Real Analysis Exchange
Vol. 29(1), 2003/2004, pp. 413–418

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A SET OF MEASURE ZERO WHICH CONTANIS A COPY OF ANY FINITE SET

Abstract

We answer a question which was stated by R. E. Svetic in [11]. The Bergelson-Hindman-Weiss lemma, which was placed in [1], is improved.

1 On Svetic's Question

In [11, p. 537], there was stated the following question: *Is it true that if a measurable set contains a copy of each finite set, then the set has positive measure*?

If one means that a copy [a similar copy of a subset of real numbers] of a subset X it is a set of the form $x + tX = \{x + ty : y \in X\}$, where x and $t \neq 0$ are some real numbers, then the question had been stated by E. Marczewski in [6] or [7] and was answered negatively by P. Erdös and S. Kakutani in [3]. More subtle examples which answered the question negatively one can find in [2], too. If one assumes that a copy means a similar copy but with $t = 1$: a set $x + X = \{x + y : y \in X\}$, where x is a real number; then the answer is negative, also. We present an answer which improves the P. Erdös and S. Kakutani result [3]. In [3] it was noted the followings.

Since for each n there holds $\sum_{m=n+1}^{\infty} \frac{m-1}{m!} = \frac{1}{n!}$, then every real $x \in [0, 1)$ is uniquely of the form $x = \sum_{n=2}^{\infty} \frac{b_n}{n!}$, where always $b_n \in \{0, 1, \ldots, n-2, n-1\}$ and infinitely many times there is $b_n \neq n - 1$.

The subset

$$S = \left\{ \sum_{n=2}^{\infty} \frac{b_n}{n!} : b_n \in \{0, 1, \ldots, n-3, n-2\} \right\} \subset [0, 1)$$

Key Words: Svetic's question, Hadwiger principle, measurably large
Mathematical Reviews subject classification: Primary 28A05; Secondary 03E05
Received by the editors March 25, 2002
Communicated by: Clifford E. Weil

has Lebesgue measure zero. It is perfect and meager, too.

And some modification of the following lemma.

Lemma 1. *Let* $n \geq m \geq 3$ *and* $\{a_n, b_n\} \in \{0, 1, \dots, n-2, n-1\}$*. If always,* $a_n + b_n \neq n-2$ *and* $a_n + b_n \neq n-1$ *and* $a_n + b_n \neq 2n-2$*, then*

$$\sum_{n=m+1}^{\infty} \frac{a_n + b_n}{n!} = \sum_{n=m}^{\infty} \frac{c_n}{n!},$$

where $c_n \in \{0, 1, \dots, n-3, n-2\}$.

PROOF. Suppose $\sum_{n=m+1}^{\infty} \frac{a_n + b_n}{n!} = \sum_{n=m}^{\infty} \frac{c_n}{n!}$, where $c_n \in \{0, 1, \dots, n-2, n-1\}$. For the digit c_3 there holds

$$\frac{c_3}{3!} \leq \sum_{n=4}^{\infty} \frac{a_n + b_n}{n!} \leq 2 \sum_{n=4}^{\infty} \frac{n-1}{n!} = \frac{2}{3!}.$$

Since for infinitely many n there holds $a_n + b_n \neq 2n-2$, then the second inequality is sharp. Therefore $c_3 < 2$.

Again use this that for infinitely many n there holds $a_n + b_n \neq 2n-2$. So, $m > 3$ implies $c_m = a_m + b_m \pmod m$ or $c_m = a_m + b_m + 1 \pmod m$. But we assume that always holds $c_m < m$. Therefore $a_m + b_m \neq m-2$ and $a_m + b_m \neq m-1$ implies that $c_m < m-1$. □

To answer Svetic's question we present the following theorem.

Theorem 1. *The subset of real numbers*

$$\bigcup_{k=1}^{\infty} k \cdot S = \left\{ k \sum_{n=2}^{\infty} \frac{b_n}{n!} : b_n \in \{0, 1, \dots, n-3, n-2\} \text{ and } k \in \{1, 2, \dots\} \right\}$$

has Lebesgue measure zero and contains a copy of any finite subsets of real numbers.

PROOF. Since Lebesgue measure of S is zero, then any set $k \cdot S = \{kx : x \in S\}$ is of Lebesgue measure zero. Also the union $\bigcup_{k=1}^{\infty} k \cdot S$ is of Lebesgue measure zero, since it is an union of countably many sets of Lebesgue measure zero.

Let d be a natural number such that $\{x_1, x_2, \dots, x_q\} \subset (0, d)$. Choose natural numbers a and m such that $m! x_i < ad$, for any $i \in \{1, 2, \dots, q\}$, and $m + 1 > 2q$. Hence

$$\frac{x_i}{ad} = \sum_{k=m+1}^{\infty} \frac{b_k^i}{k!}, \quad \text{where } b_k^i \in \{0, 1, \dots, k-1\}.$$

If $n > m$, then $n > 2q$ and one can find natural numbers $b_n^0 \in \{0, 1, \ldots, n-2, n-1\}$ such that $b_n^i + b_n^0 \neq n-1$ and $b_n^i + b_n^0 \neq n-2$ and $b_n^i + b_n^0 \neq 2n-2$, for each $i \in \{1, 2, \ldots, q\}$. By Lemma 1 there holds

$$\sum_{n=m+1}^{\infty} b_n^i n! + \sum_{n=m+1}^{\infty} \frac{b_n^0}{n!} = \sum_{n=m}^{\infty} \frac{c_n^i}{n!},$$

where $c_n^i \in \{0, 1, \ldots, n-3, n-2\}$. Therefore

$$x_i + ad \sum_{n=m+1}^{\infty} \frac{b_n^0}{n!} = ad \sum_{n=m}^{\infty} \frac{c_n^i}{n!} \in ad \cdot S.$$

This shows that $ad \cdot S \subset \bigcup_{k=1}^{\infty} k \cdot S$ contains a copy of $\{x_1, x_2, \ldots, x_q\}$. $\square$

Note that the set $ad \cdot S \subset \bigcup_{k=1}^{\infty} k \cdot S$ is an union of countably many perfect and meager sets. From the result of F. Galvin, J. Mycielski R. M. Solovay [4] it follows the following.

Theorem 2. *If a set of real numbers X is countable, then for any meager set G there exists a real x such that $(x + X) \cap G = \emptyset$.*

A proof of the above fact one can deduce from Theorem 3.5 which was placed in A. W. Miller, [8, p. 209]. Since a meager set can have the complement of Lebesgue measure zero, then any such complement has to contains a similar copy of any countable set. In other words, any dense G_δ set of Lebesgue measure zero contains a similar copy of each countable set. We have an other answer onto Svetic's question since a finite set is countable, too. But, no dense G_δ set of real numbers is an union of countably many perfect and meager sets. By this meaning, our Theorem 2 gives a more subtle answer onto Svetic's question.

2 A Uniform Density Theorem

Let E be an Euclidean space with a metric ϱ. For the Lebesgue measure λ on E and a compact set $X \subset E$ consider the following principle, where $B(X, h) = \{x \in E : \inf\{\varrho(x, y) : y \in X\} < h\}$. In [5], H. Hadwiger defined and used a principle we find useful in our context. Below, we state this principle and give a short proof.

Theorem 3 (Hadwiger Principle)**.** *For every $\varepsilon > 0$ there exists $h > 0$ such that for any $t \in B(\{0\}, h)$ it follows that*

$$\lambda(X) - \lambda(X \cap (X + t)) < \varepsilon.$$

PROOF. For any $\varepsilon > 0$ let $h > 0$ be such that $\lambda(B(X,h)) < \lambda(X) + \varepsilon$. So, for any $t \in B(\{0\},h)$ there holds $X + t \subseteq B(X,h)$, and hence

$$\lambda(X) - \lambda(X \cap (X+t)) \leq \lambda(B(X,h)) - \lambda(X) < \varepsilon.$$

In the literature one can find this principle introduced as the sentence: If a set $X \subseteq E$ is compact, then $\lim_{t\to 0} \lambda(X \cap (X+t)) = \lambda(X)$.

A set $X \subseteq E$ is called *measurably large* if X is measurable, and for every real number $h > 0$ there holds $\lambda(X \cap B(\{0\},h)) > 0$. This notion was introduced by V. Bergelson, N. Hindman and B. Weiss in [1, p. 63]. In fact, one can find it in Sz. Plewik and B. Voigt, [9, p. 138], where it was used in Theorem 1.

If X is a Lebesgue measurable set and X^* denotes its density points, then there holds the following. If $t \in X^*$ and $t + p \in X^*$, then for any real number $h > 0$ the intersection $B(\{t\},h) \cap (X - p) \cap X$ has positive Lebesgue measure. Since almost all points of X belong to X^* one has the following:

(*) *For any measurable set X there exists a measurable subset $X^* \subseteq X$ such that $\lambda(X) = \lambda(X^*)$ and if $p \in X^*$ and $t + p \in X^*$, then the intersection $(X - t - p) \cap (X - p)$ is measurably large.*

The following lemma can be found in [1, Lemma 2.2].

Lemma 2 (Bergelson-Hindman-Weiss)**.** *Let $A \subseteq (0,1]$ be measurably large. There exist* (*many*) *$t \in A$ such that $A \cap (A - t)$ is measurably large.*

We shall improve it. The word *many* is replaced by words *for almost all*. The next theorem was announced in Sz. Plewik, [10]

Theorem 4. *If X is measurably large, then for almost all $t \in X$ the intersection $X \cap (X - t)$ is measurably large.*

PROOF. Fix a measurably large set $D \subseteq X^*$ such that $D_1 = \{0\} \cup D \subseteq X$ is a compact set. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \lambda(D)$. By the Hadwiger argument there is a real number $h_1 > 0$ such that for any $t \in B(\{0\},h_1)$ there holds $\lambda(D_1) < \lambda(D_1 \cap (D_1 - t)) + \alpha_1$. Fix $t_1 \in D \cap B(\{0\},h_1)$ and put $D_2 = D_1 \cap (D_1 - t_1)$. The set D_2 is compact and $\lambda(D_1) < \lambda(D_2) + \alpha_1$.

Suppose there have been defined compact sets $D_1, D_2, \ldots, D_n$ and points $\{t_1, t_2, \ldots, t_{n-1}\} \subseteq D$ such that $D_{k+1} = D_k \cap (D_k - t_k)$ and $\lambda(D_k) < \lambda(D_{k+1}) + \alpha_k$, for $0 < k < n$. By the Hadwiger argument there is a positive real number $h_n > 0$ such that for any $t \in B(\{0\},h_n)$ there holds $\lambda(D_n) < \lambda(D_n \cap (D_n - t)) + \alpha_n$. Fix $t_n \in D \cap B(\{0\},h_n)$ and put $D_{n+1} = D_n \cap (D_n - t_n)$. The set D_{n+1} is compact and $\lambda(D_n) < \lambda(D_{n+1}) + \alpha_n$.

So, there have been defined compact sets $D_1, D_2, \ldots$ such that

$$\lambda(D) < \lambda(D_1 \cap D_2 \cap \ldots) + \sum_{n=1}^{\infty} \alpha_n.$$

We have assumed $\lambda(D) > \sum_{n=1}^{\infty} \alpha_n$, thus one infers that there exists a point $p \in D_1 \cap D_2 \cap \ldots$, where $p \neq 0$. Since

$$p \in \cap\{D_n : n = 1, 2, \ldots\} = \cap\{D_n \cap (D_n - t_n) : n = 1, 2, \ldots\}$$

there always holds $p \in D_n - t_n$. So $p + t_n \in D_n \subseteq D \subseteq X^*$. By $(*)$, because of $t_n \in D \subseteq X^*$, the intersection $(X - t_n) \cap (X - p - t_n)$ is always measurably large. Therefore $(X \cap (X - p)) - t_n$ is always measurably large, too. For a real number $h > 0$ take a set $A \subseteq B(\{0\}, \frac{h}{2}) \cap ((X \cap (X - p)) - t_n)$ such that $\lambda(A) > 0$. If $t_n \in B(\{0\}, \frac{h}{2})$, then $\lambda(A + t_n) > 0$ and

$$A + t_n \subseteq X \cap (X - p) \cap B(\{0\}, h).$$

Since $h > 0$ could be arbitrary one infers that $X \cap (X - p)$ is measurably large.

For every number $p \in D_1 \cap D_2 \cap \ldots$ the above argument works. Since the number $\sum_{n=1}^{\infty} \alpha_n < \lambda(D)$ could be arbitrarily small and $\lambda(X) = \lambda(X^*)$, then sets D_n could be chosen such that $\lambda(X \setminus (D_1 \cap D_2 \cap \ldots))$ is arbitrary small, whenever $\lambda(X) < \infty$. This follows the finish conclusion. □

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