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DIFFERENTIABILITY AS CONTINUITY

Abstract

We characterize differentiability of a map $f : \mathbb{R} \to \mathbb{R}$ in terms of continuity of a canonically associated map $\widehat{f}$. To characterize pointwise differentiability of f, both the domain and range of $\widehat{f}$ can be made topological. However, the global differentiability of f is characterized by the continuity of $\widehat{f}$ whose domain is topological but whose range is a convergence space.

1 Introduction.

A calculus student is well aware of the difference between continuity and differentiability of a map $f : \mathbb{R} \to \mathbb{R}$. For such a student, continuity is always understood as continuity for the usual topology of $\mathbb{R}$. After a first topology course, this same student may wonder if there is a way to find two topologies τ_d and τ_r such that the differentiability of a function $f : \mathbb{R} \to \mathbb{R}$ is characterized by the continuity of $f : (\mathbb{R}, \tau_d) \to (\mathbb{R}, \tau_r)$. This natural question was answered negatively by R. Geroch, E. Kronheimer and G. McCarty in [1]. This is in stark contrast with A. Machado's result [2, Propositions 2.2.1 and 2.2.2] that a map $f : \mathbb{C} \to \mathbb{C}$ is analytic if and only if $\widehat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is continuous, where the map $\widehat{f}$ is canonically associated to f. However, the structure $\widehat{\mathbb{C}}$ used by Machado is not carried by $\mathbb{C}$ but by $\mathbb{C}^2$ and is not a topology but a more general structure called *convergence* (see end of Section 2). Moreover Machado's

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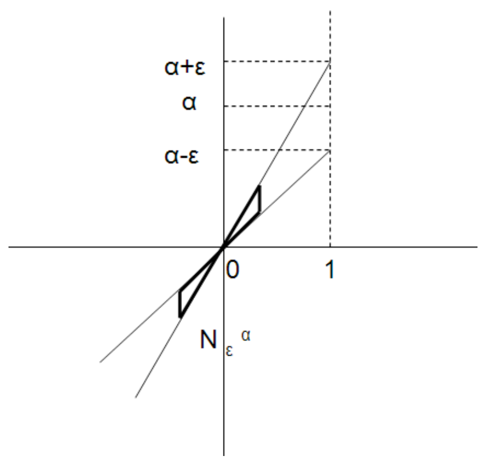
result doesn't apply to the differentiability of a map $f : \mathbb{R} \to \mathbb{R}$. Hence, this could be seen both as yet another contrast between $\mathbb{R}$-differentiability and $\mathbb{C}$-differentiability and as a striking illustration that what fails in the realm of topologies can often be fixed within more general "topological-like" structures like convergence spaces.

A closer look at the proofs in [2] gives however a different picture: the arguments can be modified (and simplified) to apply to real functions and to use only topologies, at least for pointwise differentiability. Indeed, the convergence structure used in [2] can be split into a family of topologies $(\tau_\alpha)_{\alpha\in\mathbb{R}}$, the members of which are instrumental in characterizing *pointwise* differentiability of $f : \mathbb{R} \to \mathbb{R}$ in terms of continuity of the canonically associated map $\widehat{f} = Id \times f : (\mathbb{R}^2, \tau_1) \to (\mathbb{R}^2, \tau_\alpha)$.

2 Differentiability as Continuity.

Let α be a real number. We define on $\mathbb{R} \times \mathbb{R}$ the vector space topology τ_α in which a base of neighborhoods of $(0, 0)$ is given by the sets $N_\varepsilon^\alpha = \{(\lambda, \lambda\xi) : \sup(|\lambda|, |\xi - \alpha|) < \varepsilon\}$ for $\varepsilon > 0$.



Hence a typical neighborhood of (λ_0, x_0) in τ_α is of the form

$$(\lambda_0, x_0) + N_\varepsilon^\alpha = \{(\lambda_0 + \lambda, x_0 + \lambda\xi) : \sup(|\lambda|, |\xi - \alpha|) < \varepsilon\}.$$

Theorem 1. *$f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 if and only if there exists $\alpha \in \mathbb{R}$ such that*

$$\widehat{f} = Id \times f : (\mathbb{R} \times \mathbb{R}, \tau_1) \to (\mathbb{R} \times \mathbb{R}, \tau_\alpha)$$

is continuous at (λ_0, x_0) for every $\lambda_0 \in \mathbb{R}$ (equivalently, for $\lambda_0 = 0$). Specifically α is unique and $f'(x_0) = \alpha$.

PROOF. Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0. Then $f(x) = f(x_0) + f^{'}(x_0)(x - x_0) + h(x - x_0)$ where $\lim_{x \to x_0} \frac{h(x-x_0)}{x-x_0} = 0$. Fix ε in $(0, 1)$ and $\lambda_0 \in \mathbb{R}$. We want to find $\delta > 0$ such that $\widehat{f}((\lambda_0, x_0) + N_\delta^1) \subset \widehat{f}(\lambda_0, x_0) + N_\varepsilon^{f'(x_0)}$. Suppose that $\lambda, \xi \in \mathbb{R}$; then $(\lambda, \lambda\xi) \in N_\delta^1$ provided $|\lambda| < \delta$ and $|\xi - 1| < \delta$ with $\delta > 0$ still to be chosen. We have

$$\begin{aligned}
\widehat{f}((\lambda_0, x_0) + (\lambda, \lambda\xi)) &= \widehat{f}(\lambda_0 + \lambda, x_0 + \lambda\xi) \\
&= (\lambda_0 + \lambda, f(x_0 + \lambda\xi)) \\
&= (\lambda_0, f(x_0)) + (\lambda, f(x_0 + \lambda\xi) - f(x_0)) \\
&= \widehat{f}(\lambda_0, x_0) + (\lambda, \lambda\eta)
\end{aligned}$$

where $\eta = \frac{f(x_0+\lambda\xi)-f(x_0)}{\lambda}$. Note that

$$f(x_0 + \lambda\xi) = f(x_0) + f'(x_0)\lambda\xi + h(\lambda\xi),$$

so

$$\eta - f'(x_0) = f'(x_0)(\xi - 1) + \frac{h(\lambda\xi)}{\lambda},$$

and

$$|\eta - f'(x_0)| \leq |f'(x_0)|.|\xi - 1| + \left|\frac{h(\lambda\xi)}{\lambda}\right|.$$

As $\lim_{x \to x_0} \frac{h(x-x_0)}{x-x_0} = 0$, it follows that there is $\gamma > 0$ such that $\left|\frac{h(\lambda\xi)}{\lambda\xi}\right| < \frac{\varepsilon}{3}$ when $0 < |\lambda\xi| < \gamma$. We assume that $\gamma < 1$. If $\sup(|\lambda|, |\xi - 1|) < \frac{\gamma}{2}$, then $|\lambda\xi| < \gamma$ so that

$$\left|\frac{h(\lambda\xi)}{\lambda}\right| < \frac{\varepsilon|\xi|}{3} < \frac{2\varepsilon}{3}.$$

If $f'(x_0) = 0$, then $|\eta| \leq \left|\frac{h(\lambda\xi)}{\lambda}\right|$ and $\widehat{f}((\lambda_0, x_0) + (\lambda, \lambda\xi)) \in \widehat{f}(\lambda_0, x_0) + N_\varepsilon^0$ provided that $(\lambda, \lambda\xi) \in N_\delta^1$ where $\delta = \min\{\varepsilon, \frac{\gamma}{2}\}$. If $f'(x_0) \neq 0$ and $|\xi - 1| < \frac{\varepsilon}{3|f'(x_0)|}$, then $|f'(x_0)|.|\xi - 1| < \frac{\varepsilon}{3}$. Now set $\delta = \min\{\varepsilon, \frac{\gamma}{2}, \frac{\varepsilon}{3|f'(x_0)|}\}$. If $(\lambda, \lambda\xi) \in N_\delta^1$, then it follows that $|\lambda| < \varepsilon$ and $|\eta - f'(x_0)| < \varepsilon$ so $\widehat{f}((\lambda_0, x_0) + (\lambda, \lambda\xi)) \in \widehat{f}(\lambda_0, x_0) + N_\varepsilon^{f'(x_0)}$.

Conversely, assume that $\widehat{f} = Id \times f : (\mathbb{R}\times\mathbb{R}, \tau_1) \to (\mathbb{R}\times\mathbb{R}, \tau_\alpha)$ is continuous at $(0, x_0)$ for some $\alpha \in \mathbb{R}$. We want to show that $\alpha = f'(x_0)$. By continuity, for every $\varepsilon > 0$, there exists δ such that $\varepsilon > \delta > 0$ and $\widehat{f}((0, x_0) + N^1_\delta) \subset \widehat{f}(0, x_0) + N^\alpha_\varepsilon$. In other words,

$$\{(\lambda, f(x_0+\lambda\xi)) : \sup(|\lambda|, |\xi-1|) < \delta\} \subset \{(\lambda, f(x_0)+\lambda\eta) : \sup(|\lambda|, |\eta-\alpha|) < \varepsilon\}.$$

In particular, $f(x_0+\lambda) = f(x_0)+\lambda\eta$ for some η verifying $|\eta-\alpha| < \varepsilon$, provided that $|\lambda| < \delta$. Therefore

$$\left|\frac{f(x_0+\lambda) - f(x_0)}{\lambda} - \alpha\right| < \varepsilon$$

provided that $|\lambda| < \delta$, so that $\alpha = \lim_{\lambda\to 0} \frac{f(x_0+\lambda)-f(x_0)}{\lambda} = f'(x_0)$. $\square$

A dissatisfying aspect of this result is that the topology τ_α depends on x_0. This can be remedied by using a *convergence structure* instead of a topology on the range. Recall that a *filter on* X is a family $\mathcal{F}$ of subsets of X that is stable by finite intersections ($A, B \in \mathcal{F} \Longrightarrow A\cap B \in \mathcal{F}$) and by supersets ($A \in \mathcal{F}$ and $A \subset B \Longrightarrow B \in \mathcal{F}$) and that does not contain the empty set. A family $\mathcal{A}$ of subsets of X that does not contain the empty set and is stable by finite intersections is called a *filter base.* It *generates* a filter $\mathcal{A}^\uparrow = \{B : \exists A \in \mathcal{A}, A \subset B\}$. For instance, the family $\mathcal{N}(x)$ of neighborhoods of a fixed point x of a topological space X is a filter. The family of tails $\{\{x_n : n \geq k\} : k \in \mathbb{N}\}$ of a sequence $(x_n)_{n\in\mathbb{N}}$ on X is a filter-base on X. Filters on a given set are ordered by inclusion; that is, $\mathcal{F}$ *is finer than* $\mathcal{G}$, in symbols $\mathcal{F} \geq \mathcal{G}$, if $\mathcal{F} \supset \mathcal{G}$. It is an easy exercise to verify that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if the filter generated by the filter-base of its tails is finer than $\mathcal{N}(x)$. More generally, a filter $\mathcal{F}$ on a topological space X *converges to* x if $\mathcal{F} \geq \mathcal{N}(x)$. A *convergence* ξ *on a set* X defines what are the filters convergent to each point. Formally, it is a relation between X and the set of filters on X, denoted $x \in \lim_\xi \mathcal{F}$ or $\mathcal{F} \underset{\xi}{\to} x$ whenever $(x, \mathcal{F}) \in \xi$ and verifying:

1. $\{x\}^\uparrow \to x$ for every $x \in X$;
2. $\mathcal{F} \to x$ and $\mathcal{G} \geq \mathcal{F} \Longrightarrow \mathcal{G} \to x$.

A topology is a particular convergence in which $\mathcal{F} \to x$ if and only if $\mathcal{F}$ is finer than the filter $\mathcal{N}(x)$ of neighborhoods of x and $\mathcal{N}(x)$ has a base of open sets, where $O \subset X$ is *open* if

$$\mathcal{F} \to x \in O \Longrightarrow O \in \mathcal{F}.$$

A map $f : (X, \xi) \to (Y, \sigma)$ between two convergence spaces is *continuous* if $f(\mathcal{F}) \underset{\sigma}{\to} f(x)$ whenever $\mathcal{F} \underset{\xi}{\to} x$. If $Id : (X, \xi) \to (X, \sigma)$ is continuous, we say that ξ *is finer than* σ, in symbols $\xi \geq \sigma$. If $(\xi_i)_{i \in I}$ is a family of convergences on X, the supremum and infimum of the family with respect to this order are defined by:

$$\lim\nolimits_{\vee_{i\in I}\xi_i} \mathcal{F} = \bigcap\nolimits_{i \in I} \lim\nolimits_{\xi_i} \mathcal{F};$$
$$\lim\nolimits_{\wedge_{i\in I}\xi_i} \mathcal{F} = \bigcup\nolimits_{i \in I} \lim\nolimits_{\xi_i} \mathcal{F}.$$

We call the convergence $\Gamma_c = \bigwedge_{\alpha \in \mathbb{R}} \tau_\alpha$ the *convergence along cones.* Notice that even though each τ_α is a topology, Γ_c is not.

An immediate corollary of the definitions and of Theorem 1 is the following.

Corollary 2. *$f : \mathbb{R} \to \mathbb{R}$ is differentiable if and only if*

$$\widehat{f} = Id \times f : (\mathbb{R}^2, \tau_1) \to (\mathbb{R}^2, \Gamma_c)$$

is continuous.

3 Calculus Topologically.

A cornerstone of calculus in one variable is Fermat's theorem stating that if f has a local extremum at a, then either f is not differentiable at a or $f'(a) = 0$. We show that Fermat's theorem can be proved topologically via Theorem 1.

Proposition 3. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a local extremum at $a \in \mathbb{R}$. Then for each α and λ_0 in $\mathbb{R}$ with $\alpha \neq 0$, the function $\widehat{f} : (\mathbb{R}^2, \tau_1) \to (\mathbb{R}^2, \tau_\alpha)$ is not continuous at (λ_0, a).*

PROOF. Assume that f has a local maximum at a. Given $\alpha > 0$ and λ_0, we will show that $\widehat{f}((\lambda_0, a) + N^1_\delta) \nsubseteq \widehat{f}(\lambda_0, a) + N^\alpha_{\alpha/2}$ for each $\delta > 0$, by exhibiting $h \in (0, \delta)$ such that $\widehat{f}((\lambda_0, a) + (h, h)) - \widehat{f}(\lambda_0, a) \notin N^\alpha_{\alpha/2}$. As f has a local maximum at a, we may choose $h \in (0, \delta)$ small enough that $f(a + h) \leq f(a)$. Then

$$\begin{aligned}
\widehat{f}((\lambda_0, a) + (h, h)) - \widehat{f}(\lambda_0, a) &= \widehat{f}(\lambda_0 + h, a + h) - \widehat{f}(\lambda_0, a) \\
&= (\lambda_0 + h, f(a + h)) - (\lambda_0, f(a)) \\
&= (h, f(a + h) - f(a)) \\
&= h\left(1, \frac{f(a + h) - f(a)}{h}\right).
\end{aligned}$$

Therefore, $\widehat{f}((\lambda_0, a)+(h, h))-\widehat{f}(\lambda_0, a) \notin N^{\alpha}_{\alpha/2}$ because $\frac{f(a+h)-f(a)}{h} \leq 0$ cannot be between $\alpha - \frac{\alpha}{2}$ and $\alpha + \frac{\alpha}{2}$.

A similar argument applies for $\alpha < 0$ using $h \in (-\delta, 0)$, and the proof for a local minimum is analogous. $\square$

In view of Theorem 1, we obtain the following.

Corollary 4. *(Fermat): If $f : \mathbb{R} \to \mathbb{R}$ has a local extremum at a, then either f is not differentiable at a or $f'(a) = 0$.*

Interestingly, many calculus results follow from that fact - now interpreted topologically - combined with a topological argument. For instance, Rolle's Theorem (hence the mean value theorem) follows immediately from Corollary 4, given the existence of extrema of the function on $[a, b]$, which follows by continuity of the function and compactness of $[a, b]$. The same is true, among others, for the fact that derivatives have the intermediate value property. In turn, a myriad of results are based on the mean value theorem without further use of the derivative.

References

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