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MULTIFRACTAL VARIATION MEASURES AND MULTIFRACTAL DENSITY THEOREMS

Abstract

In this paper we show that the multifractal Hausdorff measure and multifractal packing measure introduced by Olsen and Peyriere can be expressed as Henstock-Thomson "variation" measures. As an application we prove a density theorem for these two measures that extends results by Edgar and is more refined than those found in [Ol1].

1 Introduction and Statement of Results

In several recent papers Olsen [Ol1, Ol2, Ol3] and Peyriére [Pey] have proposed developing a multifractal geometry for measures which parallels the well-known fractal geometry for sets. At the heart of this suggestion are two measures which generalize the Hausdorff and packing measures. These measures have subsequently been investigated further by a large number of authors, including [BNB, BNBH, Co, Da1, Da2, FM, HRS, HY, Ol2, Ol3, O'N1, O'N2, Sc]. In this paper we show that the multifractal Hausdorff measure and multifractal packing measure can be expressed as Henstock-Thomson "variation" measures (see [He]and [Th]); see Theorem 1 and Theorem 2. This analysis follows Edgar's treatment [Ed1,Ed2] of the Hausdorff and packing measures as Henstock-Thomson "variation" measures, (cf. also [LL]).

In addition, we provide the following application of this result. Using the characterization of the multifractal Hausdorff measure and multifractal packing measure established in Theorem 1 and Theorem 2, we prove a density theorem for these measures which extends density theorems obtained by Edgar [Ed1, Ed2] and is more refined than those found in [Ol1]; see Theorem 3.

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1.1 Multifractal Hausdorff Measures and Multifractal Packing Measures.

We start by introducing the multifractal Hausdorff and packing measures. Let $E \subseteq \mathbb{R}^d$ and $\delta > 0$. A countable family $(B(x_i, r_i))_i$ of closed balls in $\mathbb{R}^d$ is called a centered δ-covering of E if $E \subseteq \cup_i B(x_i, r_i)$, $x_i \in E$ and $0 < r_i < \delta$ for all i. The family $(B(x_i, r_i))_i$ is called a centered δ-packing of E if $x_i \in E$, $0 < r_i < \delta$ and $B(x_i, r_i) \cap B(x_j, r_j) = \varnothing$ for all $i \neq j$. For $E \subseteq X$, $q, t \in \mathbb{R}$ and $\delta > 0$ write

$$\begin{aligned}
\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(E) &= \inf \Big\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t \Big| (B(x_i, r_i))_i \\
&\qquad \text{is a centered } \delta\text{-covering of } E \Big\}, E \neq \varnothing \\
\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(\varnothing) &= 0 \\
\overline{\mathcal{H}}^{q,t}_{\mu}(E) &= \sup_{\delta > 0} \overline{\mathcal{H}}^{q,t}_{\mu,\delta}(E) \\
\mathcal{H}^{q,t}_{\mu}(E) &= \sup_{F \subseteq E} \overline{\mathcal{H}}^{q,t}_{\mu}(F),
\end{aligned}$$

and

$$\begin{aligned}
\overline{\mathcal{P}}^{q,t}_{\mu,\delta}(E) &= \sup \Big\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t \Big| (B(x_i, r_i))_i \\
&\qquad \text{is a centered } \delta\text{-packing of } E \Big\}, E \neq \varnothing \\
\overline{\mathcal{P}}^{q,t}_{\mu,\delta}(\varnothing) &= 0 \\
\overline{\mathcal{P}}^{q,t}_{\mu}(E) &= \inf_{\delta > 0} \overline{\mathcal{P}}^{q,t}_{\mu,\delta}(E) \\
\mathcal{P}^{q,t}_{\mu}(E) &= \inf_{E \subseteq \cup_i E_i} \sum_i \overline{\mathcal{P}}^{q,t}_{\mu}(E_i).
\end{aligned}$$

It follows from [Ol1] that $\mathcal{H}^{q,t}_{\mu}$ and $\mathcal{P}^{q,t}_{\mu}$ are measures on the family of Borel subsets of X. The measure $\mathcal{H}^{q,t}_{\mu}$ is of course a multifractal generalization of the centered Hausdorff measure, whereas $\mathcal{P}^{q,t}_{\mu}$ is a multifractal generalization of the packing measure. In fact, it is easily seen that if $t \geq 0$, then $2^{-t}\mathcal{H}^{0,t}_{\mu} \leq \mathcal{H}^t \leq \mathcal{H}^{0,t}_{\mu}$ and $\mathcal{P}^t = \mathcal{P}^{0,t}_{\mu}$, where $\mathcal{H}^t$ denotes the t-dimensional Hausdorff measure and $\mathcal{P}^t$ denotes the t-dimensional packing measure. The reader is referred to [BNB, BNBH, Co, Da1, Da2, FM, HRS, HY, Ol2, Ol3, O'N1, O'N2, Sc] for detailed discussions of the application of these measures in multifractal analysis.

1.2 Fine Variation.

We now consider Thomson's fine variation [Th]. The variations may be defined for a general so-called derivation basis. However, we will use only the centered ball basis.

A function $h : \mathbb{R}^d \times (0,\infty) \to [0,\infty)$ is called a variation function.

A countable family $(B(x_i, r_i))_i$ of closed balls in $\mathbb{R}^d$ is called a packing if $B(x_i, r_i) \cap B(x_j, r_j) = \varnothing$ for all $i \neq j$. A fine cover (or Vitali cover) of a subset $E \subseteq \mathbb{R}^d$ is a (possibly uncountable) family $(B(x_\lambda, r_\lambda))_{\lambda\in\Lambda}$ of closed balls such that $x_\lambda \in E$ for all $\lambda \in \Lambda$, $E \subseteq \cup_{\lambda\in\Lambda} B(x_\lambda, r_\lambda)$, and for each $x \in E$ and each $\delta > 0$, there is $\lambda \in \Lambda$ with $x = x_\lambda$ and $r_\lambda < \delta$.

Let h be a variation function. For a fine cover $\mathcal{V}$ of a subset E of $\mathbb{R}^d$ we write

$$H_{\mathcal{V}}(h) = \sup\Big\{ \sum_i h(x_i, r_i) \Big| (B(x_i, r_i))_i \subseteq \mathcal{V} \text{ is a packing}\Big\}.$$

The fine variation of h is defined by

$$H(h) = \inf\{H_{\mathcal{V}}(h) \mid \mathcal{V} \text{ is a fine cover of } \mathbb{R}^d\}.$$

If the variation function h is of the special form $h(x,r) = f(x)\mu(B(x,r))^q(2r)^t$ for some positive function $f : \mathbb{R}^d \to [0,\infty)$, $q,t \in \mathbb{R}$ and a Borel probability measure μ on $\mathbb{R}^d$ we will write $H^{q,t}_{\mu,\mathcal{V}}(f) = H_{\mathcal{V}}(h)$ and $H^{q,t}_{\mu}(f) = H(h)$.

Before we can state the first main result we need to introduce the notion of a doubling measure. A Borel probability measure on $\mathbb{R}^d$ is called a doubling measure if

$$\limsup_{r\searrow 0} \sup_x \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty.$$

It is known (cf. [Ol1,PW]) that self-similar measures and self-conformal measures with totally disconnected supports are doubling measures.

Next is the first main result. It states that the fine variation measure defined by the variation function $h(x,r) = 1_E(x)\mu(B(x,r))^q(2r)^t$ for $E \subseteq \mathbb{R}^d$, $q,t \in \mathbb{R}$ and a Borel probability measure μ on $\mathbb{R}^d$ coincides with the multifractal Hausdorff measure $\mathcal{H}^{q,t}_\mu$; here 1_E denotes the indicator function on E.

Theorem 1. *Let $q,t \in \mathbb{R}$ and let μ be a Borel probability measure μ on $\mathbb{R}^d$. Assume either $q \leq 0$, or $0 < q$ and μ is a doubling measure. Then for every set $E \subseteq \mathbb{R}^d$ we have $H^{q,t}_\mu(1_E) = \mathcal{H}^{q,t}_\mu(E)$.*

1.3 Full Variation.

Next we now consider Thomson's full variation [Th].

A strictly positive function $\Phi : E \to (0,\infty)$ defined on a subset E of $\mathbb{R}^d$ is called a gauge function on E. Given a gauge function on E, a countable family $(B(x_i,r_i))_i$ of closed balls in $\mathbb{R}^d$ is called a centered Φ-packing of E if $x_i \in E$, $r_i < \Phi(x_i)$ and $B(x_i,r_i) \cap B(x_j,r_j) = \varnothing$ for all $i \neq j$.

Let h be a variation function. For a gauge function Φ on a subset E of $\mathbb{R}^d$ we write

$$P_\Phi(h) = \sup\Big\{ \sum_i h(x_i,r_i) \Big| (B(x_i,r_i))_i \text{ is a centered } \Phi\text{-packing of } E \Big\}.$$

The full variation of h is defined by

$$P(h) = \inf\{ P_\Phi(h) \big| \Phi \text{ is a gauge function on } \mathbb{R}^d \}.$$

As before, if the variation function h is of the special form $h(x,r) = f(x)\mu(B(x,r))^q(2r)^t$ for some positive function $f : \mathbb{R}^d \to [0,\infty)$, $q,t \in \mathbb{R}$ and a Borel probability measure μ on $\mathbb{R}^d$ we will write, $P^{q,t}_{\mu,\Phi}(f) = P_\Phi(h)$ and $P^{q,t}_\mu(f) = P(h)$.

Next is the second main result. It states that the full variation measure defined by the variation function $h(x,r) = 1_E(x)\mu(B(x,r))^q(2r)^t$ for $q,t \in \mathbb{R}$ and a Borel probability measure μ on $\mathbb{R}^d$ coincides with the multifractal packing measure $\mathcal{P}^{q,t}_\mu$.

Theorem 2. *Let $q,t \in \mathbb{R}$ and let μ be a Borel probability measure μ on $\mathbb{R}^d$. Then for every set $E \subseteq \mathbb{R}^d$ we have $P^{q,t}_\mu(1_E) = \mathcal{P}^{q,t}_\mu(E)$.*

1.4 Density Theorems.

As an application of Theorem 1 and Theorem 2 we prove a density theorem for the multifractal measures $\mathcal{H}^{q,t}_\mu$ and $\mathcal{P}^{q,t}_\mu$ that is more refined than those found in [Ol1]

Given two locally finite Borel measures μ and ν on $\mathbb{R}^d$, $q,t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, we define the upper and lower multifractal (q,t)-density of ν at x with respect to μ by

$$\begin{aligned} \overline{d}^{q,t}_\mu(x,\nu) &= \limsup_{r\searrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))^q(2r)^t} \quad \text{and} \\ \underline{d}^{q,t}_\mu(x,\nu) &= \liminf_{r\searrow 0} \frac{\nu(B(x,r))}{\mu(B(x,r))^q(2r)^t}, \end{aligned} \tag{1.1}$$

respectively. In [Ol1]it is shown that if E is a Borel subset of the support of μ, then the following results hold. If $\mathcal{H}_\mu^{q,t}(E) < \infty$ and μ is a doubling measure, then

$$\mathcal{H}_\mu^{q,t}(E) \inf_{x\in E} \overline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le \mathcal{H}_\mu^{q,t}(E) \sup_{x\in E} \overline{d}_\mu^{q,t}(x,\nu). \tag{1.2}$$

If $\mathcal{P}_\mu^{q,t}(E) < \infty$, then

$$\mathcal{P}_\mu^{q,t}(E) \inf_{x\in E} \underline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le \mathcal{P}_\mu^{q,t}(E) \sup_{x\in E} \underline{d}_\mu^{q,t}(x,\nu). \tag{1.3}$$

Using the characterization of the multifractal measures $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ in terms of variation measures, we improve the density results in (1.2) and (1.3).

Theorem 3. *Let μ and ν be a Borel probability measures on $\mathbb{R}^d$, $q,t \in \mathbb{R}$ and $E \subseteq \mathbb{R}^d$ be a Borel set.*

(1) Assume either $q \le 0$, or $0 < q$ and μ is a doubling measure. We have

$$\nu(E) \ge \int_E \overline{d}_\mu^{q,t}(x,\nu)\, d\mathcal{H}_\mu^{q,t}(x).$$

(2) Assume either $q \le 0$, or $0 < q$ and μ is a doubling measure. If in addition, $\mathcal{H}_\mu^{q,t}(E) < \infty$ and $\overline{d}_\mu^{q,t}(x,\nu) < \infty$ for all $x \in E$, then

$$\nu(E) = \int_E \overline{d}_\mu^{q,t}(x,\nu)\, d\mathcal{H}_\mu^{q,t}(x).$$

(3) We have

$$\nu(E) \ge \int_E \underline{d}_\mu^{q,t}(x,\nu)\, d\mathcal{P}_\mu^{q,t}(x).$$

(4) If in addition, $\mathcal{P}_\mu^{q,t}(E) < \infty$ and $\underline{d}_\mu^{q,t}(x,\nu) < \infty$ for all $x \in E$, then

$$\nu(E) = \int_E \underline{d}_\mu^{q,t}(x,\nu)\, d\mathcal{P}_\mu^{q,t}(x).$$

2 Proofs of Theorem 1 and Theorem 2

2.1 The Proof of Theorem 1

Recall that we denote the indicator function on a subset E of $\mathbb{R}^d$ by 1_E. It is easily seen that if $q,t \in \mathbb{R}$ and μ is a Borel probability measure on $\mathbb{R}^d$, then

$$H_\mu^{q,t}(f1_E) = \inf\{H_{\mu,\mathcal{V}}^{q,t}(f1_E) \mid \mathcal{V} \text{ is a fine cover of } E\},$$
$$P_\mu^{q,t}(f1_E) = \sup\{P_{\mu,\Phi}^{q,t}(f1_E) \mid \Phi \text{ is a gauge function on } E\},$$

for all positive functions $f : \mathbb{R}^d \to [0,\infty)$ and all $E \subseteq \mathbb{R}^d$; this result will be used frequently below.

Theorem 2.1. *Let h be a variation function. For a set $E \subseteq \mathbb{R}^d$, we define the variation function $h\bullet 1_E : \mathbb{R}^d \times (0,\infty) \to [0,\infty)$ by $(h\bullet 1_E)(x,r) = h(x,r)1_E(x)$. Then the set functions*

$$E \to H(h\bullet 1_E),\ E \to P(h\bullet 1_E) \text{ for } E \subseteq \mathbb{R}^d$$

are metric outer measures. In particular, it follows that if $q,t \in \mathbb{R}$ and μ is a Borel probability measure on $\mathbb{R}^d$, then the set functions

$$E \to H^{q,t}_\mu(f1_E),\ E \to P^{q,t}_\mu(f1_E) \text{ for } E \subseteq \mathbb{R}^d$$

are metric outer measures for all positive functions $f : \mathbb{R}^d \to [0,\infty)$.

Proof. This follows from [Th]. □

Next we state a version of Vitali's Covering Theorem which we will use.

Theorem 2.2. *Let μ be a Borel measure on $\mathbb{R}^d$ and let μ^* denote the exterior measure associated with μ; i.e.,*

$$\mu^*(E) = \inf\{\mu(A) \mid E \subseteq A, A \text{ is Caratheodory measurable}\}$$

for all $E \subseteq \mathbb{R}^d$. Let $E \subseteq \mathbb{R}^d$ and $\mathcal{V}$ be a fine cover of E. Then there exists a countable packing $\Pi \subseteq \mathcal{V}$ such that $\mu^*\Big(E \setminus \bigcup_{B\in\Pi} B\Big) = 0$.

Proof. It follows from Theorem 3.2 and Remark (3) in [deG] that there exists a countable subfamily Π of $\mathcal{V}$ such that $\mu^*\Big(E \setminus \cup_{B\in\Pi} B\Big) = 0$. Furthermore, the proof of Theorem 3.2 in [deG] shows that Π can be chosen to consist of pairwise disjoint sets. □

We now turn to the proof of Theorem 1.

Lemma 2.3. *Let $q,t \in \mathbb{R}$ and let μ be a Borel probability measure μ on $\mathbb{R}^d$. Fix $E \subseteq \mathbb{R}^d$. If $\mathcal{H}^{q,t}_\mu(E) = 0$, then $H^{q,t}_\mu(1_E) = 0$.*

Proof. Let $\varepsilon > 0$. For each positive integer n we have $\overline{\mathcal{H}}^{q,t}_{\mu,\frac{1}{n}}(E) = 0$, and we can thus find a centered $\frac{1}{n}$-covering $(B(x_{ni}, r_{ni}))_i$ of E such that

$$\sum_i \mu(B(x_{ni}, r_{ni}))^q (2r_{ni})^t \le \frac{\varepsilon}{2^n}.$$

For each i and n write $\mathcal{V}_{ni} = \{B(y, r_{ni}) \mid y \in E\,, |y - x_{ni}| \le r_{ni}\}$, and put $\mathcal{V} = \cup_{n,i}\mathcal{V}_{ni}$. Then $\mathcal{V}$ is a fine cover of E. Let $\Pi \subseteq \mathcal{V}$ be a packing. Since all elements of $\mathcal{V}_{ni}$ contain x_{ni}, there is at most one element of $\mathcal{V}_{ni}$ in Π. Hence,

$$\sum_{B(x,r)\in\Pi} \mu(B(x,r))^q(2r)^t \le \sum_n\sum_i \mu(B(x_{ni}, r_{ni}))^q(2r_{ni})^t \le \sum_n \frac{\varepsilon}{2^n} = \varepsilon.$$

Taking supremum over all packings $\Pi \subseteq \mathcal{V}$ gives $H^{q,t}_{\mu,\mathcal{V}}(1_E) \le \varepsilon$. Finally, letting $\varepsilon \searrow 0$ gives $H^{q,t}_{\mu}(1_E) \le H^{q,t}_{\mu,\mathcal{V}}(1_E) = 0$. □

PROOF OF THEOREM 1 "$\ge$" First we verify that $\mathcal{H}^{q,t}_{\mu}(E) \le H^{q,t}_{\mu}(1_E)$. Observe that if μ is a doubling measure, then there exists $c > 0$ such that

$$\frac{\mu(B(z,2r))}{\mu(B(y,r))} \le c \text{ for all } y, z \in \mathbb{R}^d \text{ and } r > 0 \text{ with } z \in B(y,r).$$

Let $F \subseteq E$ and $\delta > 0$. We now claim that

$$\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(F) \le H^{q,t}_{\mu}(1_F) \tag{2.1}$$

We may clearly assume that $H^{q,t}_{\mu}(1_F) < \infty$. We can thus choose a fine cover $\mathcal{V}$ of F such that $H^{q,t}_{\mu,\mathcal{V}}(1_F) < \infty$. Applying Theorem 2.2 to the fine cover $\{B(x,r) \in \mathcal{V} \mid r < \frac{\delta}{2}\}$ of F, we can conclude that there exists a countable centered packing $(B(x_i, r_i))_i \subseteq \mathcal{V}$ of F such that $r_i < \frac{\delta}{2}$ for each i, and $(\mathcal{H}^{q,t}_{\mu})^*(F \setminus \cup_i B(x_i, r_i)) = 0$ where $(\mathcal{H}^{q,t}_{\mu})^*$ denotes the exterior measure associated with $\mathcal{H}^{q,t}_{\mu}$. Fix $\varepsilon > 0$. We may thus choose a Caratheodory measurable set A such that $F \setminus \cup_i B(x_i, r_i) \subseteq A$ and $\overline{\mathcal{H}}^{q,t}_{\mu,\frac{\delta}{2}}(A) \le \mathcal{H}^{q,t}_{\mu}(A) \le \varepsilon$. Also, we can choose a centered $\frac{\delta}{2}$-covering $(B(y_i, s_i))_i$ of A satisfying

$$\sum_i \mu(B(y_i, s_i))^q(2s_i)^t \le \overline{\mathcal{H}}^{q,t}_{\mu,\frac{\delta}{2}}(A) + \varepsilon.$$

For each i with $B(y_i, s_i) \cap F \ne \varnothing$ we may choose $z_i \in B(y_i, s_i) \cap F$. Now, since $(B(x_i, r_i))_i \cup (B(z_i, 2s_i))_i$ is a centered δ-covering of F, we have that

$$\begin{aligned}\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(F) &\le \sum_i \mu(B(x_i, r_i))^q(2r_i)^t + \sum_i \mu(B(z_i, 2s_i))^q(2\cdot 2s_{ni})^t \\ &\le \begin{cases} H^{q,t}_{\mu,\mathcal{V}}(1_F) + 2^t \sum_i \mu(B(y_i, s_i))^q(2s_i)^t & \text{for } q \le 0; \\ H^{q,t}_{\mu,\mathcal{V}}(1_F) + 2^t \sum_i c^q\mu(B(y_i, s_i))^q(2s_i)^t & \text{for } 0 < q; \end{cases}\end{aligned}$$

$$\leq \begin{cases} H^{q,t}_{\mu,\mathcal{V}}(1_F) + 2^t\left(\overline{\mathcal{H}}^{q,t}_{\mu,\frac{\delta}{2}}(A) + \varepsilon\right) & \text{for } q \leq 0; \\ H^{q,t}_{\mu,\mathcal{V}}(1_F) + 2^t c^q\left(\overline{\mathcal{H}}^{q,t}_{\mu,\frac{\delta}{2}}(A) + \varepsilon\right) & \text{for } 0 < q; \end{cases}$$
$$\leq H^{q,t}_{\mu,\mathcal{V}}(1_F) + 2^{t+1}\max(1, c^q)\varepsilon.$$

Letting $\varepsilon \searrow 0$ we obtain $\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(F) \leq H^{q,t}_{\mu,\mathcal{V}}(1_F)$. Next, taking infimum over all $\mathcal{V}$ gives (2.1). Letting $\delta \searrow 0$ in (2.1) gives $\overline{\mathcal{H}}^{q,t}_{\mu}(F) \leq H^{q,t}_{\mu}(1_F) \leq H^{q,t}_{\mu}(1_E)$. The result now follows by taking supremum over all subsets F of E.

"$\leq$" Next we verify that $H^{q,t}_{\mu}(1_E) \leq \mathcal{H}^{q,t}_{\mu}(E)$. We may clearly assume that $\mathcal{H}^{q,t}_{\mu}(E) < \infty$. Fix $a > 1$ and let ν denote the restriction of $\mathcal{H}^{q,t}_{\mu}(E)$ to E; i.e., $\nu(A) = \mathcal{H}^{q,t}_{\mu}(A \cap E)$ for all $A \subseteq \mathbb{R}^d$. Write

$$F = \{x \in E \mid \overline{d}^{q,t}_{\mu}(x,\nu) \leq a^{-3}\} \text{ and } G = \{x \in E \mid \overline{d}^{q,t}_{\mu}(x,\nu) > a^{-3}\};$$

recall that the density $\overline{d}^{q,t}_{\mu}(x,\nu)$ is defined in (1.1).

We first consider the set F. We will prove that

$$H^{q,t}_{\mu}(1_F) = 0. \tag{2.2}$$

For $\in \mathbb{N}$, set

$$F_n = \left\{x \in F \middle| \frac{\nu(B(x,r))}{\mu(B(x,r))^q(2r)^t} < a^{-2} \text{ for all } r < \frac{1}{n}\right\}.$$

Fix $n \in \mathbb{N}$. We will now show that $\mathcal{H}^{q,t}_{\mu}(F_n) = 0$. For each centered $\frac{1}{n}$-covering $(B(x_i,r_i))_i$ of F_n we have

$$\sum_i \mu(B(x_i,r_i))^q(2r)^t \geq a^2\sum_i \nu(B(x_i,r_i) \geq a^2\nu\big(\bigcup_i B(x_i,r_i)\big)$$
$$\geq a^2\nu(F_n) = a^2\mathcal{H}^{q,t}_{\mu}(F_n).$$

Hence, $\overline{\mathcal{H}}^{q,t}_{\mu,\frac{1}{n}}(F_n) \geq a^2\mathcal{H}^{q,t}_{\mu}(F_n)$, which implies that $\mathcal{H}^{q,t}_{\mu}(F_n) \geq \overline{\mathcal{H}}^{q,t}_{\mu}(F_n) \geq \overline{\mathcal{H}}^{q,t}_{\mu,\frac{1}{n}}(F_n) \geq a^2\mathcal{H}^{q,t}_{\mu}(F_n)$. Now, since $a > 1$ and $\mathcal{H}^{q,t}_{\mu}(F_n) \leq \mathcal{H}^{q,t}_{\mu}(E) < \infty$, we have $\mathcal{H}^{q,t}_{\mu}(F_n) = 0$. Finally, since $F_n \nearrow F$, this implies that $\mathcal{H}^{q,t}_{\mu}(F) = 0$, and Lemma 2.3 therefore shows that $H^{q,t}_{\mu}(1_F) = 0$. This proves (2.2)

Next we consider the set G. We will prove that

$$H^{q,t}_{\mu}(1_G) \leq a^4\mathcal{H}^{q,t}_{\mu}(E). \tag{2.3}$$

Since $a^{-4} < a^{-3}$, the family

$$\mathcal{V} = \left\{B(x,r) \middle| x \in G\frac{\nu(B(x,r))}{\mu(B(x,r))^q(2r)^t} > a^{-4}\,,\, r < \frac{1}{n}\right\}$$

is a fine cover of G. Let $\Pi \subseteq \mathcal{V}$ be a packing. Then

$$\sum_{B(x,r)\in\Pi} \mu(B(x,r))^q(2r)^t \le a^4 \sum_{B(x,r)\in\Pi} \nu(B(x,r)) = a^4\nu\Big(\bigcup_{B(x,r)\in\Pi} B(x,r)\Big)$$
$$= a^4\mathcal{H}^{q,t}_\mu\Big(\bigcup_{B(x,r)\in\Pi} B(x,r)\cap E\Big) \le a^4\mathcal{H}^{q,t}_\mu(E).$$

Since this is true for all packings $\Pi \subseteq \mathcal{V}$, we conclude that $H^{q,t}_{\mu,\mathcal{V}}(1_G) \le \mathcal{H}^{q,t}_\mu(E)$. This proves (2.3)

Combining (2.2) and (2.3) (and using Theorem 2.1) we obtain

$$H^{q,t}_\mu(1_E) = H^{q,t}_\mu(1_{F\cup G}) \le H^{q,t}_\mu(1_F) + H^{q,t}_\mu(1_G) \le a^4\mathcal{H}^{q,t}_\mu(E). \qquad \square$$

Proof of Theorem 2 "$\le$" First we verify that $P^{q,t}_\mu(1_E) \le \mathcal{P}^{q,t}_\mu(E)$. Since for each $\delta > 0$, the function $\Phi(x) = \delta$ for $x \in \mathbb{R}^d$ is a gauge, we obtain $P^{q,t}_\mu(1_F) = \sup_{\Phi \text{ is a gauge}} P^{q,t}_{\mu,\Phi}(1_F) \le \inf_{\delta>0} \overline{\mathcal{P}}^{q,t}_{\mu,\delta}(F) = \overline{\mathcal{P}}^{q,t}_\mu(F)$ for all subsets F of $\mathbb{R}^d$. Hence, for $E \subseteq \cup_i E_i$ we obtain (using Theorem 2.1),

$$P^{q,t}_\mu(1_E) \le P^{q,t}_\mu(1_{\cup_i E_i}) \le \sum_i P^{q,t}_\mu(1_{E_i}) \le \sum_i \overline{\mathcal{P}}^{q,t}_\mu(E_i).$$

Taking infimum over all countable covers $(E_i)_i$ of E yields $P^{q,t}_\mu(1_E) \le \mathcal{P}^{q,t}_\mu(E)$.

"$\ge$" Next we verify that $P^{q,t}_\mu(1_E) \ge \mathcal{P}^{q,t}_\mu(E)$. Let Φ be a gauge on E. For $n \in \mathbb{N}$ let $E_n = \{x \in E \mid \Phi(x) \ge \frac{1}{n}\}$. It now follows from the definitions that

$$P^{q,t}_{\mu,\Phi}(1_E) \ge P^{q,t}_{\mu,\Phi}(1_{E_n}) \ge \overline{\mathcal{P}}^{q,t}_{\mu,\frac{1}{n}}(E_n) \ge \mathcal{P}^{q,t}_\mu(E_n)$$

for all n. Since $E_n \nearrow E$, this implies that $\mathcal{P}^{q,t}_\mu(E) \le P^{q,t}_{\mu,\Phi}(1_E)$ for all gauges Φ on E. Taking infimum over Φ yields the desired result. $\square$

3 Proof of Theorem 3

We begin with a lemma.

Lemma 3.1. *Let μ be a Borel probability measures on $\mathbb{R}^d$, $q,t \in \mathbb{R}$ and $f : \mathbb{R}^d \to \mathbb{R}$ a positive Borel function.*

(1) Assume either $q \le 0$, or $0 < q$ and μ is a doubling measure. We have $H^{q,t}_\mu(f) = \int f(x)\, d\mathcal{H}^{q,t}_\mu(x)$.

(2) We have $P^{q,t}_\mu(f) = \int f(x)\, d\mathcal{P}^{q,t}_\mu(x)$.

Proof. (1) If follows from Theorem 1 that the statement is true for indicator functions, and standard methods allow us to extend this to simple positive Borel functions. Now, if f is a positive Borel function, then there exists a sequence $(s_n)_n$ of simple positive Borel functions increasing pointwise to f. Let $0 < c < 1$ and put $E_n = \{x \in \mathbb{R}^d \mid s_n(x) \geq cf(x)\}$. It is easily seen that $H^{q,t}_\mu(s_n) \geq H^{q,t}_\mu(cf1_{E_n}) = cH^{q,t}_\mu(f1_{E_n})$. Since $E_n \nearrow \mathbb{R}^d$, this and Theorem 2.1 implies that

$$H^{q,t}_\mu(f) \geq \lim_n H^{q,t}_\mu(s_n) \geq \lim_n cH^{q,t}_\mu(f1_{E_n}) = cH^{q,t}_\mu(f1_{\cup_n E_n}) = cH^{q,t}_\mu(f).$$

Letting $c \nearrow 1$ yields $H^{q,t}_\mu(f) = \lim_n H^{q,t}_\mu(s_n)$, and the Monotone Convergence Theorem therefore implies that

$$H^{q,t}_\mu(f) = \lim_n H^{q,t}_\mu(s_n) = \lim_n \int s_n(x)\, d\mathcal{H}^{q,t}_\mu(x) = \int f(x)\, d\mathcal{H}^{q,t}_\mu(x).$$

(2) The proof of this statement is similar to the proof of the statement in (1). □

Proof of Theorem 3

(1) Since ν is finite and thus outer regular, it suffices to prove that

$$\int_E f(x)\, d\mathcal{H}^{q,t}_\mu(x) \leq \nu(U) \tag{3.1}$$

for all open sets U with $E \subseteq U$ and for all positive Borel functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $0 \leq f(x) \leq \overline{d}^{q,t}_\mu(x,\nu)$ and with strict inequality $0 \leq f(x) < \overline{d}^{q,t}_\mu(x,\nu)$ whenever $\overline{d}^{q,t}_\mu(x,\nu) > 0$. Hence, let U be an open set with $E \subseteq U$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a positive Borel function satisfying $0 \leq f(x) \leq \overline{d}^{q,t}_\mu(x,\nu)$ and with strict inequality $0 \leq f(x) < \overline{d}^{q,t}_\mu(x,\nu)$ whenever $\overline{d}^{q,t}_\mu(x,\nu) > 0$. Write

$$\mathcal{V} = \left\{ B(x,r) \middle| x \in E,\ B(x,r) \subseteq U, \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t} \geq f(x) \right\}.$$

The family $\mathcal{V}$ is clearly a fine cover of E. For each packing $\Pi \subseteq \mathcal{V}$ we have

$$\sum_{B(x,r)\in\Pi} f(x)\mu(B(x,r))^q(2r)^t \leq \sum_{B(x,r)\in\Pi} \nu(B(x,r))$$

$$= \nu\Big(\bigcup_{B(x,r)\in\Pi} B(x,r)\Big) \le \nu(U).$$

So $H^{q,t}_{\mu,\mathcal{V}}(f1_E) \le \nu(U)$. Lemma 3.1 now implies that $\int_E f(x)\,d\mathcal{H}^{q,t}_\mu(x) = H^{q,t}_\mu(f1_E) \le H^{q,t}_{\mu,\mathcal{V}}(f1_E) \le \nu(U)$. This proves (3.1)

(2) We begin by showing that

$$\nu \ll \mathcal{H}^{q,t}_\mu \llcorner E \tag{3.2}$$

where $\mathcal{H}^{q,t}_\mu \llcorner E$ denotes that restriction of $\mathcal{H}^{q,t}_\mu$ to E; i.e., $(\mathcal{H}^{q,t}_\mu \llcorner E)(A) = \mathcal{H}^{q,t}_\mu(A\cap E)$. Therefore let $F \subseteq E$ with $\mathcal{H}^{q,t}_\mu(F) = 0$. We must now prove that $\nu(F) = 0$. For $n \in \mathbb{N}$ write

$$F_n = \Big\{x \in F \Big| \frac{\nu(B(x,r))}{\mu(B(x,r))^q(2r)^t} < n \text{ for all } r < \frac{1}{n}\Big\}.$$

For any centered $\frac{1}{n}$-covering $(B(x_i,r_i)_i$ of F_n we have

$$\sum_i \mu(B(x_i,r_i))^q(2r_i)^t \ge \frac{1}{n}\sum_i \nu(B(x_i,r_i)) \ge \frac{1}{n}\nu(\cup_i B(x_i,r_i)) \ge \frac{1}{n}\nu(F_n).$$

Thus $\frac{1}{n}\nu(F_n) \le \overline{\mathcal{H}}^{q,t}_{\mu,\frac{1}{n}}(F_n) \le \overline{\mathcal{H}}^{q,t}_\mu(F_n) \le \mathcal{H}^{q,t}_\mu(F_n) = 0$, whence $\nu(F_n) = 0$. Finally, since $\overline{d}^{q,t}_\mu(x,\nu) < \infty$ for $x \in E$, we conclude that $F_n \nearrow F$, and so $\nu(F) = \sup_n \nu(F_n) = 0$. This proves (3.2)

We now prove that $\int_E \overline{d}^{q,t}_\mu(x,\nu)\,d\mathcal{H}^{q,t}_\mu(x) \ge \nu(E)$. Let $\varepsilon > 0$ and let $\mathcal{V}$ be a fine cover of E. Then

$$\mathcal{W} = \Big\{B(x,r) \in \mathcal{V} \Big| \frac{\nu(B(x,r))}{\mu(B(x,r))^q(2r)^t} \le \overline{d}^{q,t}_\mu(x,\nu) + \varepsilon\Big\}$$

is also a fine cover of E. Since E is a Borel set, Theorem 2.2 implies that there exists a packing $\Pi \subseteq \mathcal{W}$ such that $\mathcal{H}^{q,t}_\mu(E \setminus_{B\in\Pi} B) = 0$. It now follows from (3.2) that $\nu(E \setminus_{B\in\Pi} B) = 0$. Hence

$$\sum_{B(x,r)\in\Pi} \Big(\overline{d}^{q,t}_\mu(x,\nu) + \varepsilon\Big)\,\mu(B(x,r))^q(2r)^t \ge \sum_{B(x,r)\in\Pi} \nu(B(x,r))$$
$$= \nu\Big(\bigcup_{B(x,r)\in\Pi} B(x,r)\Big) \ge \nu\Big(\bigcup_{B(x,r)\in\Pi} B(x,r)\cap E\Big) + \nu\Big(E \setminus \bigcup_{B(x,r)\in\Pi} B(x,r)\Big)$$
$$= \nu(E).$$

Thus $H^{q,t}_{\mu,\mathcal{V}}((\overline{d}^{q,t}_{\mu}(\cdot,\nu)+\varepsilon)1_E) \geq H^{q,t}_{\mu,\mathcal{W}}((\overline{d}^{q,t}_{\mu}(\cdot,\nu)+\varepsilon)1_E) \geq \nu(E)$. This implies that $H^{q,t}_{\mu}((\overline{d}^{q,t}_{\mu}(\cdot,\nu)+\varepsilon)1_E) \geq \nu(E)$. Lemma 3.1 now yields

$$\begin{aligned}\int_E \overline{d}^{q,t}_{\mu}(x,\nu)\,d\mathcal{H}^{q,t}_{\mu}(x)+\varepsilon\mathcal{H}^{q,t}_{\mu}(E) &= \int_E (\overline{d}^{q,t}_{\mu}(x,\nu)+\varepsilon)\,d\mathcal{H}^{q,t}_{\mu}(x)\\ &= H^{q,t}_{\mu}((\overline{d}^{q,t}_{\mu}(\cdot,\nu)+\varepsilon)1_E) \geq \nu(E)\end{aligned}$$

and the result follows by letting $\varepsilon \searrow 0$.

(3) Since ν is finite and thus outer regular, it suffices to prove that

$$\int_E \underline{d}^{q,t}_{\mu}(x,\nu)\,d\mathcal{P}^{q,t}_{\mu}(x) \leq \frac{1}{c}\nu(U) \tag{3.3}$$

for all open sets U with $E \subseteq U$ and for all $0 < c < 1$. Therefore, let U be an open set with $E \subseteq U$ and let $0 < c < 1$. Then for each $x \in E$ it is possible to choose $\Phi(x) > 0$ such that $0 < \Phi(x) < \operatorname{dist}(x, \mathbb{R}^d \setminus U)$ and $\dfrac{\nu(B(x,r))}{\mu(B(x,r))^q(2r)^t} \geq c\underline{d}^{q,t}_{\mu}(x,\nu)$ for all $0 < r < \Phi(x)$. These conditions imply that Φ is a gauge function on E. For each centered Φ-packing Π of E we have

$$\sum_{B(x,r)\in\Pi} \underline{d}^{q,t}_{\mu}(x,\nu)\mu(B(x,r))^q(2r)^t \leq \frac{1}{c}\sum_{B(x,r)\in\Pi}\nu(B(x,r)) \leq \frac{1}{c}\nu(U).$$

Taking supremum over Π gives $P^{q,t}_{\mu,\Phi}(\underline{d}^{q,t}_{\mu}(\cdot,\nu)1_E) \leq \frac{1}{c}\nu(U)$. Lemma 3.1 now implies that

$$\int_E \underline{d}^{q,t}_{\mu}(\cdot,\nu)\,d\mathcal{P}^{q,t}_{\mu}(x) = P^{q,t}_{\mu}(\underline{d}^{q,t}_{\mu}(\cdot,\nu)1_E) \leq P^{q,t}_{\mu,\Phi}(\underline{d}^{q,t}_{\mu}(\cdot,\nu)1_E) \leq \frac{1}{c}\nu(U).$$

This proves (3.3)

(4) Let $\varepsilon > 0$ and let Φ be a gauge on E such that $P^{q,t}_{\mu,\Phi}(E) < \infty$. Then

$$\mathcal{V} = \left\{ B(x,r) \,\middle|\, x \in E, r < \Phi(x), \frac{\nu(B(x,r))}{\mu(B(x,r))^q(2r)^t} \leq \underline{d}^{q,t}_{\mu}(x,\nu)+\varepsilon \right\}$$

is a fine cover of E. Since E is Borel, Theorem 2.2 implies that there exists a packing $\Pi \subseteq \mathcal{V}$ such that $\nu(E \setminus \cup_{B\in\Pi}B) = 0$. Thus

$$\nu(E) = \nu\Big(\bigcup_{B(x,r)\in\Pi} B(x,r)\cap E\Big) + \nu\Big(E \setminus \bigcup_{B(x,r)\in\Pi} B(x,r)\Big)$$

$$\begin{aligned}&\leq \nu\Big(\bigcup_{B(x,r)\in\Pi} B(x,r)\Big) = \sum_{B(x,r)\in\Pi} \nu(B(x,r))\\&\leq \sum_{B(x,r)\in\Pi} \Big(\underline{d}_\mu^{q,t}(x,\nu)+\varepsilon\Big)\mu(B(x,r))^q(2r)^t\\&\leq P_{\mu,\Phi}^{q,t}(\underline{d}_\mu^{q,t}(\cdot,\nu)) + \varepsilon P_{\mu,\Phi}^{q,t}(E).\end{aligned}$$

Taking infimum over Φ and letting $\varepsilon \searrow$ yields $\nu(E) \leq P_\mu^{q,t}(\underline{d}_\mu^{q,t}(\cdot,\nu))$. Lemma 3.1 now implies that $\nu(E) \leq P_\mu^{q,t}(\underline{d}_\mu^{q,t}(\cdot,\nu)) = \int_E \underline{d}_\mu^{q,t}(x,\nu)\, d\mathcal{P}_\mu^{q,t}(x)$. □

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