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AN EXAMPLE OF A DARBOUX FUNCTION HAVING NO FIXED POINTS

Abstract

In this article we construct an example of a bilaterally quasicontinuous Darboux function $f : [0, 1] \to [0, 1]$, which has no fixed points.

Let $\mathbb{R}$ be the set of all reals. Denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer density $D_u(A, x)$ ($D_l(A, x)$) of the set A at the point x as

$$\limsup_{h\to 0^+} \frac{\mu_e(A \cap [x-h, x+h])}{2h}$$
$$(\liminf_{h\to 0^+} \frac{\mu_e(A \cap [x-h, x+h])}{2h} \text{respectively}).$$

A point x is said to be an outer density point (a density point) of a set A if $D_l(A, x) = 1$ (if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B, x) = 1$).

Taking the extremal limits for the expressions $\frac{A\cap[x-h,x]}{h}$ and $\frac{[x,x+h]}{h}$ we obtain respectively the left or the right upper (lower) densities of A at x.

The family T_d of all sets A for which the implication

$$x \in A \Longrightarrow x \text{ is a density point of } A$$

is true, is a topology called the density topology ([2, 6]). The sets $A \in T_d$ are Lebesgue measurable [2, 6]. Let T_e be the Euclidean topology in $\mathbb{R}$. The continuity of applications f from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$ is called the approximate continuity ([2, 6]).

Each approximately continuous function $f : [0, 1] \to [0, 1]$ is of the first Baire class and has Darboux property ([2]). It is well known that Darboux

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Baire 1 functions from $[0,1]$ to $[0,1]$ have fixed points. So, for all approximately continuous functions $f:[0,1]\to[0,1]$ there are points $x\in[0,1]$ with $f(x)=x$. This observation results also from a theorem of Brown ([1]).

Since there are approximately continuous functions which are discontinuous on sets of positive measure, in my article [3] I introduce some special conditions based also on the density topology which imply the continuity almost everywhere of considered functions. One of them is the condition (s_1).

For an arbitrary function $f:\mathbb{R}\to\mathbb{R}$ denote by $C(f)$ the set of all continuity points of f. Moreover let $D(f)=\mathbb{R}\setminus C(f)$. A function $f:\mathbb{R}\to\mathbb{R}$ has the property (s_1) at a point x ($f\in s_1(x)$) if for each positive real r and for each set $U\in T_d$ containing x there is an open interval I such that $\emptyset\neq I\cap U\subset C(f)$ and $|f(t)-f(x)|<r$ for all points $t\in I\cap U$. A function f has the property (s_1), if $f\in s_1(x)$ for every point $x\in\mathbb{R}$.

For each function f having the property (s_1) the set $D(f)=\mathbb{R}\setminus C(f)$ is nowhere dense and of Lebesgue measure 0. So, functions discontinuous on dense sets which are approximately continuous do not satisfy the condition (s_1). The characteristic function of the interval $[0,1]$ has the property (s_1), but it is not approximately continuous.

A function $f:[0,1]\to[0,1]$ is a bilaterally quasicontinuous at a point $x\in(0,1)$ if for each positive real r there are open intervals

$$I_1\subset(x-r,x)\cap[0,1] \text{ and } I_2\subset(x,x+r)\cap[0,1]$$

such that

$$f(I_1)\cup f(I_2)\subset(f(x)-r,f(x)+r).$$

Analogously we define the quasicontinuity from the right at 0 and the quasicontinuity from the left at 1. A function $f:[0,1]\to\mathbb{R}$ is said to be bilaterally quasicontinuous if it is bilaterally quasicontinuous at each point $x\in(0,1)$, quasicontinuous from the right at 0 and quasicontinuous from the left at 1 [4, 5]. Evidently each function $f:[0,1]\to\mathbb{R}$ having the property (s_1) is quasicontinuous, but it may be not bilaterally quasicontinuous (for example such is the characteristic function of the interval $[\frac{1}{3},\frac{1}{2}]$.

Some examples of Darboux functions $f:[0,1]\to[0,1]$ without the fixed point property are well known. In this article we prove the following assertion.

Theorem 1. *There is a bilaterally quasicontinuous Darboux function $f:[0,1]\to[0,1]$ satisfying condition (s_1) which has no fixed points.*

PROOF. Let $C\subset[0,1]$ be the ternary Cantor set. As well known it is a nonempty compact perfect set of measure zero. Let

$$I_{1,1}=(\frac{1}{3},\frac{2}{3}),\ \ I_{2,1}=(\frac{1}{3^2},\frac{2}{3^2}),\ \ I_{2,2}=(\frac{7}{3^2},\frac{8}{3^2}),\ldots$$

$$\ldots, I_{n,1} = (\frac{1}{3^n}, \frac{2}{3^n}), \ldots, I_{n,2^{n-1}} = (\frac{3^n-2}{3^n}, \frac{3^n-1}{3^n}), \ldots$$

be the sequence of all components of the set $[0,1] \setminus C$. For $n \geq 1$ we put

$$U_n = \bigcup_{i=1}^{2^{n-1}} I_{n,i}, \quad V_n = [0,1] \setminus U_n, \text{ and let } K_{n,1}, \ldots, K_{n,2^n}$$

be the components of the set V_n. For each interval $I_{n+1,i}$, $i \leq 2^n$, there is exactly one closed interval $K_{n,j} = K_n(I_{n+1,i})$, $j \leq 2^n$, such that $K_{n,j} \supset I_{n+1,i}$. Evidently,

$$\frac{d(I_{n+1,i})}{d(K_n(I_{n+1,i}))} = \frac{1}{3},$$

where $d(I_{n+1,i})$ denotes the length of the interval $I_{n+1,i}$. For each pair $(n+1, i)$ we find a closed interval $J_{n+1,i} \subset I_{n+1,i}$ having the same center as $I_{n+1,i}$ and such that

$$\frac{d(J_{n+1,i})}{d(K_n(I_{n+1,i})} > \frac{1}{4}. \tag{1}$$

Moreover for the interval $I_{1,1}$ we define $K_1(I_{1,1}) = [0,1]$ and find a closed interval $J_{1,1} \subset I_{1,1}$ with the center $\frac{1}{2}$ and such that $d(J_{1,1}) > \frac{1}{4}$.

Let $N_1, N_2, \ldots, N_m, \ldots$ be a sequence of pairwise disjoints infinite subsets of positive integers such that for the set $\mathbb{N}$ of all positive integers we have $\mathbb{N} = \bigcup_{n=1}^{\infty} N_n$, and let (w_n) be an enumeration of all rationals belonging to $[0,1]$ such that $w_n \neq w_m$ for $n \neq m$. For $n = 1, 2, \ldots$ put

$$P_n = \bigcup_{i \in N_n} \bigcup_{k \leq 2^{i-1}} J_{i,k} \text{ and } Q_n = \bigcup_{i \in N_n} \bigcup_{k \leq 2^{i-1}} I_{i,k}.$$

For each point $x \in C$ and for each index k there is a closed interval $K_{k,i(x)} \ni x$. So by (1) for each point $x \in A$ and for each positive integer n we obtain

$$D_u(P_n, x) \geq \frac{1}{8}. \tag{2}$$

Now for indices $n, m \geq 1$ and $k \leq 2^{n-1}$ we define functions $f_{n,k,m}, g_{n,k,m}$: $\mathrm{cl}(I_{n,k}) \to [0,1]$ (cl denotes the closure operation in the topology T_e) by the formulas: $f_{n,k,m}(x) = w_m$ for $x \in J_{n,k}$, $f_{n,k,m} = 0$ at the endpoints of $\mathrm{cl}(I_{n,k})$. $g_{n,k,m}$ is linear on the components of $\mathrm{cl}(I_{n,k} \setminus J_{n,k})$. $g_{n,k,m}(x) = w_m$ for $x \in J_{n,k}$, $g_{n,k,m} = 1$ at the endpoints of $\mathrm{cl}(I_{n,k})$. $f_{n,k,m}$ is linear on the components of $cl(I_{n,k} \setminus J_{n,k})$. Now we will define a function f on the set $[0,1] \setminus C$. For this, fix indices n, $i \in N_n$ and $k \leq 2^{i-1}$ and put $\mathrm{cl}(I_{i,k}) = [a_{i,k} b_{i,k}]$.

If $w_n < a_{i,k}$, then we put $f(x) = f_{i,k,n}(x)$ for $x \in \mathrm{cl}(I_{i,k})$.

If $w_n > b_{i,k}$, then we put $f(x) = g_{i,k,n}(x)$ for $x \in cl(I_{i,k})$.

If $w_n \in \mathrm{cl}(I_{i,k})$, then we put $f(x) = 0$ for $x \in \mathrm{cl}(I_{i,k})$.

Moreover, let $f(0) = 1$ and $f(1) = 0$. Now we will define the function f on the set

$$E = (0,1) \setminus \bigcup_{n\geq 1} \bigcup_{i\in N_n} \bigcup_{k\leq 2^{i-1}} \mathrm{cl}(I_{i,k}).$$

For this enumerate in a sequence (L_n) the set of all open intervals I with rational endpoints for which $I \cap E \neq \emptyset$. Moreover, we assume that $L_n \neq L_m$ for $n \neq m$. Now, by induction, for each positive integer n we find a nonempty perfect set

$$H_n \subset (E \cap L_n) \setminus \bigcup_{k<n} H_k,$$

which is nowhere dense in H. For each set H_n put $c_n = \inf H_n$ and $d_n = \sup H_n$. Let $z_n \in [c_n, d_n] \setminus H_n$ be a point. Then the sets $H_{n,1} = [c_n, z_n) \cap H_n$ and $H_{n,2} = (z_n, d_n] \cap H_n$ are nonempty and perfect and $H_n = H_{n,1} \cup H_{n,2}$.

There are functions $h_{n,1} : H_{n,1} \to \mathbb{R}$ and $h_{n,2} : H_{n,2} \to \mathbb{R}$ such that $h_{n,1}(H_{n,1}) = [z_n, 1]$ and $h_{n,2}(H_{n,2}) = [0, z_n)$. For $n = 1, 2, \ldots$ let $h_n(x) = h_{n,1}(x)$ if $x \in H_{n,1}$ and $h_n(x) = h_{n,2}(x)$ if $x \in H_{n,2}$. Then $h_n : H_n \to [0,1]$ is such that $h_n(H_n) = [0,1]$ and $h_n(x) \neq x$ for each $x \in H_n$. Put

$$f(x) = h_n(x) \text{ for } x \in H_n, \ \ n \geq 1, \text{ and } f(x) = 0 \text{ for } x \in E \setminus \bigcup_{n=1}^{\infty} H_n.$$

From the construction follows that the function $f : [0,1] \to [0,1]$ has no fixed points. Of course, if $x \in H$, then $f(x) \neq x$. If $x \in \mathrm{cl}(I_{n,i})$ and $\mathrm{cl}(I_{n,i}) = [a_{n,i}, b_{n,i}]$, then the following cases are possible:

(a) $w_n < a_{n,i}$ and $f(x) \leq w_n < x$ for all $x \in \mathrm{cl}(I_{n,i})$;

(b) $w_n > b_{n,i}$ and $f(x) \geq w_n > x$ for all $x \in \mathrm{cl}(I_{n,i})$;

(c) $w_n \in [a_{n,i}, b_{n.i}]$ and $f(x) = 0 < x$ for all $x \in \mathrm{cl}(I_{n,i})$.

Moreover, $f(0) = 1$ and $f(1) = 0$. Since f is continuous on each interval, $\mathrm{cl}(I_{n,i})$, $n \geq 1$ and $i \leq 2^{n-1}$, and since $f(I \cap E) = [0,1]$ for each open interval I such that $I \cap E \neq \emptyset$, the function f has the Darboux property.

We will prove that the function f satisfies the condition (s_1). If a point $x \in \mathrm{cl}(I_{n,i})$ for a pair (n,i), then f is continuous or unilaterally continuous at x and consequently $f \in s_1(x)$. So we suppose that $x \in E \cup \{0,1\}$. Let $U \ni x$ be a set belonging to T_d. Fix a positive real η. There is an index l such that $|f(x) - w_l| < \eta$. Since $D_l(U,x) = 1$ and $D_u(P_l, x) \geq \frac{1}{8}$, there is an interval $J_{l,i}$

such that $f(J_{l,i}) = \{w_l\}$ and $U \cap J_{l,i} \neq \emptyset$. So, $f(U \cap J_{l,i}) \subset (f(x)-\eta, f(x)+\eta)$ and $f \in s_1(x)$.

At the points 0 and 1 the function f is quasicontinuous, since $f \in s_1(0) \cap s_1(1)$. At points $x \in [0,1] \setminus C$ the function f is continuous, so it is bilaterally quasicontinuous at these points. If x is the right endpoint of an interval $I_{n,i}$, then f is continuous at x from the left. Fix a positive real η. From the continuity of f at x from the left, it follows that there is a real $\delta > 0$ such that

$$\delta < \eta \text{ and } f((x-\delta, x)) \subset (f(x)-\eta, f(x)+\eta).$$

Let an index m be such that $|f(x) - w_m| < \eta$. Since the set N_m is infinite, there is an integer $i \in N_m$ such that $V_i \cap (x, x+\eta) \neq \emptyset$. So there is an open interval $L \subset V_i \cap (x, x+\eta)$ such that $f(L) = \{w_m\}$. Consequently,

$$L \subset (x, x+\eta) \text{ and } f(L) \subset (f(x)-\eta, f(x)+\eta),$$

and f is bilaterally quasicontinuous at x. Similarly we can show that f is bilaterally quasicontinuous at points x, which are the left endpoints of some intervals $\mathrm{cl}(I_{n,i})$ and at points $x \in E$.

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