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Rudolf Výborný, Department of Mathematics, University of Queensland, St. Lucia, Brisbane QLD 4067, Australia. email: `rvyborny@uq.edu.au`

A REMARK ON THE DEFINITION OF THE KURZWEIL-HENSTOCK INTEGRAL

Abstract

A useful modification of the definition of the Kurzweil integral is presented.

1 Introduction.

In the Kurzweil-Henstock theory of integration one often encounters a repetitive argument in dealing with an assumption which is satisfied except for a countable set. We propose an alternative definition of the integral which handles exceptional countable sets with ease.

2 The Necessary and Sufficient Condition.

A set of couples $D \equiv \{((x_i, x_{i+1}), \xi_{i+1});\ i = 0, \ldots, n-1\}$ is called tagged partial division of a compact interval $[a, b]$ if, for $i = 0, \ldots, n-1$, the points $\xi_{i+1} \in [x_i, x_{i+1}]$, the intervals $[x_i, x_{i+1}]$ are non-degenerate, non-overlapping and $[x_i, x_{i+1}] \subset [a, b]$. If $\cup_0^{n-1}[x_i, x_{i+1}] = [a, b]$, then the partial tagged division of $[a, b]$ becomes a tagged division of $[a, b]$. A positive function will be called a gauge, a non-negative function for which the set of zeros is countable (including finite sets and the empty set) will be called countably closed gauge. For a countably closed gauge ω a tagged partial division D is said to be ω-fine if

$$\xi_{i+1} - \omega(\xi_{i+1}) < x_i \leq \xi_{i+1} \leq x_{i+1} < \xi_{i+1} + \omega(\xi_{i+1})$$

for $i = 0, \ldots, n-1$.

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Theorem. *A function $f : [a,b] \to \mathbb{R}$ is Kurzweil-Henstock integrable if and only if there exists a continuous function F such that for every $\epsilon > 0$ there is a countably closed gauge η with the property that*

$$\sum_{i=0}^{n-1} |f(\xi_{i+1})(x_{i+1} - x_i) - (F(x_{i+1} - F(x_i))| < \epsilon \tag{1}$$

whenever the tagged partial division $D \equiv \{((x_i, x_{i+1}), \xi_{i+1});\ i = 0, \dots, n-1\}$ is η fine. If the condition is satisfied then $F(b) - F(a) = \int_a^b f$.

PROOF. I. If f is Kurzweil-Henstock integrable and F its indefinite integral, then inequality (1) is satisfied by the Henstock lemma (See [1] p. 81).

II. Let the condition be satisfied with η corresponding to $\epsilon/2$ rather than ϵ and r_1, r_2, $\dots, r_n$, $\dots$ be the enumeration of the zeros of η. Then for every positive ϵ there exists $\delta(r_n) > 0$ such that

$$|F(v) - F(u)| < \frac{\epsilon}{2^{n+3}} \text{ and } |f(r_n)(v-u)| < \frac{\epsilon}{2^{n+3}} \tag{2}$$

for $|r_n - v| < \delta(r_n)$ and $|v - r_n| < \delta(r_n)$. Let $\delta(x) = \eta(x)$ if for every $n \in \mathbb{N}$ the point $x \neq r_n$. For a δ-fine tagged partial division D denote by

$$\sum{}' |F(x_{i+1}) - F(x_i) - f(\xi_{i+1})(x_{i+1} - x_i)|$$

the sum in which i ranges between 0 and $n-1$ but is never equal to any r_n and by $\sum{}'' |F(x_{i+1}) - F(x_i) - f(\xi_{i+1})(x_{i+1} - x_i)|$ for the remaining indices i. It follows from the inequalities (2) that[1]

$$\sum{}'' |F(x_{i+1} - F(x_i) - f(\xi_{i+1})(x_{i+1} - x_i)| < 4\epsilon \sum_1^\infty \frac{1}{2^{n+3}} = \frac{\epsilon}{2}.$$

Moreover, by assumption

$$\sum{}' |F(x_{i+1} - F(x_i) - f(\xi_{i+1})(x_{i+1} - x_i)| < \frac{\epsilon}{2}.$$

[1] For a given r_n there might be a ξ_i equal to r_n. This increases the factor on the right hand side of the following inequality by 2.

Consequently

$$\begin{aligned}&\Big|F(b) - F(a) - \sum_{0}^{n-1} f(\xi_{i+1})(x_{i+1} - x_i)\Big|\\ &\leq \sum_{i=0}^{n-1} |F(x_{i+1} - F(x_i) - f(\xi_{i+1})(x_{i+1} - x_i)|\\ &= \sum{}' + \sum{}'' < \frac{\epsilon}{2} + \frac{\epsilon}{2}.\end{aligned}$$

This proves that $F(b) - F(a)$ is the value of the Kurzweil-Henstock integral over $[a, b]$. □

The theorem can be extended to infinite intervals in a similar fashion as the Kurzweil-Henstock definition is. Also $\mathbb{R}$ can be replaced by a Banach space.

3 Two Examples.

It is an immediate consequence of our Theorem that if f and g are two functions differing on a countable set and one is Kurzweil-Henstock integrable, then so is the other.

The second example concerns the proof of the Fundamental Theorem. Let F be continuous on $[a, b]$ and $F'(x) = f(x)$ for $x \in [a, b]$ except a countable set N. For $\xi \notin N$ there exists, by the alternative definition of the derivative (see [1] p. 46) a positive $\eta(\xi)$ such that

$$|F(v) - F(u) - f(\xi)(v - u)| < \frac{\epsilon}{b - a}(v - u) \tag{3}$$

whenever $\xi - \eta(\xi) < u \leq \xi \leq v < \xi + \eta(\xi)$. Define $\eta(x) = 0$ for $x \in N$. Clearly η is a countably closed gauge and if D is a tagged partial division of $[a, b]$, then it follows easily from (3) that inequality (1) holds for a η-fine tagged partial division D.

References

[1] Lee Peng Yee and Rudolf Výborný, *The Integral: An easy approach after Kurzweil and Henstock*, Cambridge University Press, Cambridge, UK, 2000.