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A SUMMABILITY FACTOR THEOREM FOR GENERALIZED ABSOLUTE SUMMABILITY

Abstract

In this paper, we establish a summability factor theorem for summability $|A, \delta|_k$ as defined in (1) where A is a lower triangular matrix with non-negative entries satisfying certain conditions. Our paper is an extension of the main result of [1] using definition (1) below.

Recently, Bor and Seyhan [1] proved a theorem on $|\bar{N}, p, \delta|_k$ summability factor under weaker conditions by using an almost increasing sequence. Unfortunately they used an incorrect definition, (for detail, see, [3]). In this paper, we generalize their result by using the correct definition and a lower triangular matrix with non-negative entries satisfying certain conditions.

Let A be a lower triangular matrix, $\{s_n\}$ a sequence. Then

$$A_n := \sum_{\nu=0}^{n} a_{n\nu} s_\nu.$$

A series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |A_n - A_{n-1}|^k < \infty.$$

and it is said to be summable $|A, \delta|_k, k \geq 1$ and $\delta \geq 0$ if (see [2])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |A_n - A_{n-1}|^k < \infty. \tag{1}$$

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We may associate with A two lower triangular matrices $\overline{A}$ and $\hat{A}$ defined by

$$\bar{a}_{n\nu} = \sum_{r=\nu}^{n} a_{nr},\ n, \nu = 0, 1, 2, \ldots,$$

and

$$\hat{a}_{n\nu} = \bar{a}_{n\nu} - \bar{a}_{n-1,\nu},\ n = 1, 2, 3, \ldots\ .$$

A positive sequence $\{b_n\}$ is said to be almost increasing if there exists an increasing sequence $\{c_n\}$ and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$. Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = e^{(-1)^n} n$.

Given any sequence $\{x_n\}$, the notation $x_n \asymp O(1)$ means $x_n = O(1)$ and $1/x_n = O(1)$. For any matrix entry $a_{n\nu}$, $\Delta_\nu a_{n\nu} := a_{n\nu} - a_{n\nu+1}$.

Theorem 1. *Let $\{X_n\}$ be an almost increasing sequence and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences such that:*

(i) $|\Delta\lambda_n| \leq \beta_n$,

(ii) $\lim \beta_n = 0$,

(iii) $\sum_{n=1}^{\infty} n|\Delta\beta_n| X_n < \infty$, *and*

(iv) $|\lambda_n| X_n = O(1)$.

Let A be a lower triangular matrix with non-negative entries satisfying

(v) $na_{nn} \asymp O(1)$,

(vi) $a_{n-1,\nu} \geq a_{n\nu}$ *for* $n \geq \nu + 1$,

(vii) $\bar{a}_{n0} = 1$ *for all* n,

(viii) $\sum_{\nu=1}^{n-1} a_{\nu\nu}\hat{a}_{n\nu+1} = O\Big(a_{nn}\Big)$,

(ix) $\sum_{n=\nu+1}^{m+1} n^{\delta k} |\Delta_\nu \hat{a}_{n\nu}| = O\Big(\nu^{\delta k} a_{\nu\nu}\Big)$, *and*

(x) $\sum_{n=\nu+1}^{m+1} n^{\delta k} \hat{a}_{n\nu+1} = O\Big(\nu^{\delta k}\Big)$.

If

(xi) $\sum_{n=1}^{m} n^{\delta k-1}|t_n|^k = O(X_m)$,

where $t_n := \dfrac{1}{n+1}\displaystyle\sum_{k=1}^{n} ka_k$, *then the series* $\sum a_n\lambda_n$ *is summable* $|A,\delta|_k,\ k \geq 1$, $0 \leq \delta < 1/k$.

The following lemma is essential for the proof of Theorem 1.

Lemma 1. ([1]) *Under the conditions on* $\{X_n\},\{\beta_n\}$ *and* $\{\lambda_n\}$ *as taken from the statement of the theorem if (iii) is satisfied, then*

(1) $n\beta_n X_n = O(1)$

(2) $\displaystyle\sum_{n=1}^{\infty}\beta_n X_n < \infty.$

Proof. Let (y_n) be the *nth* term of the A-transform of the partial sums of $\sum_{i=0}^{n}\lambda_i a_i$. Then

$$y_n := \sum_{i=0}^{n} a_{ni}s_i = \sum_{i=0}^{n} a_{ni}\sum_{\nu=0}^{i}\lambda_\nu a_\nu = \sum_{\nu=0}^{n}\lambda_\nu a_\nu \sum_{i=\nu}^{n} a_{ni} = \sum_{\nu=0}^{n}\bar{a}_{n\nu}\lambda_\nu a_\nu$$

and

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n}(\bar{a}_{n\nu} - \bar{a}_{n-1,\nu})\lambda_\nu a_\nu = \sum_{\nu=0}^{n}\hat{a}_{n\nu}\lambda_\nu a_\nu.$$

We may write (Note that (vii) implies that $\hat{a}_{n0} = 0$.)

$$\begin{aligned}
Y_n &= \sum_{\nu=1}^{n}\Big(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\Big)\nu a_\nu = \sum_{\nu=1}^{n}\Big(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\Big)\Big[\sum_{r=1}^{\nu} ra_r - \sum_{r=1}^{\nu-1} ra_r\Big] \\
&= \sum_{\nu=1}^{n-1}\Delta_\nu\Big(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\Big)\sum_{r=1}^{\nu} ra_r + \frac{\hat{a}_{nn}\lambda_n}{n}\sum_{r=1}^{n} ra_r \\
&= \sum_{\nu=1}^{n-1}(\Delta_\nu\hat{a}_{n\nu})\lambda_\nu\frac{\nu+1}{\nu}t_\nu + \sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}(\Delta\lambda_\nu)\frac{\nu+1}{\nu}t_\nu \\
&\quad + \sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}\lambda_{\nu+1}\frac{1}{\nu}t_\nu + \frac{(n+1)a_{nn}\lambda_n t_n}{n} \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4},\ \text{say.}
\end{aligned}$$

To complete the proof, by Minkowski's inequality, it suffices to show that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |T_{nr}|^k < \infty, \text{ for } r = 1, 2, 3, 4.$$

From the definition of $\hat{A}$ and by (vi) and (vii)

$$\begin{aligned}\hat{a}_{n,\nu+1} &= \bar{a}_{n,\nu+1} - \bar{a}_{n-1,\nu+1} = \sum_{i=\nu+1}^{n} a_{ni} - \sum_{i=\nu+1}^{n-1} a_{n-1,i} \\ &= 1 - \sum_{i=0}^{\nu} a_{ni} - 1 + \sum_{i=0}^{\nu} a_{n-1,i} = \sum_{i=0}^{\nu} \Big(a_{n-1,i} - a_{n,i} \Big) \geq 0.\end{aligned}$$

By Hölder's inequality

$$\begin{aligned} I_1 := \sum_{n=1}^{m} n^{\delta k+k-1} |T_{n1}|^k &= \sum_{n=1}^{m} n^{\delta k+k-1} \Big| \sum_{\nu=1}^{n-1} \Delta_\nu \hat{a}_{n\nu} \lambda_\nu \frac{\nu+1}{\nu} t_\nu \Big|^k \\ &= O(1) \sum_{n=1}^{m+1} n^{\delta k+k-1} \Big(\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| |\lambda_\nu| |t_\nu| \Big)^k \\ &= O(1) \sum_{n=1}^{m+1} n^{\delta k+k-1} \Big(\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| |\lambda_\nu|^k |t_\nu|^k \Big) \times \Big(\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| \Big)^{k-1} \end{aligned}$$

$$\begin{aligned}\Delta_\nu \hat{a}_{n\nu} &= \hat{a}_{n\nu} - \hat{a}_{n,\nu+1} = \bar{a}_{n\nu} - \bar{a}_{n-1,\nu} - \bar{a}_{n,\nu+1} + \bar{a}_{n-1,\nu+1} \\ &= a_{n\nu} - a_{n-1,\nu} \leq 0.\end{aligned}$$

Thus, by (vii)

$$\sum_{\nu=0}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| = \sum_{\nu=0}^{n-1} (a_{n-1,\nu} - a_{n\nu}) = 1 - 1 + a_{nn} = a_{nn}.$$

Since $\{X_n\}$ is an almost increasing sequence, condition (iv) implies that $\{\lambda_n\}$

is bounded. Then, by $(v), (ix), (xi)$, and (i) and condition (2) of Lemma 1,

$$\begin{aligned}
I_1 &= O(1)\sum_{n=1}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|\lambda_\nu|^k|t_\nu|^k|\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1)\sum_{n=1}^{m+1} n^{\delta k}\Big(\sum_{\nu=1}^{n-1}|\lambda_\nu|^{k-1}|\lambda_\nu||\Delta_\nu \hat{a}_{n\nu}||t_\nu|^k\Big) \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu||t_\nu|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}|\Delta_\nu \hat{a}_{n\nu}| = O(1)\sum_{\nu=1}^{m}\nu^{\delta k}|\lambda_\nu|a_{\nu\nu}|t_\nu|^k \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu|\Big[\sum_{r=1}^{\nu}a_{rr}|t_r|^k r^{\delta k}-\sum_{r=1}^{\nu-1}a_{rr}|t_r|^k r^{\delta k}\Big] \\
&= O(1)\Big[\sum_{\nu=1}^{m}|\lambda_\nu|\sum_{r=1}^{\nu}a_{rr}|t_r|^k r^{\delta k}-\sum_{\nu=0}^{m-1}|\lambda_{\nu+1}|\sum_{r=1}^{\nu}a_{rr}|t_r|^k r^{\delta k}\Big] \\
&= O(1)\Big[\sum_{\nu=1}^{m-1}\Delta(|\lambda_\nu|)\sum_{r=1}^{\nu}a_{rr}|t_r|^k r^{\delta k}+|\lambda_m|\sum_{r=1}^{m}a_{rr}|t_r|^k r^{\delta k}\Big] \\
&= O(1)\Big[\sum_{\nu=1}^{m-1}\Delta(|\lambda_\nu|)\sum_{r=1}^{\nu}r^{\delta k-1}|t_r|^k+|\lambda_m|\sum_{r=1}^{m}r^{\delta k-1}|t_r|^k\Big] \\
&= O(1)\sum_{\nu=1}^{m-1}|\Delta\lambda_\nu|X_\nu+O(1)|\lambda_m|X_m \\
&= O(1)\sum_{\nu=1}^{m}\beta_\nu X_\nu+O(1)|\lambda_m|X_m = O(1).
\end{aligned}$$

By (i) and Hölder's inequality,

$$\begin{aligned}
I_2 &:= \sum_{n=2}^{m+1} n^{\delta k+k-1}|T_{n2}|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\Big|\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}(\Delta\lambda_\nu)\frac{\nu+1}{\nu}t_\nu\Big|^k \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}\beta_\nu|t_\nu|\Big]^k \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\sum_{\nu=1}^{n-1}\beta_\nu|t_\nu|^k\hat{a}_{n,\nu+1}\Big]\times\Big[\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}\beta_\nu\Big]^{k-1}.
\end{aligned}$$

Using the definition of $\hat{A}$ and $\bar{A}$ and conditions (vi) and (vii)

$$\begin{aligned}
J := \sum_{\nu=1}^{n-1} \hat{a}_{n,\nu+1}\beta_\nu &= \sum_{\nu=1}^{n-1} (\bar{a}_{n,\nu+1} - \bar{a}_{n-1,\nu+1})\beta_\nu \\
&= \sum_{\nu=1}^{n-1} \Big(\sum_{i=\nu+1}^{n} a_{ni} - \sum_{i=\nu+1}^{n-1} a_{n-1,i} \Big)\beta_\nu \\
&= \sum_{\nu=1}^{n-1} \Big(1 - \sum_{i=0}^{\nu} a_{ni} - 1 + \sum_{i=0}^{\nu} a_{n-1,i} \Big)\beta_\nu \\
&= \sum_{\nu=1}^{n-1} \Big(\sum_{i=0}^{\nu} (a_{n-1,i} - a_{n,i}) \Big)\beta_\nu \\
&= \sum_{\nu=1}^{n-1} (a_{n-1,0} - a_{n,0})\beta_\nu + \sum_{\nu=1}^{n-1}\sum_{i=1}^{\nu} (a_{n-1,i} - a_{n,i})\beta_\nu \\
&= (a_{n-1,0} - a_{n,0}) \sum_{\nu=1}^{n-1} \beta_\nu + \sum_{i=1}^{n-1} (a_{n-1,i} - a_{n,i}) \sum_{\nu=i}^{n-1} \beta_\nu \\
&\le \sum_{i=0}^{\nu} (a_{n-1,i} - a_{n,i}) \sum_{\nu=1}^{n-1} \beta_\nu .
\end{aligned}$$

Since $\{X_n\}$ is almost increasing, condition (2) of the Lemma implies that $\sum_{\nu=0}^{\infty} \beta_\nu$ converges. Therefore there exists a positive constant M such that $\sum_{\nu=0}^{\infty} \beta_\nu \le M$ and we obtain $J \le M a_{nn}$. Using (v) we have

$$\begin{aligned}
I_2 &= O(1) \sum_{n=2}^{m+1} n^{\delta k} (n a_{nn})^{k-1} \sum_{\nu=1}^{n-1} \hat{a}_{n,\nu+1} \beta_\nu |t_\nu|^k \\
&= O(1) \sum_{\nu=1}^{m} \beta_\nu |t_\nu|^k \sum_{n=\nu+1}^{m+1} n^{\delta k} \hat{a}_{n,\nu+1} .
\end{aligned}$$

Therefore, from (x),

$$I_2 = O(1) \sum_{\nu=1}^{m} \nu^{\delta k} \beta_\nu |t_\nu|^k = O(1) \sum_{\nu=1}^{m} \nu \beta_\nu \frac{|t_\nu|^k}{\nu} \nu^{\delta k} .$$

Using summation by parts, (iii), (xi) and conditions (1) and (2) of Lemma 1,

$$\begin{aligned}
I_2 :&= O(1)\sum_{\nu=1}^{m}\nu\beta_\nu\Big[\sum_{r=1}^{\nu} r^{\delta k-1}|t_r|^k - \sum_{r=1}^{\nu-1} r^{\delta k-1}|t_r|^k\Big]\\
&= O(1)\Big[\sum_{\nu=1}^{m}\nu\beta_\nu\sum_{r=1}^{\nu} r^{\delta k-1}|t_r|^k - \sum_{\nu=1}^{m-1}(\nu+1\beta_{\nu+1})\sum_{r=1}^{\nu} r^{\delta k-1}|t_r|^k\Big]\\
&= O(1)\sum_{\nu=1}^{m-1}\Delta(\nu\beta_\nu)\sum_{r=1}^{\nu} r^{\delta k-1}|t_r|^k + O(1)m\beta_m\sum_{r=1}^{m} r^{\delta k-1}|t_r|^k\\
&= O(1)\sum_{\nu=1}^{m-1}|\Delta(\nu\beta_\nu)|X_\nu + O(1)m\beta_m X_m\\
&= O(1)\sum_{\nu=1}^{m-1}\nu|\Delta(\beta_\nu)|X_\nu + O(1)\sum_{\nu=1}^{m-1}\beta_{\nu+1}X_\nu + O(1) = O(1).
\end{aligned}$$

By (viii) and Hölder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{k-1}|T_{n3}|^k &= \sum_{n=2}^{m+1} n^{\delta k+k-1}\Big|\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}\lambda_{\nu+1}\frac{1}{\nu}t_\nu\Big|^k\\
&\le \sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|\frac{\hat{a}_{n,\nu+1}}{\nu}|t_\nu|\Big]^k\\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}||\hat{a}_{n,\nu+1}|t_\nu|a_{\nu\nu}\Big]^k\\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^k a_{\nu\nu}|t_\nu|^k\hat{a}_{n,\nu+1}\Big]\\
&\quad\times\Big[\sum_{\nu=1}^{n-1}a_{\nu\nu}|\hat{a}_{n,\nu+1}|\Big]^{k-1}
\end{aligned}$$

By (iv), (v), (x) and the boundedness of $(a_{\nu\nu})$,

$$\begin{aligned}
I_3 &= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^k a_{\nu\nu}|t_\nu|^k \hat{a}_{n,\nu+1}\\
&= O(1)\sum_{n=2}^{m+1} \nu^{\delta k}\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^{k-1}|\lambda_{\nu+1}|a_{\nu\nu}|t_\nu|^k \hat{a}_{n,\nu+1}\\
&= O(1)\sum_{\nu=1}^{m}|\lambda_{\nu+1}|a_{\nu\nu}|t_\nu|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}\hat{a}_{n,\nu+1}\\
&= O(1)\sum_{\nu=1}^{m}|\lambda_{\nu+1}|\nu^{\delta k}a_{\nu\nu}|t_\nu|^k = O(1),
\end{aligned}$$

as in the proof of I_1.

Finally, by (iv) and (v) we have

$$\begin{aligned}
\sum_{n=1}^{m} n^{\delta k+k-1}|T_{n4}|^k &= \sum_{n=1}^{m} n^{\delta k+k-1}\left|\frac{(n+1)a_{nn}\lambda_n t_n}{n}\right|^k\\
&= O(1)\sum_{n=1}^{m} n^{\delta k+k-1}|a_{nn}|^k|\lambda_n|^k|t_n|^k\\
&= O(1)\sum_{n=1}^{m} n^{\delta k}(na_{nn})^{k-1}a_{nn}|\lambda_n|^{k-1}|\lambda_n||t_n|^k\\
&= O(1)\sum_{n=1}^{m} n^{\delta k}a_{nn}|\lambda_n||t_n|^k = O(1),
\end{aligned}$$

as in the proof of I_1. □

Setting $\delta = 0$ in the theorem yields the following corollary.

Corollary 1. *Let $\{X_n\}$ be an almost increasing sequence and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences satisfying conditions (i)-(iv) of Theorem 1. Let A be a triangle satisfying conditions (v)-(viii) of Theorem 1. If*

(vii) $\sum_{n=1}^{m}\frac{1}{n}|t_n|^k = O(X_m)$,

where $t_n := \dfrac{1}{n+1}\displaystyle\sum_{k=1}^{n} ka_k$, *then the series* $\sum a_n\lambda_n$ *is summable* $|A|_k, k \geq 1$.

Corollary 2. *Let $\{p_n\}$ be a positive sequence such that $P_n := \sum_{k=0}^{n} p_k \to \infty$, and satisfying:*

(v) $np_n = O(P_n)$ *and*

(vi) $\displaystyle\sum_{n=\nu+1}^{m+1} n^{\delta k}\left|\frac{p_n}{P_nP_{n-1}}\right| = O\Big(\frac{\nu^{\delta k}}{P\ _\nu}\Big).$

Let $\{X_n\}$ *be an almost increasing sequence, let* $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be sequences satisfying conditions (i)-(iv) of Theorem 1. If*

(vii) $\sum_{n=1}^{m} n^{\delta k-1}|t_n|^k = O(X_m)$,

where $t_n := \dfrac{1}{n+1}\displaystyle\sum_{k=1}^{n} ka_k$, *then the series* $\sum a_n\lambda_n$ *is summable* $|\bar{N}, p, \delta|_k,\ k \geq 1$ *for* $0 \leq \delta < 1/k$.

PROOF. Conditions (i)-(iv) and (vii) of Corollary 2 are, respectively, conditions (i)-(iv) and (xi) of Theorem 1. Conditions (vi), (vii) and (viii) of Theorem 1 are automatically satisfied for any weighted mean method. Condition (v) of Theorem 1 becomes condition (v) of Corollary 2, and conditions (ix) and (x) of Theorem 1 become condition (vi) of Corollary 2. □

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