Real Analysis Exchange
Vol. 28(2), 2002/2003, pp. 573–578

Zbigniew Grande, Institute of Mathematics, Bydgoszcz Academy, Plac Weyssenhoffa 11, 85-072 Bydgoszcz, Poland. e:mail: `grande@ab-byd.edu.pl`

EXTENDING SOME FUNCTIONS TO FUNCTIONS SATISFYING CONDITION $(\mathcal{A}_3)$

Abstract

A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $(\mathcal{A}_3)$ if for each real $r > 0$, for each x and for each set $U \ni x$ belonging to the density topology there is an open interval I such that $C(f) \supset I \cap U \neq \emptyset$ and $f(U \cap I) \subset (f(x) - r, f(x) + r)$, where $C(f)$ denotes the set of all continuity points of f. In this article we investigate the sets A such that each almost continuous function may be extended from A to a function having property $(\mathcal{A}_3)$.

Let $\mathbb{R}$ be the set of all reals. Denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer density $D_u(A, x)$ $(D_l(A, x))$ of the set A at the point x as

$$\limsup_{h \to 0^+} \frac{\mu_e(A \cap [x - h, x + h])}{2h}$$

$$(\liminf_{h \to 0^+} \frac{\mu_e(A \cap [x - h, x + h])}{2h} \text{ respectively}).$$

A point x is said to be an outer density point (a density point) of a set A if $D_l(A, x) = 1$ (if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B, x) = 1$).

The family T_d of all sets A for which the implication

$$x \in A \Longrightarrow x \text{ is a density point of } A$$

is true, is a topology called the density topology ([1, 6]).

The sets $A \in T_d$ are Lebesgue measurable ([1, 6]).

In [5] O'Malley investigates the topology

$$T_{ae} = \{A \in T_d; \mu(A \setminus \text{int}(A)) = 0\},$$

Key Words: Density topology, condition $(\mathcal{A}_3)$, extension, continuity.
Mathematical Reviews subject classification: 26A05, 26A15
Received by the editors June 10, 2002

where $\mathrm{int}(A)$ denotes the interior of the set A.

Let T_e be the Euclidean topology in $\mathbb{R}$. Continuity of functions f from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$ is called approximate continuity ([1, 6]). For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ denote by $C(f)$ the set of all continuity points of f and by $A(f)$ the set of all approximate continuity points of f. Moreover let $D(f) = \mathbb{R} \setminus C(f)$ and $D_{ap}(f) = \mathbb{R} \setminus A(f)$. In [5] it is proved that a function $f : \mathbb{R} \to \mathbb{R}$ is T_{ae}-continuous (i.e. continuous as an function from $(\mathbb{R}, T_{ae})$ to $(\mathbb{R}, T_e)$) if and only if it is T_d-continuous (i.e. approximately continuous) everywhere and $\mu(D(f)) = 0$.

In [2] the following properties are investigated.

A function $f : \mathbb{R} \to \mathbb{R}$ has property $\mathcal{A}_3$ at a point x ($f \in \mathcal{A}_3(x)$) if for each positive real r and for each set $U \in T_d$ containing x there is an open interval I such that $\emptyset \neq I \cap U \subset C(f)$ and $|f(t) - f(x)| < r$ for all points $t \in I \cap U$.

A function f has property $\mathcal{A}_3$, if $f \in \mathcal{A}_3(x)$ for every point $x \in \mathbb{R}$.

A function $f : \mathbb{R} \to \mathbb{R}$ has property $\mathcal{A}_5$ if for each nonempty open set $U \in T_d$ there is an open interval I such that $\emptyset \neq I \cap U \subset C(f)$.

Evidently each function f having property $\mathcal{A}_3$ also has property $\mathcal{A}_5$.

For each function f having property $\mathcal{A}_5$ the set $D(f) = \mathbb{R} \setminus C(f)$ is nowhere dense and of Lebesgue measure 0. But the closure $\mathrm{cl}(D(f))$ for some functions f having the property $\mathcal{A}_3$ may be of positive measure. For example, if $A \subset [0,1]$ is a Cantor set of positive measure, (I_n) is a sequence of all components of the set $[0,1] \setminus A$ with $I_n \neq I_m$ for $n \neq m$ and (J_n) is a sequence of closed nondegenerate intervals $J_n \subset I_n$ with the same centers as I_n and such that $\frac{\mu(J_n)}{\mu(I_n)} < \frac{1}{n}$ for $n = 1, 2, \dots$, then the function

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x \in J_n, \quad n = 1, 2, \dots \\ f(x) = 0 & \text{otherwise on } \mathbb{R} \end{cases}$$

has property $\mathcal{A}_3$ but $\mu(\mathrm{cl}(D(f))) > 0$.

Each approximately continuous function $f : \mathbb{R} \to \mathbb{R}$ is of the first Baire class ([1]). In [4] the authors investigates the family Φ_{ap} of all nonempty sets A such that for every Baire 1 function $g : \mathbb{R} \to \mathbb{R}$ there is an approximately continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f \restriction A = g \restriction A$. They prove there that $A \in \Phi_{ap}$ if and only if $\mu(A) = 0$. In [3] I investigate the family Φ_{ae} of all nonempty sets A such that for every Baire 1 function $g : \mathbb{R} \to \mathbb{R}$ there is a T_{ae}-continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f \restriction A = g \restriction A$. I show in this article that a nonempty set $A \in \Phi_{ae}$ if and only if $\mu(\mathrm{cl}(A)) = 0$, where $\mathrm{cl}(A)$ denotes the closure of the set A.

In this paper I investigate the families $\Phi_{\mathcal{A}_i}$ $(i = 3, 5)$ of all nonempty sets A such that for every almost everywhere continuous function $g : \mathbb{R} \to \mathbb{R}$ there is a function $f : \mathbb{R} \to \mathbb{R}$ having property $(\mathcal{A}_i)$ such that $f \restriction A = g \restriction A$.

Theorem 1. *The equality $\Phi_{\mathcal{A}_3} = \Phi_{\mathcal{A}_5}$ is true. Moreover, a nonempty set $A \subset \mathbb{R}$ belongs to $\Phi_{\mathcal{A}_3}$ if and only if $\mu(\mathrm{cl}(A)) = 0$.*

PROOF. Since the property $\mathcal{A}_3$ implies the property $\mathcal{A}_5$, we have $\Phi_{\mathcal{A}_3} \subset \Phi_{\mathcal{A}_5}$. To show $\Phi_{\mathcal{A}_5} \subset \Phi_{\mathcal{A}_3}$, it will be first be shown that the sets $A \in \Phi_{\mathcal{A}_5}$ satisfy $\mu(\mathrm{cl}(A)) = 0$. Let a set $A \subset \mathbb{R}$ be such that $\mu(\mathrm{cl}(A)) > 0$. Then the set

$$B = \{x \in \mathrm{cl}(A); D_l(\mathrm{cl}(A), x) = 1\} \in T_d$$

and is nonempty. Let $E = \{a_1, \dots, a_n, \dots\} \subset B$ be a countable set dense in B. Let

$$g(x) = \begin{cases} 0 & \text{for } x \le \inf E \\ \sum_{a_n < x} \frac{1}{2^n} & \text{for } x > \inf E. \end{cases}$$

Then $D(g) = E$ and $\mu(D(g)) = 0$, so g is almost everywhere continuous. Now assume that there is a function $f : \mathbb{R} \to \mathbb{R}$ having property $\mathcal{A}_5$ and such that $f \restriction A = g \restriction A$. Since f has property $\mathcal{A}_5$ and the set $B \neq \emptyset$ belongs to T_d, there is an open interval I such that

$$\emptyset \neq I \cap B \subset C(f). \qquad (*)$$

But the set E is dense in B; so there is a positive integer k with $a_k \in I \cap B$. As a density point of the set $\mathrm{cl}(A)$ the point a_k is a bilateral accumulation point of A. Moreover $f \restriction A = g \restriction A$ and at the point a_k we have

$$g(a_k-) = \lim_{x \to a_k^-} g(x) < g(a_k+) = \lim_{x \to a_k^+} g(x).$$

So $a_k \in D(f)$ and we obtain a contradiction to (*), which shows that A is not in the family $\Phi_{\mathcal{A}_5}$. So for each set $A \in \Phi_{\mathcal{A}_5}$ we have $\mu(\mathrm{cl}(A)) = 0$.

Now we suppose that A is a nonempty set such that $\mathrm{cl}(A)$ is a compact set of measure zero and that $g : \mathbb{R} \to \mathbb{R}$ is an almost everywhere continuous function. In the first step observe that there are pairwise disjoint open intervals $I_{1,1}, I_{1,2}, \dots, I_{1,i(1)}$ such that

$$U_1 = \bigcup_{x \in A} (x-1, x+1) = I_{1,1} \cup \ldots \cup I_{1,i(1)},$$

and $A \cap I_{1,j} \neq \emptyset$ for $j \le i(1)$. There are also pairwise disjoint nondegenerate closed intervals $L_{1,1}, \dots, L_{1,k(1)} \subset U_1 \setminus A$ with the endpoints belonging to $C(g)$ such that for every positive integer $j \le i(1)$

$$\frac{\mu(I_{1,j} \cap \bigcup_{i \le k(1)} L_{1,i})}{\mu(I_{1,j})} > \frac{1}{2}.$$

In the second step put

$$r_2 = \frac{\inf\{|x-y|; x \in A,\ y \in \bigcup_{i \le k(1)} L_{1,i}\}}{2},$$

and observe that there are pairwise disjoint open intervals $I_{2,1}, I_{2,2}, \dots, I_{2,i(2)}$ such that

$$U_2 = \bigcup_{x \in A} (x - r_2, x + r_2) = I_{2,1} \cup \dots \cup I_{2,i(2)},$$

and $I_{2,k} \cap A \neq \emptyset$ for $k \le i(2)$. Next we find pairwise disjoint nondegenerate closed intervals $L_{2,1}, \dots, L_{2,k(2)} \subset U_2 \setminus A$ with the endpoints belonging to $C(g)$ such that for every positive integer $j \le i(2)$

$$\frac{\mu(I_{2,j} \cap \bigcup_{i \le k(2)} L_{2,i})}{\mu(I_{2,j})} > 1 - \frac{1}{2^2}.$$

In general in the n^{th} step $(n > 2)$ we define the positive real

$$r_n = \frac{\inf\{|x-y|; x \in A,\ y \in \bigcup_{i \le k(n-1)} L_{n-1,i}\}}{2},$$

and pairwise disjoint open intervals $I_{n,1}, I_{n,2}, \dots, I_{n,i(n)}$ such that

$$U_n = \bigcup_{x \in A} (x - r_n, x + r_n) = I_{n,1}, \cup \dots \cup I_{n,i(n)},$$

and $I_{n,j} \cap A \neq \emptyset$ for $j \le i(n)$. Next we find pairwise disjoint nondegenerate closed intervals $L_{n,1}, \dots, L_{n,k(n)} \subset U_n \setminus A$ with the endpoints belonging to $C(g)$ such that for each positive integer $j \le i(n)$

$$\frac{\mu(I_{n,j} \cap \bigcup_{i \le k(n)} L_{n,i})}{\mu(I_{n,j})} > 1 - \frac{1}{2^n} \tag{**}$$

Let $N_1, N_2, \dots, N_m, \dots$ be a sequence of pairwise disjoints infinite subsets of positive integers and let $N_k = \{n_{k,1}, n_{k,2}, \dots\}$, where $n_{k,i} < n_{k,j}$ for $i < j$. For $i = 1, 2, \dots$ let

$$(K_{i,j})_j = (L_{n_{i,1},1}, \dots, L_{n_{i,1},k(n_{i,1})}, L_{n_{i,2},1}, \dots, L_{n_{i,2},k(n_{i,2})}, \dots).$$

Then by (**) the family of pairwise disjoint closed intervals $\{K_{i,j}; i, j = 1, 2, \dots\}$ contained in $\mathbb{R} \setminus A$ is such that for each point $x \in A$ and for each positive integer i, $D_u(\bigcup_{j=1}^{\infty} K_{i,j}, x) = 1$.

In the interiors $\mathrm{int}(K_{i,j})$ we find closed intervals $J_{i,j} \subset \mathrm{int}(K_{i,j})$ such that for each point $x \in A$ and for each integer $i = 1, 2, \dots,\ D_u(\bigcup_{j=1}^{\infty} J_{i,j}, x) = 1$. Put

$$h(x) = \begin{cases} g(x) & \text{for } x \in \mathrm{cl}(A) \\ g(\inf A) & \text{for } x \le \inf A \\ g(\sup A) & \text{for } x \ge \sup A \\ \text{linear} & \text{on the components of } [\inf A, \sup A] \setminus \mathrm{cl}(A). \end{cases}$$

Order the rationals as a (w_i) and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} w_i & \text{for } x \in J_{i,j}, \quad i, j = 1, 2, \dots \\ h(x) & \text{for } x \in \mathbb{R} \setminus \bigcup_{i,j=1}^{\infty} \mathrm{int}(K_{i,j}), \end{cases}$$

and let f be linear on all components of the sets $K_{i,j} \setminus \mathrm{int}(J_{i,j})$, $i, j = 1, 2, \dots$. Then $f = h = g$ on $\mathrm{cl}(A)$. Moreover for each point $x \in \mathbb{R} \setminus \mathrm{cl}(A)$ there is an open interval $J \ni x$ disjoint from $\mathrm{cl}(A)$ and such that the set of all pairs (i, j) for which $J \cap K_{i,j} \ne \emptyset$ is empty or finite. So f is continuous on the complement $\mathbb{R} \setminus \mathrm{cl}(A)$ and, consequently $f \in \mathcal{A}_3(x)$ for each point $x \in \mathbb{R} \setminus \mathrm{cl}(A)$.

We will prove that f also has property $\mathcal{A}_3$ at all $x \in \mathrm{cl}(A)$. For this, fix a positive real r, a point $x \in \mathrm{cl}(A)$ and a set $U \in T_d$ containing x. There is a positive integer m with $|f(x) - w_m| = |g(x) - w_m| < r$. Since $D_l(U, x) = 1$ and $D_u(\bigcup_{j=1}^{\infty} J_{m,j}, x) = 1$, there is an open interval $I \subset \bigcup_{j=1}^{\infty} J_{m,j}$ such that $I \cap U \ne \emptyset$. For all points $u \in I \cap U$ we have $|f(u) - f(x)| = |w_m - f(x)| < r$; so $f \in \mathcal{A}_3(x)$, and consequently $A \in \Phi_{\mathcal{A}_3} \subset \Phi_{\mathcal{A}_5}$ in this case.

Up to now we have supposed that $\mathrm{cl}(A)$ is bounded. Now we consider the general case. We have $\mathbb{R} = \bigcup_{k=-\infty}^{\infty} [x_k, x_{k+1}]$, where $x_k \in \mathbb{R} \setminus \mathrm{cl}(A)$ and

$$-\infty \leftarrow x_{-k} < x_{-k+1} < \cdots < x_0 < \cdots x_k < x_{k+1} \to \infty.$$

For every function $g_k = g \restriction [x_k, x_{k+1}]$ there is a function $f_k : [x_k, x_{k+1}] \to \mathbb{R}$ having property $\mathcal{A}_3$ such that $g_k \restriction (A \cap [x_k, x_{k+1}]) = f \restriction (A \cap [x_k, x_{k+1}])$. For each $k = 0, 1, -1, 2, -2, \dots$ let

$$r_k = \frac{\min\{|x_k - t|; t \in \mathrm{cl}(A)\}}{3},$$

and $J_k = (x_k - r_k, x_k + r_k)$. Putting

$$f(x) = \begin{cases} f_k(x) & \text{for } x \in [x_k, x_{k+1}] \setminus (J_k \cup J_{k+1}), \quad k = 0, 1, -1, 2, -2, \dots \\ \text{linear} & \text{on } \mathrm{cl}(J_k), \quad k = 0, 1, -1, 2, -2, \dots \end{cases}$$

we obtain a function f having property $\mathcal{A}_3$ such that $f \restriction A = g \restriction A$. So, $A \in \Phi_{\mathcal{A}_3} \subset \Phi_{\mathcal{A}_5}$ and the proof is finished. $\square$

Corollary 1. *Let $A \subset \mathbb{R}$ be a set. The following conditions are equivalent:*

(1) $\mu(\mathrm{cl}(A)) = 0$,

(2) $A \in \Phi_{\mathcal{A}_3}$,

(3) for each function $g : \mathbb{R} \to \mathbb{R}$ there is a function $f : \mathbb{R} \to \mathbb{R}$ having property $\mathcal{A}_3$ and such that $f \restriction A = g \restriction A$.

Proof. The equivalence of (1) and (2) follows from Theorem 1. Evidently (3) implies (2). For the proof of the implication (2) $\Longrightarrow$ (3) it suffices to observe that for each function $g : \mathbb{R} \to \mathbb{R}$ and a set A with $\mu(\mathrm{cl}(A)) = 0$ the function

$$h(x) = \begin{cases} g(x) & \text{on } \mathrm{cl}(A) \\ 0 & \text{on } \mathbb{R} \setminus \mathrm{cl}(A \end{cases}$$

is almost everywhere continuous and, consequently there is a function f having property $\mathcal{A}_3$ with $f \restriction A = h \restriction A = g \restriction A$. □

References

[1] A. M. Bruckner, *Differentiation of real functions*, Lectures Notes in Math. 659, Springer-Verlag, Berlin 1978.

[2] Z. Grande, *On some special notions of approximate quasi-continuity*, Real Anal. Exch., **24** (1998–99), 171–184.

[3] Z. Grande, *Sur le prolongement des fonctions*, Acta Math. Acad. Sci. Hungar., **34** (1979), 43–45.

[4] M. Laczkovich, G. Petruska, *Baire 1 functions, approximately continuous functions and derivatives*, Acta Math. Acad. Sci. Hungar., **25** (1974), 189–212.

[5] R. J. O'Malley, *Approximately differentiable functions. The r topology*, Pacific J. Math., **72** (1977), 207–222.

[6] F. D. Tall, *The density topology*, Pacific J. Math., **62** (1976), 275–284.