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ON THE INDEPENDENCE OF A GENERALIZED STATEMENT OF EGOROFF'S THEOREM FROM ZFC AFTER T. WEISS

Abstract

We consider a generalized version (GES) of the well-known Severini–Egoroff theorem in real analysis, first shown to be undecidable in ZFC by Tomasz Weiss in [6]. This independence is easily derived from suitable hypotheses on some cardinal characteristics of the continuum like $\mathfrak{b}$ and $\mathrm{non}(\mathcal{N})$.

In this paper, we will consider the following *Generalized Egoroff Statement*, which is a version "without regularity assumptions" of the well-known Severini–Egoroff theorem from real analysis:

GES Given a sequence $(f_n : n \in \mathbb{N})$ of arbitrary functions $[0,1] \to \mathbb{R}$ converging pointwise to 0, for each $\eta > 0$ there is a subset $A \subseteq [0,1]$ of outer measure $\mu^*(A) > 1 - \eta$ such that (f_n) converges uniformly on A.

This conjecture first emerged from some questions about the behaviour of bounded harmonic functions on the unit disc in $\mathbb{C}$; in particular, it has been used in [2] to show the independence from ZFC of a strong Littlewood-type statement about tangential approaches. In [6], the author shows the following:

1. In a model M obtained by an $\aleph_2$-iteration with countable supports of Laver forcing over a countable standard model M_0 of $\mathsf{ZFC} + \mathsf{CH}$, GES holds [6, theorem 1].

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2. If the continuum hypothesis CH holds [6, theorem 2], and more generally if $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$ [6, theorem 7], then there is a counterexample to GES (here $\mathfrak{c}$ is the cardinality of the continuum and $\mathrm{add}(\mathcal{M})$ is the additivity of the ideal of first category subsets of $\mathbb{R}$).

The aim of this paper is to give a simple combinatorial version of GES (theorem 5), which can be then applied to prove either GES or ¬GES under suitable hypotheses on the cardinal characteristics of the continuum: corollary 7 shows the consistency of GES without an "ad-hoc" analysis of forcing extensions, whereas proposition 8 and corollary 9 provide finer criteria for the failure of GES, also involving $\mathfrak{c}$-Lusin sets.

There are other variants and generalizations of the Egoroff theorem which have been studied in the literature. Nakano and Luxemburg isolated simple combinatorial properties of Boolean algebras and of Riesz spaces which are related to the classical Egoroff theorem; the simplest form in the case of a Boolean algebra $\mathscr{B}$ is the following "Egoroff property":

EP Given a doubly indexed sequence (b_k^n) of elements of $\mathscr{B}$, such that $(b_k^n)_k$ converges monotonically to 1 for all n, there is a sequence (b_m) converging monotonically to 1 and such that $\forall m, n \exists k\ (b_m \leq b_k^n)$.

The article [5] explains in detail these abstract Egoroff conditions and presents a proof of the equivalence of the following two statements:

i. The Severini–Egoroff theorem holds for a measure space $(X, \mathcal{E}, \mu)$.

ii. The measure algebra of $(X, \mathcal{E}, \mu)$ (quotient of the σ-field $\mathcal{E}$ modulo the ideal of μ-nullsets) has the EP.

Forgetting about measures, in [4] the authors completely characterize the sets X for which the field of all subsets of X has the EP: these are precisely the sets of cardinality $|X| < \mathfrak{b}$. Our approach uses some ideas which are similar to the ones exploited in the proofs cited above, although our combinatorial translation of GES is directly adapted for outer measures: to every pointwise converging sequence of functions $X \to \mathbb{R}$, one associates an X-indexed family of sequences $\mathbb{N} \to \mathbb{N}$ (via the order of convergence), and then one has to study *all* subfamilies indexed by sets $Y \subseteq X$ of maximal outer measure $\mu^*(Y) = \mu^*(X)$.

As a preliminary remark, notice that in GES it is necessary to consider Lebesgue *outer* measure to avoid simple counterexamples in ZFC:

Proposition 1. *There is a decreasing sequence $(f_n : n \in \mathbb{N})$ of functions $[0,1] \to \mathbb{R}$, converging pointwise to zero, such that every subset $A \subseteq [0,1]$ on which (f_n) converges uniformly has Lebesgue inner measure zero.*

PROOF. By a theorem of Lusin and Sierpiński, there exists a partition of $[0,1]$ into countably many (in fact, even continuum many) pieces $\{B_n : n \in \mathbb{N}\}$ each having full outer measure. Consider then the sequence (f_n) where, for every $n \in \mathbb{N}$, f_n is the characteristic function of the subset $B_{\geq n} = \bigcup_{k \geq n} B_k$ of the unit interval. Clearly, $(f_n(x))$ converges monotonically to zero on every point $x \in [0,1]$; if (f_n) converges uniformly on a subset A, A has to be disjoint from $B_{\geq \bar{n}}$ for some $\bar{n} \in \mathbb{N}$, so $\mu_*(A) \leq 1 - \mu^*(B_{\geq \bar{n}}) = 0$. □

Fix once and for all a decreasing vanishing sequence $\varepsilon = (\varepsilon_n)_{n \in \mathbb{N}}$ of positive real numbers, e.g., $\varepsilon_n = 2^{-n}$; consider the following function, mapping a sequence of reals to its (ε-)*order of convergence* to zero:

$$\begin{aligned} \mathrm{oc} \ : \ c_0 \ &\to \ {}^{\mathbb{N}}\mathbb{N}{\uparrow}, \quad \text{defined on each } a = (a_n) \in c_0 \text{ as} \\ (\mathrm{oc}\, a)_n \ &= \ \min\left\{m \ : \ \forall l \geq m \ \left(|a_l| \leq \varepsilon_n\right)\right\}, \end{aligned} \tag{1}$$

where c_0 denotes the set of infinitesimal real-valued sequences and ${}^{\mathbb{N}}\mathbb{N}{\uparrow} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is the set of nondecreasing sequences of natural numbers.

Using the natural identification of ${}^{\mathbb{N}}({}^{X}\mathbb{R})$ with ${}^{X}({}^{\mathbb{N}}\mathbb{R})$, we can view a sequence of real-valued functions $X \to \mathbb{R}$ converging pointwise to zero as a single function $F : X \to c_0$, and then study the associated order of convergence, $\mathrm{oc}\, F = \mathrm{oc} \circ F : X \to {}^{\mathbb{N}}\mathbb{N}{\uparrow}$.

Lemma 2. *F converges uniformly to zero if and only if the range of $\mathrm{oc}\, F$ is bounded in $({}^{\mathbb{N}}\mathbb{N}, \leq)$, where $\leq$ is the partial order of everywhere domination: $\alpha \leq \beta$ iff $\forall n \ (\alpha_n \leq \beta_n)$.*

PROOF. This is just a restatement of the definition of uniform convergence:

$$\begin{aligned} &F \text{ converges uniformly to } 0 \quad \leftrightarrow \\ \leftrightarrow \quad &\forall n \ \exists m \ \forall x \in X \ \forall l \geq m \ \left(|F_l(x)| \leq \varepsilon_n\right) \quad \leftrightarrow \\ \leftrightarrow \quad &\exists\, (m_n) \in {}^{\mathbb{N}}\mathbb{N} \ \forall n \ \forall x \in X \ \left((\mathrm{oc}\, F(x))_n \leq m_n\right). \end{aligned}$$

□

Lemma 3. *For all $\varphi : X \to {}^{\mathbb{N}}\mathbb{N}{\uparrow}$, there exists a sequence F of real-valued functions on X converging pointwise to 0 with order $\mathrm{oc}\, F = \varphi$.*

PROOF. It is sufficient to prove the lemma pointwise. Given a nondecreasing sequence of natural numbers $\alpha \in {}^{\mathbb{N}}\mathbb{N}{\uparrow}$, we construct a sequence $a \in c_0$ converging to 0 with order α. For that, just let

$$a = (a_n)_{n \in \mathbb{N}} \quad \text{where} \quad a_n \ = \ \inf\{\varepsilon_k \ : \ \alpha_k \leq n\};$$

it is straightforward to check that this works; i.e., $\mathrm{oc}\, a = \alpha$. □

Let μ^* be an upward continuous outer measure on a set X; i.e., an outer measure $\operatorname{Pow} X \to [0, +\infty]$ satisfying

$$A = \bigcup_{n\in\mathbb{N}} A_n \quad \to \quad \mu^*(A) = \lim_{n\to\infty} \mu^*\Big(\bigcup_{k<n} A_k\Big).$$

For every sequence F of real-valued functions on X converging pointwise to zero, consider the statement

$\mathsf{GES}(X, \mu^*, F)$ for each $M < \mu^*(X)$, there is a subset $A \subseteq X$ such that $\mu^*(A) > M$ and F converges uniformly on A.

The Generalized Egoroff Statement relative to the space (X, μ^*) is the formula

$$\mathsf{GES}(X, \mu^*) = \forall F \ \mathsf{GES}(X, \mu^*, F).$$

Clearly, our original statement GES is just $\mathsf{GES}([0,1], m^*)$, where m^* is Lebesgue outer measure on the unit interval $[0,1] \subseteq \mathbb{R}$. Denote by $\mathcal{K}_\sigma$ the σ-ideal generated by the bounded subsets of $({}^{\mathbb{N}}\mathbb{N}, \leq)$; equivalently, $\mathcal{K}_\sigma$ is the family of those subsets which are bounded with respect to the order $\leq^*$ of eventual domination,

$$\alpha \leq^* \beta \ \leftrightarrow \ \forall^\infty n \ (\alpha_n \leq \beta_n) \ \leftrightarrow \ \exists n \ \forall k \geq n \ (\alpha_k \leq \beta_k) \qquad (\alpha, \beta \in {}^{\mathbb{N}}\mathbb{N}),$$

and $\mathcal{K}_\sigma$ is also the σ-ideal generated by the compact subsets of the Baire space ${}^{\mathbb{N}}\mathbb{N}$ (see [3]).

Lemma 4. $\mathsf{GES}(X, \mu^*, F)$ *holds iff there is a subset* $Y \subseteq X$ *such that* $\mu^*(Y) = \mu^*(X)$ *and* $\operatorname{oc} F[Y] \in \mathcal{K}_\sigma$.

Proof. Fix an increasing sequence of positive real numbers (M_n) with limit $\mu^*(X)$. Assume $\mathsf{GES}(X, \mu^*, F)$: by lemma 2, for every $n \in \mathbb{N}$, there is a subset $A_n \subseteq X$ such that $\mu^*(A_n) > M_n$ and $\operatorname{oc} F[A_n]$ is bounded in ${}^{\mathbb{N}}\mathbb{N}$. Taking $Y = \bigcup_{n\in\mathbb{N}} A_n$, Y has outer measure equal to $\mu^*(X)$ and $\operatorname{oc} F[Y] = \bigcup_{n\in\mathbb{N}} \operatorname{oc} F[A_n]$ is σ-bounded, as required. Conversely, suppose that $\mu^*(Y) = \mu^*(X)$ and $\operatorname{oc} F[Y] \subseteq \bigcup_{n\in\mathbb{N}} B_n$, where each B_n is a bounded subset of $({}^{\mathbb{N}}\mathbb{N}, \leq)$, and put

$$A_n = (\operatorname{oc} F)^{-1}[B_0 \cup \ldots \cup B_{n-1}].$$

Since $\operatorname{oc} F[A_n]$ is bounded, F converges uniformly on every A_n (lemma 2); moreover, as μ^* is continuous and $Y \subseteq \bigcup_{n\in\mathbb{N}} A_n$, for all m there is some n such that $\mu^*(A_n) > M_m$, that is, $\mathsf{GES}(X, \mu^*, F)$ holds. □

Theorem 5. $\mathsf{GES}(X, \mu^*)$ *holds if and only if for all functions* $\varphi : X \to {}^{\mathbb{N}}\mathbb{N}$, *there is a subset* $Y \subseteq X$ *such that* $\mu^*(Y) = \mu^*(X)$ *and* $\varphi[Y] \in \mathcal{K}_\sigma$.

This theorem provides a translation of GES into a purely set-theoretical statement.

Proof. The "if" direction follows directly from lemma 4 using $\varphi = \operatorname{oc} F$. For the converse, consider the function Θ which maps a sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ to the nondecreasing sequence $\left(\sum_{k\leq n} \alpha_k\right)_{n\in\mathbb{N}}$: it is a bijective order morphism $({}^{\mathbb{N}}\mathbb{N}, \leq) \to ({}^{\mathbb{N}}\mathbb{N}{\uparrow}, \leq)$ satisfying $\alpha \leq \Theta(\alpha)$. Therefore, for all $Y \subseteq {}^{\mathbb{N}}\mathbb{N}$, $\Theta[Y]$ is (σ-)bounded iff Y is (σ-)bounded. Assume $\mathsf{GES}(X, \mu^*)$ and let φ be a function $X \to {}^{\mathbb{N}}\mathbb{N}$. By lemma 3, there exists a sequence F of real-valued functions converging pointwise to 0 with $\operatorname{oc} F = \Theta \circ \varphi$, so there is a set $Y \subseteq X$ of such that $\mu^*(Y) = \mu^*(X)$ and $\Theta[\varphi[Y]] = \operatorname{oc} F[Y] \in \mathcal{K}_\sigma$ (lemma 4); i.e., $\varphi[Y] \in \mathcal{K}_\sigma$ as desired. $\square$

Remark. Theorem 5 is still valid for measure spaces (X, μ) and the classical Egoroff Statement, provided that we only consider measurable maps φ and measurable subsets $Y \subseteq X$. Thus, theorem 5 entails the Severini–Egoroff theorem: if μ is finite and $\varphi : X \to {}^{\mathbb{N}}\mathbb{N}$ is measurable, the image measure $\varphi_*\mu$ is a finite Borel measure on ${}^{\mathbb{N}}\mathbb{N}$, hence it is regular, and it is always supported by a σ-compact subset.

Recall that the *bounding number* $\mathfrak{b} = \operatorname{non}(\mathcal{K}_\sigma)$ (see [3]) is the smallest possible size of a subset of ${}^{\mathbb{N}}\mathbb{N}$ not belonging to $\mathcal{K}_\sigma$. Let us also denote with $\mathfrak{o}(X, \mu^*)$ the least cardinality of a subset of X having outer measure equal to $\mu^*(X)$; when $X = [0, 1]$ and $\mu^* = m^*$ (Lebesgue outer measure), this cardinal is well-known.

Lemma 6. $\mathfrak{o}([0, 1], m^*) = \operatorname{non}(\mathcal{N})$*, that is,* $\mathfrak{o}([0, 1], m^*)$ *is the least size of a Lebesgue non-null subset of* $[0, 1]$.

Proof. The inequality $\operatorname{non}(\mathcal{N}) \leq \mathfrak{o}([0, 1], m^*)$ is obvious. Consider a non-null set $A \subseteq [0, 1]$ of size $\operatorname{non}(\mathcal{N})$; the sum (modulo 1) $\tilde{A} = A + \mathbb{Q}$ has cardinality $\operatorname{non}(\mathcal{N})$ too, and we claim that it is a set of full outer measure, thus proving the reverse inequality $\operatorname{non}(\mathcal{N}) \geq \mathfrak{o}([0, 1], m^*)$. In fact, let E be any measurable set containing $\tilde{A}$: the set

$$\tilde{E} = \bigcap_{q\in\mathbb{Q}} (q + E) \subseteq E$$

is measurable, $\mathbb{Q}$-invariant and non-null (since it contains $\tilde{A}$), so it has measure 1 by the Zero-One law (see [1]); it follows that E has measure 1 too, and therefore $m^*(\tilde{A}) = 1$. $\square$

Corollary 7. *Assuming* $\mathfrak{o}(X,\mu^*) < \mathfrak{b}$*,* $\mathsf{GES}(X,\mu^*)$ *holds. In particular,* $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$ *implies* GES[1].

Proof. Fix a subset $Y \subseteq X$ of maximal outer measure with $|Y| = \mathfrak{o}(X,\mu^*)$; then every function $\varphi : X \to {}^{\mathbb{N}}\mathbb{N}$ maps Y onto a set of cardinality less than $\mathfrak{b}$, hence $\varphi[Y] \in \mathcal{K}_\sigma$. □

We can also invoke theorem 5 to prove sufficient conditions for the failure of GES. Precisely, we infer $\neg\mathsf{GES}(X,\mu^*)$ by constructing (under suitable hypotheses) a set $Z \subseteq {}^{\mathbb{N}}\mathbb{N}$ of cardinality $|Z| \geq |X|$ such that all subsets of Z belonging to $\mathcal{K}_\sigma$ have size less than $\mathfrak{o}(X,\mu^*)$. Once this is achieved, if φ is any injection $X \to Z$, no subset $Y \subseteq X$ of outer measure $\mu^*(Y) = \mu^*(X)$ can be mapped onto an element of $\mathcal{K}_\sigma$, because $|\varphi[Y]| = |Y| \geq \mathfrak{o}(X,\mu^*)$. In order to state the next proposition, we recall that the *dominating number* $\mathfrak{d} \geq \mathfrak{b}$ is the least cardinality of a cofinal subset of $({}^{\mathbb{N}}\mathbb{N}, \leq^*)$ and that a *κ-Lusin set* is a subset $L \subseteq \mathbb{R}$ of cardinality κ whose meager (i.e., Baire first category) subsets have size less than κ.

Proposition 8. *Assume* $\mathfrak{o}(X,\mu^*) = |X| = \kappa$*; then* $\mathsf{GES}(X,\mu^*)$ *fails in each of the following cases:*

1. $\kappa = \mathfrak{b}$*;*
2. $\kappa = \mathfrak{d}$*;*
3. *there exists a* κ*-Lusin set.*

Proof. Following the plan outlined before stating the proposition, we try to build a "κ-Lusin set" Z for the ideal $\mathcal{K}_\sigma$ instead of the ideal of meager sets. This is automatic under hypothesis (3): every (true) κ-Lusin set has the required properties, since all compact subsets of ${}^{\mathbb{N}}\mathbb{N}$ have empty interior and thus every $\mathcal{K}_\sigma$ set is meager.

Assume $\kappa = \mathfrak{b}$ and let $\left\{\alpha^\xi\right\}_{\xi<\mathfrak{b}}$ be an unbounded family in $({}^{\mathbb{N}}\mathbb{N}, \leq^*)$. By transfinite recursion, we build a well ordered unbounded chain $Z = \left\{\beta^\xi\right\}_{\xi<\mathfrak{b}}$ of length $\mathfrak{b}$. After the construction of all β^η for $\eta < \xi$, pick β^ξ among the strict $\leq^*$-upper bounds of the set $\left\{\alpha^\xi\right\} \cup \left\{\beta^\eta\right\}_{\eta<\xi}$ (which has size less than $\mathfrak{b}$ and thus is $\leq^*$-bounded). It is clear that no $\leq^*$-bounded subset of Z can be cofinal in Z, hence all $\mathcal{K}_\sigma$ subsets of Z have cardinality $< \mathfrak{b}$.

[1] The fact that GES holds when there is a subset Y of $[0,1]$ of full outer measure and size less than $\mathfrak{b}$ has also been pointed out by I. Recław (see [6]). This is also a consequence of [4, theorem 2], since the field of all subsets of such a Y has the abstract Egoroff property EP, which means that for all pointwise converging sequences of functions $Y \to \mathbb{R}$, Y is the union of a countable increasing sequence of subsets Y_n where the convergence is uniform.

Finally, suppose $\kappa = \mathfrak{d}$ and let $\left\{\alpha^{\xi}\right\}_{\xi<\mathfrak{d}}$ be a cofinal family in $({}^{\mathbb{N}}\mathbb{N}, \leq^*)$. We build a set $Z = \left\{\beta^{\xi}\right\}_{\xi<\mathfrak{d}}$ of cardinality $\mathfrak{d}$ by transfinite recursion as follows: after the construction of all β^{η} for $\eta < \xi$, pick an element β^{ξ} which is not $\leq^*$ any element of the set $\{\alpha^{\eta}\}_{\eta\leq\xi} \cup \{\beta^{\eta}\}_{\eta<\xi}$ (which has size less than $\mathfrak{d}$ and thus is not $\leq^*$-cofinal). Z has the desired properties: $\left(\beta^{\xi}\right)_{\xi<\mathfrak{d}}$ is a sequence without repetitions, hence $|Z| = \mathfrak{d}$, and moreover, if $A \subseteq Z$ is in $\mathcal{K}_{\sigma}$, some α^{ξ} has to eventually dominate all elements of A, which implies that $A \subseteq \{\beta^{\eta}\}_{\eta<\xi}$ has cardinality less than $\mathfrak{d}$. □

Corollary 9. GES *fails whenever at least one of the following hypotheses is satisfied:*

1. $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$*;*
2. *there exists a* $\mathfrak{c}$*-Lusin set and* $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$*;*
3. *there exists a* $\mathfrak{c}$*-Lusin set and* $\mathfrak{c}$ *is a regular cardinal.*

The last two conditions provide an affirmative answer (at least when $\mathfrak{c}$ is regular or it coincides with $\mathrm{non}(\mathcal{N})$) to a question posed by T. Weiss about the failure of GES under the assumption that there are $\mathfrak{c}$-Lusin sets; he also noticed that there are models of ZFC (e.g. the iterated Mathias real model, where $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$) which contain no $\mathfrak{c}$-Lusin sets, but nevertheless satisfy ¬GES.

PROOF. Assumptions (1) and (2) are just particular instances of cases (2) and (3), respectively, of proposition 8. Moreover, hypothesis (3) is stronger than both (1) and (2). If κ is a regular cardinal and there is a κ-Lusin set, then $\mathrm{cov}(\mathcal{M}) \geq \kappa$, and thus $\min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\} \geq \mathrm{cov}(\mathcal{M}) \geq \kappa$ (see [1] for the relevant definitions of these cardinal characteristics associated to the σ-ideals $\mathcal{M}$ of meager sets and $\mathcal{N}$ of Lebesgue nullsets, as well as for the proofs in ZFC of the stated inequalities). □

Corollary 10 (T. Weiss)**.** GES *is undecidable in* ZFC*.*

PROOF. The hypothesis of corollary 7, and therefore GES, holds in the iterated Laver real model (see [1] and the proof of theorem 1 in [6]). On the other hand, $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$ is certainly true (thus ¬GES holds) under the Continuum Hypothesis CH or just Martin's Axiom MA, which are consistent with ZFC. □

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