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A GENERALIZATION OF THE DENSITY TOPOLOGY WITH RESPECT TO CATEGORY

Abstract

This paper deals with the generalization of the density points with respect to category. Topologies using this concept of density are introduced, and their properties are investigated.

The concept of a density point with respect to category as the analogue of a density point with respect to Lebesgue measure was first introduced in 1982 (see papers [6], [5]). In this paper, we will present a more general approach to the density point with respect to category. It is a generalization of the concept presented in [5] and [3].

Let $\mathbb{R}$ be the set of reals and $\mathbb{N}$ stand for the set of natural numbers. Let $\mathcal{I}$ be the σ-ideal of first category sets in $\mathbb{R}$, $\mathcal{S}$ be the σ-algebra of sets having the Baire property in $\mathbb{R}$, and $\mathcal{T}_{nat}$ be the natural topology in $\mathbb{R}$.

According to paper [5], we say that 0 is a density point with respect to category of a subset A of reals having the Baire property if the sequence $\{f_n\}_{n\in\mathbb{N}} = \{\chi_{nA\cap[-1,1]}\}_{n\in\mathbb{N}}$ converges with respect to the σ-ideal of the first category sets to the characteristic function $\chi_{[-1,1]}$. It means that every subsequence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ contains a subsequence converging to the function $\chi_{[-1,1]}$ everywhere except for a set of the first category. For $J_n = [a_n, b_n]$, $n \in \mathbb{N}$, let us put

$$s(J_n) = \frac{1}{2}(a_n + b_n),$$
$$h(A, J_n)(x) = \chi_{\frac{2}{|J_n|}(A-s(J_n))\cap[-1,1]}(x),$$

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where $A+z=\{a+z : a\in A\}$, $\alpha A=\{\alpha a : a\in A\}$ for $z,\alpha\in\mathbb{R}, A\subset\mathbb{R}$. By $J=\{J_n\}_{n\in\mathbb{N}}$, we shall denote a sequence of intervals tending to zero, that means

$$\lim_{n\to\infty} s(J_n)=0 \qquad \wedge \qquad \lim_{n\to\infty}|J_n|=0.$$

We will identify sequences which differ in finite numbers of their terms.

Definition 1. The point 0 is called an $\mathcal{I}(J)-$density point of a set $A\in\mathcal{S}$ if

$$h(A,J_n)(x)\xrightarrow[n\to\infty]{\mathcal{I}}\chi_{[-1,1]}(x),$$

which means that

$$\underset{\{n_k\}_{k\in\mathbb{N}}}{\forall}\quad \underset{\{n_{k_m}\}_{m\in\mathbb{N}}}{\exists}\quad \underset{\Theta\in\mathcal{I}}{\exists}\quad \underset{x\notin\Theta}{\forall}\quad h(A,J_{n_{k_m}})(x)\xrightarrow[m\to\infty]{}\chi_{[-1,1]}(x).$$

It is obvious that 0 is an $\mathcal{I}(J)-$density point of a set $A\in\mathcal{S}$ if and only if

$$\underset{\{n_k\}_{k\in\mathbb{N}}}{\forall}\quad \underset{\{n_{k_m}\}_{m\in\mathbb{N}}}{\exists}\quad \limsup_{m\to\infty}\Big([-1,1]\setminus(A-s(J_{n_{k_m}}))\frac{2}{|J_{n_{k_m}}|}\Big)\in\mathcal{I}.$$

We shall say that a point $x_0\in\mathbb{R}$ is an $\mathcal{I}(J)-$density point of a set $A\in\mathcal{S}$ if and only if 0 is an $\mathcal{I}(J)-$density point of the set $A-x_0$.

A point $x_0\in\mathbb{R}$ is an $\mathcal{I}(J)-$dispersion point of a set $A\in\mathcal{S}$ if and only if x_0 is an $\mathcal{I}(J)-$density point of the complementary set A'.

It is easy to see that if $J_n=\left[-\frac{1}{n},\frac{1}{n}\right]$, for $n\in\mathbb{N}$, then x_0 is an $\mathcal{I}(J)-$density point of a set $A\in\mathcal{S}$ if and only if x_0 is an $\mathcal{I}-$density point of A (see [5]).

Moreover, if $J_n=\left[-\frac{1}{s_n},\frac{1}{s_n}\right]$, where $s=\{s_n\}_{n\in\mathbb{N}}$ is an unbounded and nondecreasing sequence of positive real numbers, then x_0 is an $\mathcal{I}(J)-$density point of a set $A\in\mathcal{S}$ if and only if x_0 is an $\langle s\rangle-$density point of A (see [3]).

If $A\in\mathcal{S}$, then we let

$$\Phi_{\mathcal{I}(J)}(A)=\{x\in\mathbb{R} : x \text{ is an } \mathcal{I}(J)-\text{density point of } A\}.$$

Theorem 1. *For any sets $A,B\in\mathcal{S}$ we have :*

1. $\Phi_{\mathcal{I}(J)}(\emptyset)=\emptyset$, $\Phi_{\mathcal{I}(J)}(\mathbb{R})=\mathbb{R}$;
2. $A\bigtriangleup B\in\mathcal{I}\Rightarrow\Phi_{\mathcal{I}(J)}(A)=\Phi_{\mathcal{I}(J)}(B)$;
3. $A\bigtriangleup\Phi_{\mathcal{I}(J)}(A)\in\mathcal{I}$;
4. $\Phi_{\mathcal{I}(J)}(A\cap B)=\Phi_{\mathcal{I}(J)}(A)\cap\Phi_{\mathcal{I}(J)}(B)$.

PROOF. Properties 1 and 2 are obvious. Let us prove property 3. Let $A \in \mathcal{S}$. Then there exist an open set G and a set $P \in \mathcal{I}$ such that $A = G \triangle P$. We shall show that $A \setminus \Phi_{\mathcal{I}(J)}(A) \in \mathcal{I}$. Let us take a point $x \in G$. Then there exists a number $n_0 \in \mathbb{N}$ such that $x + J_n \subset G$ for $n \geq n_0$, and hence $J_n \subset G - x$. So we obtain

$$\frac{2}{|J_n|}\Big(A - (x + s\,(J_n))\Big) \supset \frac{2}{|J_n|}\Big((G \setminus P) - (x + s\,(J_n))\Big) =$$
$$\frac{2}{|J_n|}\Big((G - x) - s(J_n)\Big) \setminus \Big(P - (x + s(J_n))\Big) \supset$$
$$\frac{2}{|J_n|}\Big(J_n - s(J_n)\Big) \setminus \Big(P - (x + s(J_n))\Big) = [-1,1] \setminus \frac{2}{|J_n|}\Big(P - (x + s(J_n))\Big).$$

If $P \in \mathcal{I}$, then $\frac{2}{|J_n|}\Big(P - (x + s(J_n))\Big) \in \mathcal{I}$. Hence, for $x \in G$, we obtain that

$$h(A - x, J_n)(x) \underset{n\to\infty}{\longrightarrow} \chi_{[-1,1]}(x),$$

so that $A \setminus \Phi_{\mathcal{I}(J)}(A) \in \mathcal{I}$.

We prove that $\Phi_{\mathcal{I}(J)}(A) \setminus A \in \mathcal{I}$. Observe that $\Phi_{\mathcal{I}(J)}(A) \subset \mathbb{R} \setminus \Phi_{\mathcal{I}(J)}(A')$. Then

$$\Phi_{\mathcal{I}(J)}(A) \setminus A = \Phi_{\mathcal{I}(J)}(A) \cap A' \subset (\mathbb{R} \setminus \Phi_{\mathcal{I}(J)}(A')) \cap A' = A' \setminus \Phi_{\mathcal{I}(J)}(A') \in \mathcal{I}.$$

This fact ends the proof of property 3.

It remains to show 4. Observe that

$$\Phi_{\mathcal{I}(J)}(A \cap B) \subset \Phi_{\mathcal{I}(J)}(A), \quad \Phi_{\mathcal{I}(J)}(A \cap B) \subset \Phi_{\mathcal{I}(J)}(B).$$

Hence, $\Phi_{\mathcal{I}(J)}(A \cap B) \subset \Phi_{\mathcal{I}(J)}(A) \cap \Phi_{\mathcal{I}(J)}(B)$.

Suppose that $x_0 \in \Phi_{\mathcal{I}(J)}(A) \cap \Phi_{\mathcal{I}(J)}(B)$, and let us take a sequence $\{n_k\}_{k\in\mathbb{N}}$. From the assumption that $x_0 \in \Phi_{\mathcal{I}(J)}(A)$, there exists a subsequence $\{n_{k_m}\}_{m\in\mathbb{N}}$ such that

$$\limsup_{m\to\infty}\Big([-1,1] \setminus \big(A - s(J_{n_{k_m}})\big)\frac{2}{|J_{n_{k_m}}|}\Big) \in \mathcal{I}.$$

Using the assumption that $x_0 \in \Phi_{\mathcal{I}(J)}(B)$, we can choose a subsequence $\{n_{k_{m_l}}\}_{l\in\mathbb{N}}$ of $\{n_{k_m}\}_{m\in\mathbb{N}}$ such that

$$\limsup_{l\to\infty}\Big([-1,1] \setminus (B - s(J_{n_{k_{m_l}}}))\frac{2}{|J_{n_{k_{m_l}}}|}\Big) \in \mathcal{I}.$$

For the sequence $\{n_{k_{m_l}}\}_{l\in\mathbb{N}}$, we obtain that

$$\limsup_{l\to\infty}\Big([-1,1]\setminus\big((A\cap B)-s(J_{n_{k_{m_l}}})\big)\frac{2}{|J_{n_{k_{m_l}}}|}\Big)=$$
$$\limsup_{l\to\infty}\bigg(\Big([-1,1]\setminus\big(A-s(J_{n_{k_{m_l}}})\big)\frac{2}{|J_{n_{k_{m_l}}}|}\Big)\cup$$
$$\Big([-1,1]\setminus\big(B-s(J_{n_{k_{m_l}}})\big)\frac{2}{|J_{n_{k_{m_l}}}|}\Big)\bigg)=$$
$$\limsup_{l\to\infty}\Big([-1,1]\setminus\big(A-s(J_{n_{k_{m_l}}})\big)\frac{2}{|J_{n_{k_{m_l}}}|}\Big)\cup$$
$$\limsup_{l\to\infty}\Big([-1,1]\setminus\big(B-s(J_{n_{k_{m_l}}})\big)\frac{2}{|J_{n_{k_{m_l}}}|}\Big)\in\mathcal{I}.$$

Hence, $x\in\Phi_{\mathcal{I}(J)}(A\cap B)$. □

Theorem 2. *Let J be a sequence of intervals tending to zero. Then*

$$\mathcal{T}_{\mathcal{I}(J)}=\{A\in\mathcal{S}:A\subset\Phi_{\mathcal{I}(J)}(A)\}$$

is a topology on $\mathbb{R}$, which will be called $\mathcal{I}(J)-$density topology.
Moreover, we have $\mathcal{T}_{nat}\subset\mathcal{T}_{\mathcal{I}(J)}$.

Proof. From the previous theorem, it follows that the operator $\Phi_{\mathcal{I}(J)}:\mathcal{S}\to\mathcal{S}$ is a lower density operator, and the pair $(\mathcal{S},\mathcal{I})$ fulfills countable chain condition. By the general lifting theorem (see [4]), we obtain that $\mathcal{T}_{\mathcal{I}(J)}$ is a topology on $\mathbb{R}$. □

Theorem 3. *Let $J=\{J_n\}_{n\in\mathbb{N}}$, where $J_n=[a_n,b_n]$ for $n\in\mathbb{N}$, be a sequence tending to zero. Then 0 is an $\mathcal{I}(J)-$density point of the set*

$$A=\{0\}\cup\bigcup_{n\in\mathbb{N}}int(J_n).$$

Moreover, $A\in\mathcal{T}_{\mathcal{I}(J)}$.

Proof. Observe that for every $n\in\mathbb{N}$,

$$\big(A-s(J_n)\big)\frac{2}{|J_n|}\supset(-1,1).$$

Hence, $h(A,J_n)\xrightarrow[n\to\infty]{\mathcal{I}}\chi_{[-1,1]}$. It implies that $0\in\Phi_{\mathcal{I}(J)}(A)$. Moreover, $\mathcal{T}_{nat}\subset\mathcal{T}_{\mathcal{I}(J)}$, so we obtain that $A\in\mathcal{T}_{\mathcal{I}(J)}$. □

The following theorem shows that we have obtained an essential extension of $\mathcal{I}$-density points.

Theorem 4. *For every sequence $J = \{J_n\}_{n\in\mathbb{N}}$ of intervals tending to zero, there exists a sequence $K = \{K_n\}_{n\in\mathbb{N}}$ of intervals tending to zero such that*

$$\mathcal{T}_{\mathcal{I}(J)} \setminus \mathcal{T}_{\mathcal{I}(K)} \neq \emptyset \quad \wedge \quad \mathcal{T}_{\mathcal{I}(K)} \setminus \mathcal{T}_{\mathcal{I}(J)} \neq \emptyset.$$

PROOF. Let $J_n = [a_n, b_n]$ for $n \in \mathbb{N}$. Let us consider that $b_n > 0$. We can assume that $b_n > b_{n+1}$ (eventually we choose subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that $b_{n_k} > b_{n_{k+1}}$). Let us denote $c_n = \max\{a_n, b_{n+1}\}$. Now we define intervals K_n for $n \in \mathbb{N}$ in the following way:

$$K_n = \Big[b_n - \frac{1}{n}(b_n - c_n), b_n\Big].$$

The sequence $K = \{K_n\}_{n\in\mathbb{N}}$ tends to zero. Putting

$$A = \{0\} \cup \Big(\bigcup_{n\in\mathbb{N}} int(J_n) \setminus \bigcup_{n\in\mathbb{N}} int(K_n)\Big), \quad B = \{0\} \cup \bigcup_{n\in\mathbb{N}} int(K_n),$$

we obtain from the previous theorem that $B \in \mathcal{T}_{\mathcal{I}(K)}$. Moreover, $B \notin \mathcal{T}_{\mathcal{I}(J)}$ and $A \notin \mathcal{T}_{\mathcal{I}(K)}$. Let us notice that for every $n \in \mathbb{N}$,

$$\big(A - s(J_n)\big)\frac{2}{|J_n|} \supset \Big(-1, 1 - \frac{1}{n}\Big].$$

Thus, we obtain $A \in \mathcal{T}_{\mathcal{I}(J)}$.

If there exists sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $b_{n_k} > 0$, then we do analogous construction for this subsequence. Otherwise, we do similar construction for the sequence $-J = \{-J_n\}_{n\in\mathbb{N}}$. Therefore, the theorem holds. □

From definition 1 we have:

Proposition 5. *Let $J = \{J_n\}_{n\in\mathbb{N}}$ and $K = \{K_n\}_{n\in\mathbb{N}}$ be sequences tending to zero. If for every $n \in \mathbb{N}$ there exists $k(n) \in \mathbb{N}$ such that $J_n = K_{k(n)}$, then $\mathcal{T}_{\mathcal{I}(K)} \subset \mathcal{T}_{\mathcal{I}(J)}$.*

Theorem 6. *Let $J = \{J_n\}_{n\in\mathbb{N}}$, where $J_n = [a_n, b_n]$ for $n \in \mathbb{N}$, be a sequence tending to zero. Let us fix $l_0 \in \mathbb{N}$ and $i_0 \in \{1, \ldots, l_0\}$. Putting*

$$K_n = \left[a_n + \frac{i_0 - 1}{l_0}(b_n - a_n), a_n + \frac{i_0}{l_0}(b_n - a_n)\right]$$

for $n \in \mathbb{N}$, we have that the sequence $K = \{K_n\}_{n\in\mathbb{N}}$ tends to zero and $\mathcal{T}_{\mathcal{I}(J)} \subset \mathcal{T}_{\mathcal{I}(K)}$.

Proof. The fact that the sequence $K = \{K_n\}_{n\in\mathbb{N}}$ tends to zero is a consequence of the assumption that the sequence $J = \{J_n\}_{n\in\mathbb{N}}$ tends to zero.

Now we prove that $\mathcal{T}_{\mathcal{I}(J)} \subset \mathcal{T}_{\mathcal{I}(K)}$. Let $A \in \mathcal{S}$ be such a set that $0 \in \Phi_{\mathcal{I}(J)}(A)$. We will show that $0 \in \Phi_{\mathcal{I}(K)}(A)$. Let us take a sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$. From the assumption there exists a subsequence $\{n_{k_m}\}_{m\in\mathbb{N}}$ and a set $\Theta_0 \in \mathcal{I}$ such that

$$h(A, J_{n_{k_m}})(x) \underset{m\to\infty}{\longrightarrow} \chi_{[-1,1]}(x) \text{ for } x \notin \Theta_0. \tag{1}$$

Let us put $\Theta = l_0\Theta_0 - 2i_0 + 1 + l_0$. Obviously, $\Theta \in \mathcal{I}$. We will show that

$$h(A, K_{n_{k_m}})(x) \underset{m\to\infty}{\longrightarrow} \chi_{[-1,1]}(x) \text{ for } x \notin \Theta.$$

It is sufficient to consider the case when $x \in [-1, 1] \setminus \Theta$. Then

$$\frac{1}{l_0}x + \frac{2i_0 - 1 - l_0}{l_0} \in \left[\frac{2i_0 - 2 - l_0}{l_0}, \frac{2i_0 - l_0}{l_0}\right] \setminus \Theta_0 \subset [-1, 1] \setminus \Theta_0.$$

From condition (1), there exists a natural number m_x such that for every $m \geq m_x$, we have that

$$h(A, J_{n_{k_m}})\left(\frac{1}{l_0}x + \frac{2i_0 - 1 - l_0}{l_0}\right) = 1.$$

This is equivalent to the condition :

$$\frac{1}{l_0}x + \frac{2i_0 - 1 - l_0}{l_0} \in \big(A - s(J_{n_{k_m}})\big)\frac{2}{|J_{n_{k_m}}|} \cap [-1, 1].$$

Simultaneously,

$$s(K_{n_{k_m}}) - s(J_{n_{k_m}}) = \frac{1}{2}(2i_0 - 1 - l_0)|K_{n_{k_m}}| \quad \wedge \quad x \in \left[\frac{2i_0 - 2 - l_0}{l_0}, \frac{2i_0 - l_0}{l_0}\right].$$

Hence, we have that

$$
\begin{aligned}
&h(A, J_{n_{k_m}})\left(\frac{1}{l_0}x+\frac{2i_0-1-l_0}{l_0}\right)=1 \Leftrightarrow \\
&\qquad \left(\frac{1}{l_0}x+\frac{2\big(s(K_{n_{k_m}})-s(J_{n_{k_m}})\big)}{l_0|K_{n_{k_m}}|}\right) \in \Big(A-s(J_{n_{k_m}})\Big)\frac{2}{|J_{n_{k_m}}|}\cap \\
&\qquad \left[\frac{2i_0-2-l_0}{l_0}, \frac{2i_0-l_0}{l_0}\right] \Leftrightarrow \left(x+\frac{2\big(s(K_{n_{k_m}})-s(J_{n_{k_m}})\big)}{|K_{n_{k_m}}|}\right) \in \\
&\qquad \Big(A-s(J_{n_{k_m}})\Big)\frac{2l_0}{|J_{n_{k_m}}|}\cap\Big[2i_0-2-l_0, 2i_0-l_0\Big] \Leftrightarrow \\
&\qquad \left(x+\frac{2\big(s(K_{n_{k_m}})-s(J_{n_{k_m}})\big)}{|K_{n_{k_m}}|}\right) \in \Big(A-s(J_{n_{k_m}})\Big)\frac{2}{|K_{n_{k_m}}|}\cap \\
&\qquad \left[\frac{2\big(s(K_{n_{k_m}})-s(J_{n_{k_m}})\big)}{|K_{n_{k_m}}|}-1, \frac{2\big(s(K_{n_{k_m}})-s(J_{n_{k_m}})\big)}{|K_{n_{k_m}}|}+1\right] \Leftrightarrow \\
&\qquad x \in \Big(A-s(K_{n_{k_m}})\Big)\frac{2}{|K_{n_{k_m}}|}\cap[-1,1] \Leftrightarrow h(A, K_{n_{k_m}})(x)=1.
\end{aligned}
$$

Therefore, we obtain that

$$h(A, K_{n_{k_m}})(x) \xrightarrow[m\to\infty]{\mathcal{I}} \chi_{[-1,1]},$$

and finally we have proved that $0 \in \Phi_{\mathcal{I}(K)}(A)$. □

Observe that, if in the above theorem we take sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ such that $0 < b_{n+1} < a_n$, for every $n \in \mathbb{N}$, $l_0 > 1$, then we obtain that $\mathcal{T}_{\mathcal{I}(J)} \neq \mathcal{T}_{\mathcal{I}(K)}$. It suffices to take the set

$$A = \{0\} \cup \bigcup_{n\in\mathbb{N}} int(K_n).$$

Then, $A \in \mathcal{T}_{\mathcal{I}(K)}$ and $0 \notin \Phi_{\mathcal{I}(J)}(A)$.

Moreover, if $0 < b_{n+1} < a_n$, and we put $K_n = \big[a_n, a_n + \frac{1}{n}(b_n - a_n)\big]$, then we obtain that the inclusion $\mathcal{T}_{\mathcal{I}(J)} \subset \mathcal{T}_{\mathcal{I}(K)}$ does not hold. Indeed, let us put

$$A = \{0\} \cup \left(\bigcup_{n\in\mathbb{N}} int(J_n) \setminus \bigcup_{n\in\mathbb{N}} K_n\right).$$

Then $A \in \mathcal{T}_{\mathcal{I}(J)}$, and $0 \notin \Phi_{\mathcal{I}(K)}(A)$.

Theorem 7. *Let $J = \{J_n\}_{n\in\mathbb{N}}$, where $J_n = [a_n, b_n]$, be a sequence tending to zero. Let us put*

$$K_n^i = \left[a_n + \frac{i-1}{l}(b_n - a_n), a_n + \frac{i}{l}(b_n - a_n)\right],$$

for $n \in \mathbb{N}$, $l \in \mathbb{N}$, $i \in \{1, \dots, l\}, n \in \mathbb{N}$. Then the family $\{K_n^i\}_{i\in\{1,\dots,l\},n\in\mathbb{N}}$ ordered in the sequence

$$K = \{K_1^1, K_1^2, \dots K_1^l, K_2^1, K_2^2, \dots, K_2^l, \dots\}$$

tends to zero and $\mathcal{T}_{\mathcal{I}(J)} = \mathcal{T}_{\mathcal{I}(K)}$.

PROOF. The fact that K tends to zero is the consequence of the assumption that the sequence $J = \{J_n\}_{n\in\mathbb{N}}$ tends to zero . The inclusion $\mathcal{T}_{\mathcal{I}(J)} \subset \mathcal{T}_{\mathcal{I}(K)}$ follows from the previous theorem.

It suffices to show the reverse inclusion. Let $A \in \mathcal{S}$ be such a set that $0 \in \Phi_{\mathcal{I}(K)}(A)$. We will prove that $0 \in \Phi_{\mathcal{I}(J)}(A)$. Let $\{n_k\}_{k\in\mathbb{N}}$ be a sequence of natural numbers. From the assumption there exists a subsequence $\{n_{1,k}\}_{k\in\mathbb{N}} \subset \{n_k\}_{k\in\mathbb{N}}$ and a set $\Theta_1 \in \mathcal{I}$ such that

$$h(A, K^1_{n_{1,k}})(x) \underset{k\to\infty}{\longrightarrow} \chi_{[-1,1]}(x) \text{ for } x \notin \Theta_1.$$

Similarly, for every $i \in \{2, \dots, l\}$, we find a subsequence $\{n_{i,k}\}_{k\in\mathbb{N}}$ of $\{n_{i-1,k}\}_{k\in\mathbb{N}}$ and such a set $\Theta_i \in \mathcal{I}$ that

$$h(A, K^i_{n_{i,k}})(x) \underset{k\to\infty}{\longrightarrow} \chi_{[-1,1]}(x) \text{ for } x \notin \Theta_i.$$

Let $\{n_{k_m}\}_{m\in\mathbb{N}} = \{n_{l,k}\}_{k\in\mathbb{N}}$. Then for every $x \notin \bigcup_{i=1}^l \Theta_i$ and $j \in \{1, \dots, l\}$,

$$h(A, K^j_{n_{k_m}})(x) \underset{m\to\infty}{\longrightarrow} \chi_{[-1,1]}(x). \tag{2}$$

Let us put

$$\Theta = \bigcup_{i=1}^{l} \left(\frac{1}{l}\Theta_i - 1 + \frac{2(i-1)+1}{l}\right) \cup \{1\}.$$

Obviously, $\Theta \in \mathcal{I}$.

We will show that for every $x \notin \Theta$, we have that

$$h(A, J_{n_{k_m}})(x) \underset{m\to\infty}{\longrightarrow} \chi_{[-1,1]}(x).$$

It is sufficient to consider the case when $x \in [-1, 1] \setminus \Theta$. Then there exists i_0 such that

$$\frac{2(i_0 - 1)}{l} - 1 \le x < \frac{2i_0}{l} - 1.$$

Observe also that

$$x \in \left[\frac{2(i_0-1)}{l}-1, \frac{2i_0}{l}-1\right] \setminus \Theta \Rightarrow \left(x-\frac{2(i_0-1)+1}{l}+1\right) l \in [-1,1] \setminus \Theta_{i_0}.$$

From condition (2), there exists a natural number $m_x \in \mathbb{N}$ such that for every $m \geq m_x$, we have that

$$h(A, K^{i_0}_{n_{k_m}})\left(\left(x-\frac{2(i_0-1)+1}{l}+1\right)l\right) = 1.$$

Simultaneously,

$$\begin{aligned}
&h(A, K^{i_0}_{n_{k_m}})\left(\left(x-\frac{2(i_0-1)+1}{l}+1\right)l\right) = 1 \Leftrightarrow \\
&\quad\left(\left(x-\frac{2(i_0-1)+1}{l}+1\right)l\right) \in \left(A - s(K^{i_0}_{n_{k_m}})\right)\frac{2}{|K^{i_0}_{n_{k_m}}|} \cap [-1,1] \Leftrightarrow \\
&\quad\left(x-\frac{2(i_0-1)+1}{l}+1\right) \in \left(A - s(K^{i_0}_{n_{k_m}})\right)\frac{2}{l|K^{i_0}_{n_{k_m}}|} \cap \left[\frac{-1}{l}, \frac{1}{l}\right] \Leftrightarrow \\
&\quad\left(x-\frac{2(i_0-1)+1}{l}+1\right) \in \left(A - s(K^{i_0}_{n_{k_m}})\right)\frac{2}{|J_{n_{k_m}}|} \cap \left[\frac{-1}{l}, \frac{1}{l}\right] \Leftrightarrow \\
&\quad x \in \left(A - s(K^{i_0}_{n_{k_m}}) + \left(\frac{2(i_0-1)+1}{l}-1\right)\frac{|J_{n_{k_m}}|}{2}\right)\frac{2}{|J_{n_{k_m}}|} \cap \\
&\quad\left[\frac{-1}{l} + \left(\frac{2(i_0-1)+1}{l}-1\right), \frac{1}{l} + \left(\frac{2(i_0-1)+1}{l}-1\right)\right].
\end{aligned}$$

From the equality,

$$s(K^{i_0}_{n_{k_m}}) = s(J_{n_{k_m}}) - \frac{|J_{n_{k_m}}|}{2}\left(1-\frac{2(i_0-1)+1}{l}\right),$$

we obtain that the last assertion is equivalent to the following one:

$$\begin{aligned}
&x \in \left(A - s(J_{n_{k_m}})\right)\frac{2}{|J_{n_{k_m}}|} \cap \left[\frac{2(i_0-1)}{l}-1, \frac{2(i_0)}{l}-1\right] \Leftrightarrow \\
&x \in \left(A - s(J_{n_{k_m}})\right)\frac{2}{|J_{n_{k_m}}|} \cap [-1,1] \Leftrightarrow h(A, J_{n_{k_m}})(x) = 1.
\end{aligned}$$

So for every $m \geq m_x$, we have that

$$h(A, J_{n_{k_m}})(x) = 1.$$

It implies that $h(A, J_{n_{k_m}})(x) \underset{m\to\infty}{\longrightarrow} \chi_{[-1,1]}(x)$ for $x \notin \Theta$. Therefore, $0 \in \Phi_{\mathcal{I}(J)}(A)$. $\square$

The next theorem gives us an example of a situation when the topologies generated by sequences of intervals are identical.

Theorem 8. *Let $J = \{J_n\}_{n\in\mathbb{N}}$ and $K = \{K_n\}_{n\in\mathbb{N}}$ be sequences of intervals tending to zero. If*

$$\lim_{n\to\infty} \frac{|J_n \triangle K_n|}{|J_n \cap K_n|} = 0,$$

then $\quad \mathcal{T}_{\mathcal{I}(J)} = \mathcal{T}_{\mathcal{I}(K)}$.

Proof. Let $A \in \mathcal{S}$ be a set such that $0 \in \Phi_{\mathcal{I}(J)}(A)$. We show that $0 \in \Phi_{\mathcal{I}(K)}(A)$. Let us take a sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers. From the assumption that $0 \in \Phi_{\mathcal{I}(J)}(A)$, there exists a subsequence $\{n_{k_m}\}_{m\in\mathbb{N}}$ and a set $\Theta_1 \in \mathcal{I}$ such that

$$h(A, J_{n_{k_m}})(x) \underset{k\to\infty}{\longrightarrow} \chi_{[-1,1]}(x) \text{ for } x \notin \Theta_1. \tag{3}$$

It is easy to see that

$$\Theta = \{-1,1\} \cup \bigcup_{m\in\mathbb{N}} \left(\Theta_1 \frac{|J_{n_{k_m}}|}{|K_{n_{k_m}}|} - \frac{2(s(K_{n_{k_m}}) - s(J_{n_{k_m}}))}{|K_{n_{k_m}}|} \right) \in \mathcal{I}.$$

From the assumption, we obtain that

$$\frac{|K_n|}{|J_n|} \underset{n\to\infty}{\longrightarrow} 1 \qquad \wedge \qquad \frac{2\Big(s(K_n) - s(J_n)\Big)}{|K_n|} \underset{n\to\infty}{\longrightarrow} 0.$$

Indeed,

$$\begin{aligned} 1 = \lim_{n\to\infty} \frac{|J_n \cap K_n|}{|J_n \cap K_n|} &= \limsup_{n\to\infty} \frac{|J_n \setminus (J_n \triangle K_n)|}{|J_n \cap K_n|} \geq \\ \limsup_{n\to\infty} \frac{|J_n|}{|J_n \cap K_n|} - \lim_{n\to\infty} \frac{|J_n \triangle K_n|}{|J_n \cap K_n|} &= \limsup_{n\to\infty} \frac{|J_n|}{|J_n \cap K_n|} \geq \limsup_{n\to\infty} \frac{|J_n|}{|K_n|}. \end{aligned}$$

Similarly, we can show that

$$\limsup_{n\to\infty} \frac{|K_n|}{|J_n|} \leq 1,$$

and hence,

$$\liminf_{n\to\infty} \frac{|J_n|}{|K_n|} \geq 1.$$

Thus, we obtain that

$$\lim_{n\to\infty}\frac{|J_n|}{|K_n|}=1.$$

Suppose conversely that there exists $1>\alpha>0$ and a natural number $n_0\in\mathbb{N}$ such that for every $n\geq n_0$, we have

$$\frac{|s(K_n)-s(J_n)|}{|K_n|}>\alpha.$$

There exists $n_1\geq n_0$ such that for every $n\geq n_1$, we have

$$(1-\alpha)|K_n|<|J_n|<(1+\alpha)|K_n|.$$

Then we obtain that

$$\frac{|J_n\bigtriangleup K_n|}{|J_n\cap K_n|}\geq\frac{\alpha|K_n|}{|K_n|}=\alpha.$$

It contradicts with the assumption of the theorem. Indeed, if intervals J_n, K_n are mutually disjoint, then for every $n\in\mathbb{N}$, we have $|J_n\bigtriangleup K_n|\geq|K_n|$. In the opposite case, we have $|J_n\bigtriangleup K_n|\geq|s(K_n)-s(J_n)|$.

Let $x\in(-1,1)\setminus\Theta$. For sufficiently large $m\in\mathbb{N}$, we have that

$$x\in\left[-\frac{|J_{n_{k_m}}|}{|K_{n_{k_m}}|}+\frac{2\big(s(J_{n_{k_m}})-s(K_{n_{k_m}})\big)}{|K_{n_{k_m}}|},\frac{|J_{n_{k_m}}|}{|K_{n_{k_m}}|}+\frac{2\big(s(J_{n_{k_m}})-s(K_{n_{k_m}})\big)}{|K_{n_{k_m}}|}\right].$$

Therefore,

$$\frac{|K_{n_{k_m}}|}{|J_{n_{k_m}}|}\left(x+\frac{2\big(s(K_{n_{k_m}})-s(J_{n_{k_m}})\big)}{|K_{n_{k_m}}|}\right)\in[-1,1]\setminus\Theta_1.$$

From condition (3), there exists a natural number $m_0\geq m_1$ such that for every $m\geq m_0$, we have that

$$h(A,J_{n_{k_m}})\left(\frac{|K_{n_{k_m}}|}{|J_{n_{k_m}}|}\left(x+\frac{2\big(s(K_{n_{k_m}})-s(J_{n_{k_m}})\big)}{|K_{n_{k_m}}|}\right)\right)=1.$$

Simultaneously,

$$h(A,J_{n_{k_m}})\left(\frac{|K_{n_{k_m}}|}{|J_{n_{k_m}}|}\left(x+\frac{2\big(s(K_{n_{k_m}})-s(J_{n_{k_m}})\big)}{|K_{n_{k_m}}|}\right)\right)=1 \Leftrightarrow$$

$$\frac{|K_{n_{k_m}}|}{|J_{n_{k_m}}|}\left(x+\frac{2\big(s(K_{n_{k_m}})-s(J_{n_{k_m}})\big)}{|K_{n_{k_m}}|}\right)\in\big(A-s(J_{n_{k_m}})\big)\frac{2}{|J_{n_{k_m}}|}\cap[-1,1]$$

$$\Leftrightarrow\left(\frac{|K_{n_{k_m}}|}{2}x+s(K_{n_{k_m}})-s(J_{n_{k_m}})\right)\in\big(A-s(J_{n_{k_m}})\big)$$

$$\cap\left[-\frac{|J_{n_{k_m}}|}{2},\frac{|J_{n_{k_m}}|}{2}\right]\Leftrightarrow\left(\frac{|K_{n_{k_m}}|}{2}x+s(K_{n_{k_m}})\right)$$

$$\in A\cap\left[-\frac{|J_{n_{k_m}}|}{2}+s(J_{n_{k_m}}),\frac{|J_{n_{k_m}}|}{2}+s(J_{n_{k_m}})\right]$$

$$\Leftrightarrow\frac{|K_{n_{k_m}}|}{2}x\in\big(A-s(K_{n_{k_m}})\big)\cap$$

$$\left[-\frac{|J_{n_{k_m}}|}{2}+s(J_{n_{k_m}})-s(K_{n_{k_m}}),\frac{|J_{n_{k_m}}|}{2}+s(J_{n_{k_m}})-s(K_{n_{k_m}})\right]$$

$$\Leftrightarrow x\in\big(A-s(K_{n_{k_m}})\big)\frac{2}{|K_{n_{k_m}}|}\cap$$

$$\left[-\frac{|J_{n_{k_m}}|}{|K_{n_{k_m}}|}+\frac{2\big(s(J_{n_{k_m}})-s(K_{n_{k_m}})\big)}{|K_{n_{k_m}}|},\frac{|J_{n_{k_m}}|}{|K_{n_{k_m}}|}+\frac{2(s(J_{n_{k_m}})-s(K_{n_{k_m}}))}{|K_{n_{k_m}}|}\right]$$

$$\Rightarrow x\in\big(A-s(K_{n_{k_m}})\big)\frac{2}{|K_{n_{k_m}}|}\cap[-1,1].$$

Hence, we obtain that $h(A,K_{n_{k_m}})(x)=1$ for $m\geq m_0$, so

$$h(A,K_{n_{k_m}})(x)\underset{m\to\infty}{\longrightarrow}\chi_{[-1,1]}\text{ for }x\notin\Theta.$$

It follows that $0\in\Phi_{\mathcal{I}(K)}(A)$.

Similarly, we prove that $\mathcal{T}_{\mathcal{I}(K)}\subset\mathcal{T}_{\mathcal{I}(J)}$. This ends the proof. $\square$

The following theorems show some properties of the family of $\mathcal{I}(J)$ type topologies. Let $\mathcal{J}$ be the family of all sequences tending to zero.

Theorem 9. *Let $\mathcal{T}_H=\{V\setminus P : V$ is open, $P\in\mathcal{I}\}$. Then*

$$\bigcap_{J\in\mathcal{J}}\mathcal{T}_{\mathcal{I}(J)}=\mathcal{T}_H.$$

Proof. It is obvious that $\mathcal{T}_H \subset \bigcap_{J\in\mathcal{J}} \mathcal{T}_{\mathcal{I}(J)}$. Let us suppose to the contrary that there exists a set $A \in \bigcap_{J\in\mathcal{J}} \mathcal{T}_{\mathcal{I}(J)} \setminus \mathcal{T}_H$. Observe that for every set $A \in \mathcal{S}$, there exist sets G, F such that G is regular open, $F \in \mathcal{I}$, and $A = G \triangle F$. Since $A \notin \mathcal{T}_H$, there exists a point $x_0 \in F \setminus G$. Let $B = (G \setminus F) \cup \{x_0\}$. We have that $B \in \bigcap_{J\in\mathcal{J}} \mathcal{T}_{\mathcal{I}(J)}$ because $B \triangle A \in \mathcal{I}$ and $B \subset A$. If $x_0 \notin FrG$, then $d(x_0, G) > 0$ and for every sequence $J = \{J_n\}_{n\in\mathbb{N}} \in \mathcal{J}$, there exists n_0 such that for every $n \geq n_0$, we have $(B - x_0) \cap J_n \subset \{0\}$. Hence, x_0 is not an $\mathcal{I}(J)-$density point of B, for any sequence $J \in \mathcal{J}$ and $B \notin \bigcap_{J\in\mathcal{J}} \mathcal{T}_{\mathcal{I}(J)}$. If $x_0 \in FrG$, then there exists a sequence $J = \{J_n\}_{n\in\mathbb{N}} \in \mathcal{J}$ such that $J_n \subset \mathbb{R} \setminus (G - x_0)$ for every $n \in \mathbb{N}$. Therefore, we have $(B - x_0) \cap J_n \subset \{0\}$ for every $n \geq n_0$. In this case, we obtain that x_0 is not an $\mathcal{I}(J)$-density point of B and $B \notin \bigcap_{J\in\mathcal{J}} \mathcal{T}_{\mathcal{I}(J)}$. □

Theorem 10. *Let $\mathcal{T}^*$ be the topology generated by $\bigcup_{J\in\mathcal{J}} \mathcal{T}_{\mathcal{I}(J)}$. Then*

$$\mathcal{T}^* = 2^{\mathbb{R}} \quad \text{and} \quad \bigcup_{J\in\mathcal{J}} \mathcal{T}_{\mathcal{I}(J)} \neq \mathcal{T}^*.$$

Proof. From theorem 6 in [2], it follows that $\mathcal{T}^* = 2^{\mathbb{R}}$. It is obvious that $\bigcup_{J\in\mathcal{J}} \mathcal{T}_{\mathcal{I}(J)} \neq \mathcal{T}^*$ because $\{0\} \notin \mathcal{T}_{\mathcal{I}(J)}$ for any $J \in \mathcal{J}$. □

Theorem 11. *For any sequence J of intervals tending to zero, the space $(\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)})$ is Hausdorff but not regular.*

Proof. Let J be a sequence of intervals tending to zero. The fact that the space $(\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)})$ is Hausdorff is evident because $\mathcal{T}_{\mathcal{I}(J)}$ contains the natural topology. The concept of a proof that $(\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)})$ is not regular is based on the proof of theorem 2.6.2 from [1]. Observe that if $V \in \mathcal{T}_{\mathcal{I}(J)}$ is a dense set in the natural topology, then V is residual. Suppose to the contrary that $X \setminus V$ is second category. Then there exists a nonempty open set G such that $G \triangle (X \setminus V) \in \mathcal{I}$. Hence, $G \subset \Phi_{\mathcal{I}(J)}(G) = \Phi_{\mathcal{I}(J)}(X \setminus V) \neq \emptyset$. So we obtain that $V \cap \Phi_{\mathcal{I}(J)}(X \setminus V) \subset \Phi_{\mathcal{I}(J)}(V) \cap \Phi_{\mathcal{I}(J)}(X \setminus V) = \emptyset$. On the other hand, V is a dense set so $\emptyset \neq V \cap G \subset V \cap \Phi_{\mathcal{I}(J)}(X \setminus V)$. This contradiction proves that the set $X \setminus V$ must be of the first category. Finally, we have that the set of all rational numbers $\mathbb{Q}$, which is dense with respect to $\mathcal{T}_{nat}$ and $\mathcal{T}_{\mathcal{I}(J)}$-closed, cannot be separated from any point $x \in \mathbb{R} \setminus \mathbb{Q}$ with respect to topology $\mathcal{T}_{\mathcal{I}(J)}$. □

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