n\}$ of $\{x_n\}$ and have a sequence of integers m_j such that for each j

$$H_{m_j+1}(x) > b.$$

This contradicts the assumption that f is consistently polygonally approximable with respect to D.

Next, note that if Case 2 holds, then f cannot be polygonally recoverable with respect to any ordering of the support set D, since for any such ordering, it will follow that for all sufficiently large n we have $H_n(x) > d$, where $d = \min\{(f|_D^-(x) + f(x))/2, (f|_D^+(x) + f(x))/2\}$. This completes the proof that $(A) \Rightarrow (B)$.

$(B) \Rightarrow (C):$ This is obvious since if $\overline{x}$ is any ordering of D, Condition (B) guarantees that f is first-return recoverable with respect to $\overline{x}$.

$(C) \Rightarrow (D):$ Suppose that f is consistently recoverable with respect to D. Suppose there is a point $s \in I \setminus D$ at which f is discontinuous. Hence there is an $\epsilon > 0$ and a sequence $\{s_k\}$ converging to s such that $|s_{k+1} - s| < |s_k - s|$ and $|f(s_k) - f(s)| > \epsilon$ for all k. Let $\overline{x} = \{x_j\}$ be an arbitrary but fixed ordering of D. We shall inductively define a rearrangement $\{y_k\}$ of $\{x_j\}$ which fails to first-return recover f at s.

Since $f|D$ is dense in f, there is a point $y_1 \in D$ such that $|y_1 - s_1| < |s_2 - s_1|$ and $|f(y_1) - f(s_1)| < \epsilon/2$. There are finitely many, say K integers j less than $\overline{x}^{-1}(y_1)$ for which $|x_j - s| \geq |y_1 - s|$. List these in any order as $y_2, y_3, \ldots, y_{k_1}$, where $k_1 = K + 1$.

Next, suppose that for a natural number n, an integer k_n has been chosen, and $y_1, y_2, \ldots, y_{k_n}$ have been defined. There is a point $y_{k_n+1} \in D$ such that $|y_{k_n+1} - s_{n+1}| < |s_{n+2} - s_{n+1}|$ and $|f(y_{k_n+1}) - f(s_{n+1})| < \epsilon/2$. There are finitely many, say P integers j less than $\overline{x}^{-1}(y_{k_n+1})$ for which $|x_j - s| \geq |y_{k_n+1} - s|$ and $x_j \notin \{y_k : k \leq k_n + 1\}$. List these in any order as $y_{k_n+2}, y_{k_n+3}, \ldots, y_{k_{n+1}}$, where $k_{n+1} = P + k_n + 1$.

In this inductive manner we have defined a rearrangement $\{y_k\}$ of $\{x_j\}$. Furthermore, with respect to the trajectory $\{y_k\}$ we have that the first-return route to s contains the sequence $\{y_{k_n+1}\}$. For each n we have

$$|f(y_{k_n+1}) - f(s)| \geq |f(s_n) - f(s)| - |f(y_{k_n+1}) - f(s_n)| > \epsilon/2,$$

indicating that f is not first-return recoverable with respect to $\{y_k\}$ at s and completing the proof that $(B) \Rightarrow (C)$.

$(D) \Rightarrow (A):$ This is also obvious since if $\overline{x}$ is any ordering of D, Condition (D) guarantees that f is polygonally approximable with respect to $\overline{x}$. $\square$

The following is then immediate.

Corollary 1. *The following are equivalent for a function $f : I \to \mathbb{R}$:*

(I) f is consistently first-return recoverable.

(II) f is consistently polygonally approximable.

(III) f is continuous except at countably many points.

3 Stronger Consistency Notions

In addition to the notions of first-return recoverability and polygonal approximation, the stronger concepts of first-return approachabilty, first-return continuity, and strong polygonal approximation have all been found to be useful. The first-return approachable functions have been shown to be the Baire one functions having no isolated points on their graphs [2]. Both the first-return continuous functions [3] and the strongly polygonally approximable functions [1] have been shown to characterize the Baire one, Darboux functions. Thus, we would be remiss in not mentioning the consistent versions of these properties. The story will be short, however, since all three of the new classes will easily be seen to coincide with the class of continuous functions.

First, the definitions:

Definition 6. For each $x \in I$ the *first return approach to x based on* a trajectory $\overline{x} = \{x_n\}$, $\mathcal{A}(\overline{x})_x = \{u_k(x)\}$, is defined recursively via

$$u_1(x) = r(\overline{x}, (0,1) \setminus \{x\}), \text{ and } u_{k+1}(x) = r(\overline{x}, B_{|x-u_k(x)|}(x) \setminus \{x\}).$$

We say that f is *first return approachable at x with respect to the trajectory $\overline{x}$* provided

$$\lim_{\substack{u \to x \\ u \in \mathcal{A}(\overline{x})_x}} f(u) = f(x).$$

We say that f is *first return approachable with respect to $\overline{x}$* provided it is first return approachable with respect to $\overline{x}$ at each $x \in I$. Likewise, f is said to be *first return approachable* provided there exists a trajectory with respect to which f is first return approachable. Given a support set D we say that f is *consistently first-return approachable with respect to D* if f is first-return approachable with respect to each ordering of D. Finally, f is *consistently first-return approachable* if there exists a support set with respect to which it is consistently first-return approachable.

Definition 7. Let $\overline{x} = \{x_n\}$ be a trajectory. For $0 < x \leq 1$, the *left first return path to* x *based on* $\overline{x}$, $\mathcal{P}(\overline{x})^l_x = \{t_k(x)\}$, is defined recursively via

$$t_1(x) = r(\overline{x}, (0, x)), \text{ and } t_{k+1}(x) = r(\overline{x}, (t_k(x), x)).$$

For $0 \leq x < 1$, the *right first return path to* x *based on* $\overline{x}$, $\mathcal{P}(\overline{x})^r_x = \{s_k(x)\}$, is defined analogously. We say that f is *first return continuous from the left [right] at* x *with respect to the trajectory* $\overline{x}$ provided

$$\lim_{\substack{t \to x \\ t \in \mathcal{P}(\overline{x})^l_x}} f(t) = f(x) \quad \left[\lim_{\substack{s \to x \\ s \in \mathcal{P}(\overline{x})^r_x}} f(s) = f(x) \right].$$

We say that for any $x \in (0, 1)$, f is *first-return continuous at* x *with respect to the trajectory* $\overline{x}$ provided it is both left and right first-return continuous at x with respect to the trajectory $\overline{x}$. (For $x = 0$ or $x = 1$ we only require the appropriate one-sided first-return continuity.)

We say that f is *first-return continuous with respect to* $\overline{x}$ provided it is first-return continuous with respect to $\overline{x}$ at each $x \in I$. Likewise, f is said to be *first return continuous* provided there exists a trajectory with respect to which f is first return continuous. Given a support set D we say that f is *consistently first-return continuous with respect to* D if f is first-return continuous with respect to each ordering of D. Finally, f is *consistently first-return continuous* if there exists a support set with respect to which it is consistently first-return continuous.

Definition 8. Let D be a support set. We say that a function $f : I \to \mathbb{R}$ is *consistently strongly polygonally approximable with respect to* D if for every ordering $\overline{x}$ of D, the sequence $\{H_{n,\overline{x}}\}$ strongly approximates f. Likewise, we say that f is *consistently strongly polygonally approximable* provided there exists a support set D with respect to which f is consistently strongly polygonally approximable.

Theorem 2. *The following are equivalent for a function* $f : I \to \mathbb{R}$*:*

(i) f *is continuous.*

(ii) f *is consistently strongly polygonally approximable.*

(iii) f *is consistently first-return continuous.*

(iv) f *is consistently first-return approachable.*

Proof. Clearly, $(i) \Rightarrow (ii)$. Also, $(ii) \Rightarrow (iii)$ since if f is strongly polygonally approximable with respect to a trajectory $\overline{x}$, then f is clearly first-return continuous with respect to $\overline{x}$. It is also immediate that $(iii) \Rightarrow (iv)$. To see that $(iv) \Rightarrow (i)$, suppose that f is consistently first-return approachable with respect to the support set D. Suppose there is a point $s \in I$ at which f fails to be continuous. One may now read the proof of $(C) \Rightarrow (D)$ in Theorem 1, ignoring the restriction that $s \in I \setminus D$ there. The ordering $\{y_n\}$ produced there has the property that f is not first return approachable at s with respect to $\{y_n\}$. □

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