DARBOUX-INTEGRABILITY AND
UNIFORM CONVERGENCE

Abstract

In 1992, Šikić gives a characterization of Riemann-integrable functions as uniform limits of simple functions. The aim of this article is to prove an extension to the case of functions defined on a basic space \((X, D, \mu)\) and with values in any Banach space \(F\).

0 Introduction

In the article [6], the author gives a characterization of Riemann-integrable functions as uniform limits of simple functions; more exactly, he proves the following assertion:

Theorem (Šikić). The function \(f : [a, b] \mapsto \mathbb{R}\) is Riemann-integrable if and only if \(f\) is the uniform limit of a sequence of functions

\[ f_n = \sum_{i=1}^{l_n} a_{i,n} \cdot 1_{A_{i,n}} \]

where \(A_{i,n} \in A\), the algebra of subsets of \([a, b]\) formed by the Lebesgue-measurable subsets \(A\) of \([a, b]\) with \(\Lambda(\text{Fr}(A)) = 0\), where \(\text{Fr}\) denotes the boundary and \(\Lambda\) is the Lebesgue measure.

Note that exercise 116 of §7 from [2] presents a generalization of this result to the case of functions with values in a Banach space of finite dimension. The aim of this article is to prove Theorem 2.5, which gives an extension to the case of functions defined on a basic space \((X, D, \mu)\) and with values in any Banach space \(F\). We precise that the proofs - most of them are simple - of the results quoted in this paper are in the thesis [1].

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1 Preliminaries

1.1 Conventions and Notation

If \( a_i \) denotes an element of a vector space and \( A_i \) a subset of a set, we use the following conventions: \( \sum_{i \in \emptyset} a_i = 0 \), and \( \bigcup_{i \in \emptyset} A_i = \emptyset \). Moreover, the notation \( \coprod \) denotes disjoint union.

The Banach spaces we consider are over the field \( \mathbb{R} \) of real numbers. Let \( F \) be a Banach space with norm \( \| \cdot \| \), and \( P \) a non-empty subset of \( F \); we call diameter of \( P \) the quantity \( \text{diam}(P) = \sup_{y,z \in P} \|y - z\| \).

1.2 Semi-Ring

Given a set \( X \), a semi-ring \( D \) of subsets of \( X \) is a family of subsets of \( X \) such that

- \( \emptyset \in D \);
- if \( A, B \in D \), then \( A \cap B \in D \);
- if \( A, B \in D \), then \( A \setminus B = A \cap B^c = \coprod_{j=1}^n C_j \), where \( C_j \in D, 1 \leq j \leq n \).

Note that, generally, \( A \setminus B \notin D \).

1.3 Finite \( D \)-Partition

Given a non-empty set \( X \) and \( D \) a semi-ring of subsets of \( X \), every finite family \( \pi = \{ D_1, \ldots, D_m \} \) of non-empty disjoint elements of \( D \) and such that \( X = \coprod_{j=1}^n D_j \), is called a finite \( D \)-partition of \( X \). We write \( \Pi_X \) for the set of all the finite \( D \)-partitions of \( X \).

1.4 Fineness on \( \Pi_X \)

Suppose that \( \pi_1 \) is a finite \( D \)-partition of \( X \); a finite \( D \)-partition \( \pi_2 \) of \( X \) is said to be finer than \( \pi_1 \), which we note by \( \pi_2 \gg \pi_1 \), if every element of \( \pi_1 \) is the union of elements of \( \pi_2 \).

1.5 Remark

Given \( \pi_1 \) and \( \pi_2 \) any two finite \( D \)-partitions of \( X \), there exists a finite \( D \)-partition \( \pi \) of \( X \) finer than \( \pi_1 \) and \( \pi_2 \). Indeed, if \( \pi_1 = \{ D_1, \ldots, D_m \} \) and \( \pi_2 = \{ E_1, \ldots, E_n \} \), it suffices to consider the set of the \( D_i \cap E_j \) which are non-empty, \( 1 \leq i \leq m, 1 \leq j \leq n \).
1.6 Lemma

Let $X$ be a non-empty set, $\mathcal{D}$ a semi-ring of subsets of $X$ such that there exists a finite $\mathcal{D}$-partition of $X$. Then,

$$\mathcal{A}(\mathcal{D}) = \left\{ \bigcap_{i=1}^{n} D_i : D_i \in \mathcal{D}, 1 \leq i \leq n, n \in \mathbb{N}^* \right\}$$

is the algebra (of subsets of $X$) generated by $\mathcal{D}$.

1.7 Remark

(to be used in the proof of Theorem 2.5)

In the hypothesis of Lemma 1.6, if $m \in \mathbb{N}^*$ and $D_1, \ldots, D_m \in \mathcal{D} \setminus \{\emptyset\}$ with $D_i \cap D_j = \emptyset$ if $i \neq j$, then there exists $\pi \in \Pi_X$ such that every $D_i \in \pi$, $1 \leq i \leq m$. Indeed, if $A = \bigcup_{i=1}^{m} D_i = X$, then $\pi = \{D_1, \ldots, D_m\}$. And if $A = \bigcup_{i=1}^{m} D_i \neq X$, then $A^c \in \mathcal{A}(\mathcal{D})$ and $A^c \neq \emptyset$, thus there exists $D_{m+1}, \ldots, D_n \in \mathcal{D} \setminus \{\emptyset\}$ such that $A^c = \bigcap_{i=m+1}^{n} D_i$; so that $\pi = \{D_1, \ldots, D_n\} \in \Pi_X$.

1.8 Functions $\mathcal{D}$-Simple

Let $\mathcal{D}$ be a semi-ring of subsets of a set $X$ (such that $\Pi_X \neq \emptyset$), and $F$ a Banach space. Consider $V = \mathbb{R}_+$ or $V = F$, and let

$$S_V(\mathcal{D}) = \left\{ \sum_{i=1}^{m} v_i \cdot 1_{D_i} : v_i \in V, \{D_1, \ldots, D_m\} \in \Pi_X \right\},$$

where $1_D$ denotes the indicator function of $D$. The elements of $S_V(\mathcal{D})$ are called $\mathcal{D}$-simple functions with values in $V$.

1.9 (Jordan) Content

Given $X$ a non-empty set, and $\mathcal{D}$ a semi-ring of subsets of $X$ such that there exists a finite $\mathcal{D}$-partition of $X$, we call (Jordan) content, any monotone function of sets $\mu$ defined on $\mathcal{A}(\mathcal{D})$ which is finite, positive and additive, that is $\mu : \mathcal{A}(\mathcal{D}) \rightarrow \mathbb{R}_+$, $\mu(\emptyset) = 0$, $\mu(A) \leq \mu(B)$ if $A \subset B$, and

$$\mu\left( \bigcap_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} \mu(A_i), n \in \mathbb{N}^*.$$
1.10 Basic Space

We call basic space any triple \((X, D, \mu)\), where \(X\) is a non-empty set provided with a semi-ring \(D\) of subsets of \(X\) such that there exists a finite \(D\)-partition of \(X\), and \(\mu\) is a (Jordan) content defined on \(A(D)\).

1.11 Lemma

Let \((X, D, \mu)\) be a basic space. Then, \(\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i)\), for every \(A_i \in A(D)\), \(1 \leq i \leq n, n \in \mathbb{N}^*\).

1.12 Example

Consider \(a < b \in \mathbb{R}\), and \(X = [a, b]\); let \(D = \{[a, b] \cap [\alpha, \beta] : \alpha < \beta \in \mathbb{R}\} \) 
\(= \{[a, \beta] : a \leq \beta \leq b\} \cup \{[\alpha, b] : a \leq \alpha \leq \beta \leq b\}\), with \(\mu([a, \beta]) = (\beta - a)\) and \(\mu([\alpha, b]) = (\beta - a)\). Then, \((X, D, \mu)\) is a basic space.

1.13 Darboux-Integrability

Consider \((X, D, \mu)\) a basic space and \(F\) a Banach space. A function \(f : X \mapsto F\) is said to be Darboux-integrable, what we will note by \(D\)-integrable or \(D\)-\(F\)-integrable if there is a risk of confusion, if

(a) \(\text{diam}(f(X)) < \infty\) (what is equivalent to \(f\) bounded);

(b) for every \(\varepsilon > 0\), there exists \(\pi_{\varepsilon} = \{D_1, \ldots, D_n\}\) a finite \(D\)-partition of \(X\) such that \(\sum_{i=1}^{n} \text{diam}(f(D_i))\mu(D_i) < \varepsilon\).

1.14 Lemma

The set \(\mathcal{I}_D(X, D, \mu; F)\) of the \(D\)-(\(X, D, \mu; F\))-integrable functions is a vector subspace of \(B(X; F)\), the set of the bounded functions from \(X\) to \(F\), and moreover \(S_F(D)\) is a subset of \(\mathcal{I}_D(X, D, \mu; F)\).

The proposition 1.16 below will allow us to give the definition of the Darboux-integral of a Darboux-integrable function.
1.15 Notations and Remarks

Let $(X, \mathcal{D}, \mu)$ be a basic space, $F$ a Banach space, and $f : X \mapsto F$ a bounded function. For each $D \in \mathcal{D}$ with $D \neq \emptyset$, let

$$E_f(D) = \{y \in F : \exists a \in \text{Conv}(f(D)) \text{ such that } ||y - a|| \leq \text{diam}(f(D))\},$$

where Conv$(f(D))$ is the convex hull of $f(D)$, that is,

$$\text{Conv}(f(D)) = \left\{\sum_{i=1}^{m} \lambda_i f(x_i) : 0 \leq \lambda_i \leq 1, \sum_{i=1}^{m} \lambda_i = 1, x_i \in D, 1 \leq i \leq m, m \in \mathbb{N}^*\right\}.$$

Note that $f(D) \subset \text{Conv}(f(D)) \subset E_f(D)$; moreover, for every $x \in D$, we have $\{y \in F : ||y - f(x)|| \leq \text{diam}(f(D))\} \subset E_f(D)$. In addition, we observe that if $f(D) = \{a\}$, then $E_f(D) = \{a\}$.

1.16 Proposition

Given $(X, \mathcal{D}, \mu)$ a basic space and $F$ a Banach space, a bounded function $f : X \mapsto F$ is $\mathcal{D}$-$F$-integrable if and only if there exists $I \in F$ such that for every $\varepsilon > 0$, there exists $\pi_\varepsilon = \{D_1, \ldots, D_l\}$ a finite $\mathcal{D}$-partition of $X$ such that

$$\left|\left|\sum_{i=1}^{l} y_i \cdot \mu(D_i) - I\right|\right| < \varepsilon$$

for every $y_i \in E_f(D_i), 1 \leq i \leq l$. Moreover, in that case, $I$ is unique.

1.17 Darboux-Integral

Let $f : X \mapsto F$ be a $\mathcal{D}$-$F$-integrable function. Then, the unique element $I = I(f)$ established in Proposition 1.16 is called the Darboux-integral of $f$ and is noted $\mathcal{D} \int_X f(x) d\mu(x)$ or $I_{\mathcal{D}}(f)$ (to simplify the writing).

1.18 Remark

Given $f$ a $\mathcal{D}$-$F$-integrable function and $\varepsilon_n \downarrow 0$, if $\pi_n = \{D_{1,n}, \ldots, D_{l_n,n}\} \in \Pi_X$ with $\sum_{i=1}^{l_n} \text{diam}(f(D_{i,n}))\mu(D_{i,n}) < \varepsilon_n$, then if $x_{i,n} \in D_{i,n}, 1 \leq i \leq l_n, n \geq 1$, we obtain

$$I_{\mathcal{D}}(f) = \lim_{n \to \infty} \left(\sum_{i=1}^{l_n} f(x_{i,n})\mu(D_{i,n})\right).$$
1.19 Proposition

Let \((X, D, \mu)\) be a basic space and \(\Theta\) a topology on \(X\). Suppose that for every \(D \in D\) with \(D \neq \emptyset\) and for every \(\delta > 0\), there exists \(E_1, \ldots, E_{K(D, \delta)} \in D \setminus \{\emptyset\}\) (which depend on \(D\) and \(\delta\)) pairwise disjoint, \(E_k \subset D, 1 \leq k \leq K(D, \delta)\), with

\[
\sum_{k=1}^{K(D, \delta)} \mu(E_k) < \delta, \quad \text{and} \quad \left(D \setminus \bigcap_{k=1}^{K(D, \delta)} E_k\right) \subset D.
\]

Consider a Banach space \(F\) and \(f : X \mapsto F\) a bounded function. Then, \(f\) is \(\mathcal{D}\)-\(F\)-integrable if and only if for every \(\varepsilon > 0\), there exists \(\pi_\varepsilon = \{D_1, \ldots, D_m\} \in \Pi_X\) such that \(\sum_{i=1}^{m} \text{diam}(f(D_i))\mu(D_i) < \varepsilon\).

Moreover, in that case, \(I_\mathcal{D}(f) = \lim_{n \to \infty} \left(\sum_{i=1}^{n} f(x_{i,n})\mu(D_{i,n})\right)\), where \(x_{i,n} \in D_{i,n}, 1 \leq i \leq l_n, \) and \(\sum_{i=1}^{l_n} \text{diam}(f(D_{i,n}))\mu(D_{i,n}) < \varepsilon_n \downarrow 0\).

1.20 An Application

Consider the basic space \((X = [a, b], D, \mu)\) of Example 1.12, and \(X\) provided with the usual topology.

If \(D = [\alpha, \beta]\), then \(\overline{D} = D\). If \(D = [\alpha, \beta]\) with \(\alpha < \beta\), and if \(\delta > 0\), let \(E_\delta = [\alpha, \gamma_\delta]\), where \(\gamma_\delta = \min\{\alpha + \frac{\delta}{2}; \beta - \frac{\delta}{2}\}\); then \(\overline{D} \setminus E_\delta = [\gamma_\delta, \beta] \subset [\alpha, \beta] = D\), and moreover \(\mu(E_\delta) \leq \frac{\delta}{2} < \delta\).

Consequently, by Proposition 1.19, if \(f : X \mapsto F\) is a bounded function with values in a Banach space \(F\), then \(f\) is \(\mathcal{D}\)-\(F\)-integrable if and only if for every \(\varepsilon > 0\), there exists \(\pi_\varepsilon = \{D_1, \ldots, D_m\} \in \Pi_X\) with \(\sum_{i=1}^{m} \text{diam}(f(D_i))\mu(D_i) < \varepsilon\).

In particular, if \(F = \mathbb{R}\), then \(f\) is \(\mathcal{D}\)-\(\mathbb{R}\)-integrable if and only if for each \(\varepsilon > 0\), there exists \(\pi_\varepsilon = \{D_1, \ldots, D_m\} \in \Pi_X\) such that

\[
\varepsilon > \sum_{i=1}^{m} \text{diam}(f(D_i) = [\alpha_i, \beta_i])\mu(D_i) = \sum_{i=1}^{m} \sup_{x,y \in [\alpha_i, \beta_i]} |f(x) - f(y)| (\beta_i - \alpha_i)
\]

\[
= \sum_{i=1}^{m} \sup_{x \in [\alpha_i, \beta_i]} (f(x) - \inf_{x \in [\alpha_i, \beta_i]} f(x)) (\beta_i - \alpha_i),
\]

in other words, \(f\) is \(\mathcal{D}\)-\(\mathbb{R}\)-integrable if and only if \(f\) is Riemann-integrable.

Moreover, we have \(I_\mathcal{D}(f) = \int_{a}^{b} f(x)dx\).
1.21 Proposition

Consider a basic space \((X, \mathcal{D}, \mu)\) and \(F\) a Banach space. Then, a bounded function \(f : X \mapsto F\) is \(\mathcal{D}-F\)-integrable if and only if there exists a sequence \(\pi_n = \{D_{1,n}, \ldots, D_{k_n,n}\}\) of finite \(\mathcal{D}\)-partitions of \(X\) such that 
\[\pi_{n+1} \gg \pi_n, \quad n \in \mathbb{N}^*,\]
and such that for every \(\varepsilon > 0\), 
\[\lim_{n \to \infty} \mu(A_n(f; \varepsilon)) = 0,\]
where 
\[A_n(f; \varepsilon) = \bigcup_{j \in J_n(\varepsilon)} D_{j,n},\]
with 
\[J_n(\varepsilon) = \{1 \leq j \leq k_n : \text{diam}(f(D_{j,n})) > \varepsilon\}, \quad n \in \mathbb{N}^*.\]

2 Darboux-Integrability and Uniform Convergence

The aim of this paragraph is Theorem 2.5. However we first give some preliminary results. Add that Lemma 2.1 can be proved in a classical way, but Corollary 3.4 gives another proof.

2.1 Lemma

Given a basic space \((X, \mathcal{D}, \mu)\) and a Banach space \(F\), let \((f_n)_{n \geq 1}\) be a sequence of \(\mathcal{D}-F\)-integrable functions and \(f\) be a function such that \(f\) is the uniform limit of the \(f_n\). Then, \(f\) is \(\mathcal{D}-F\)-integrable.

2.2 Remark

If \(f\) is the uniform limit of \(C\)-simple functions, where \(C\) is a semi-ring of subsets of \(X\), then \(f(X)\) is totally bounded and then, as \(F\) is a Banach space, we deduce that \(f(X)\) is compact. Indeed, let \(\varepsilon > 0\); we have \(\|f - f_n\|_\infty < \varepsilon, \quad n \geq n_0 = n_0(\varepsilon) \in \mathbb{N}^*\), where 
\[f_n = \sum_{i=1}^{l_n} c_{i,n} \cdot 1_{C_{i,n}} \in \mathcal{S}_F(C).\]
Then, as \(\{C_{1,n_0}, \ldots, C_{l_n,n_0}\} \in \Pi_X\), we have \(\overline{f(X)} \subset \bigcup_{i=1}^{l_n} B(c_{i,n_0}, \varepsilon).\)

2.3 Definition of the Algebra \(\mathcal{B}\) (of Subsets of \(X\))

Given \(B \subset X\) and \(\pi = \{D_1, \ldots, D_n\}\) a finite \(\mathcal{D}\)-partition of \(X\), let 
\[\Delta_{\pi, B} = \{1 \leq i \leq n : D_i \cap B \neq \emptyset \quad \text{and} \quad D_i \cap B^c \neq \emptyset\}.\]
Let \(\mathcal{B} = \{B \subset X \quad \text{such that for every} \quad \varepsilon > 0, \quad \text{there exists} \quad \pi_\varepsilon = \{D_1, \ldots, D_n\} \quad \text{a finite} \quad \mathcal{D}\)-partition of \(X \quad \text{such that} \quad \sum_{i \in \Delta_{\pi_\varepsilon, B}} \mu(D_i) < \varepsilon\}.\)
2.4 Lemma

(a) The family $B$ is an algebra (of subsets of $X$) containing $\mathcal{D}$.

(b) If $F$ is a Banach space, then $\mathcal{S}_F(\mathcal{B})|||\infty \subset I_\mathcal{D}(X, \mathcal{D}, \mu; F)$.

2.5 Theorem

Given a basic space $(X, \mathcal{D}, \mu)$ and a Banach space $F$, let $f : X \mapsto F$ be a function. Then, $f \in \mathcal{S}_F(\mathcal{B})|||\infty$ if and only if $f(X)$ is compact and $f$ is $\mathcal{D}$-$F$-integrable.

Proof.

Necessity. From (b) of Lemma 2.4, $f$ is $\mathcal{D}$-$F$-integrable; moreover, from Remark 2.2, we deduce that $f(X)$ is compact.

Sufficiency. Suppose that $f$ is $\mathcal{D}$-$F$-integrable; from Proposition 1.21, there exists a sequence $\left(\pi_n = \{D_{1,n}, \ldots, D_{n,n}\}\right)_{n \geq 1}$ of finite $\mathcal{D}$-partitions of $X$ such that $\pi_{n+1} \Rightarrow \pi_n$, $n \in \mathbb{N}^*$, and such that for every $\varepsilon > 0$, $\lim_{n \to \infty} \mu(A_n(\varepsilon)) = 0$, where

$$A_n(\varepsilon) = \bigcap_{j \in J_n(\varepsilon)} D_{j,n},$$

where $J_n(\varepsilon) = \{1 \leq j \leq l_n : \operatorname{diam}(f(D_{j,n})) > \varepsilon\}$.

Consider $\varepsilon > 0$ and let $B_\varepsilon = \bigcap_{n=1}^{\infty} A_n(\varepsilon)$. Prove that $B \subset B_\varepsilon$. Now, for each $\eta > 0$, there exists $n_0 = n_0(\eta) \in \mathbb{N}^*$ such that $\mu(A_n(\varepsilon)) < \eta$ for every $n \geq n_0$; consider $A_{n_0}(\varepsilon)$.

As $B \subset A_{n_0}(\varepsilon)$, we deduce $\Delta_{\pi_{n_0}, B} \subset J_{n_0}(\varepsilon)$ (because if $D_{j,n_0} \cap B \neq \emptyset$, then $D_{j,n_0} \cap A_{n_0}(\varepsilon) \neq \emptyset$ and therefore $D_{j,n_0} \subset A_{n_0}(\varepsilon)$). It follows that $\sum_{j \in J_{n_0}} \mu(D_{j,n_0}) \leq \mu(A_{n_0}(\varepsilon)) < \eta$. Thus, as $\eta > 0$ is arbitrary, we obtain $B \subset B_\varepsilon$.

Considering first (if necessary) $g = f - f(x_0)$, where $x_0 \in X$, we can suppose, without loss of generality, that there exists $x \in X$ with $f(x) = 0$. As $f(X)$ is compact, there exists $a_1, \ldots, a_p \in F$ such that $f(X) \subset \bigcup_{i=1}^p B(a_i; \varepsilon) = \bigcap_{j=1}^q V_j$, where $q \leq p$, $V_j \neq \emptyset$, and $||y - z|| < 2\varepsilon$ if $y, z \in V_j, 1 \leq j \leq q$, (where $B(a_i; \varepsilon)$ denotes the open ball of center $a_i$ and radius $\varepsilon$). Indeed, let $U_m = \bigcup_{i=1}^p B(a_i; \varepsilon), 1 \leq m \leq p$. Then,

$$\bigcup_{i=1}^p B(a_i; \varepsilon) = U_1 \prod_{m=2}^p (U_m \setminus U_{m-1})$$
and we have the existence of the $V_j$.

Let $f = f_1 + f_2$, where $f_1 = f \cdot 1_{B_j}$ and $f_2 = f \cdot 1_{B_j^c}$. For every $1 \leq j \leq q$, let $B_j = f_j^{-1}(V_j)$. There exists (one and only one) $j_0 \in \{1, \ldots, q\}$ with $0 \in V_{j_0}$. So, for every $1 \leq j \leq q$ with $j \neq j_0$, we have $B_j \subset B_{j_0}$; therefore, $B_j \in \mathcal{B}$, $j \neq j_0$. As $\mathcal{B}$ is an algebra and from the fact that $X = \bigcup_{j=1}^{q} B_j$, it follows that $B_{j_0} \in \mathcal{B}$.

For each $1 \leq j \leq q$, consider $b_j \in V_j$ and let $\varphi_{e} = \sum_{j=1}^{q} b_j \cdot 1_{B_j}$. We obtain $\varphi_{e} \in \mathcal{S}_{F}(\mathcal{B})$ and $\|f_1 - \varphi_{e}\|_{\infty} \leq 2\varepsilon$. Consider the case of $f_2 = f \cdot 1_{B_j^c}$. Observe that $B_{j} = \bigcup_{n=1}^{\infty} \left( A_n(\varepsilon) \right)^c = (A_1(\varepsilon))^c \prod_{n=1}^{\infty} \left( A_n(\varepsilon) \setminus A_{n+1}(\varepsilon) \right)$. Now, we have $(A_1(\varepsilon))^c = D_{i_{1}(1,1)} \prod_{j=1}^{p} D_{i_{j}(1,1)} = E_1 \prod_{j=1}^{p} E_{k_j}$ with $E_k = D_{i_{k}(1,1)} \in \mathcal{D}$ (maybe $\emptyset$, but only if $(A_1(\varepsilon))^c = \emptyset$), and $\text{diam}(f(D_{i_{k}(1,1)})) \leq \varepsilon$ ($1 \leq k \leq k_{1}$); with the convention $\text{diam}(\emptyset) = 0$; and for every $n \in \mathbb{N}^{*}$, we have

$$A_n(\varepsilon) \setminus A_{n+1}(\varepsilon) = D_{i_{1}(n+1,1)} \prod_{j=1}^{p} D_{i_{j}(n+1,1)} = E_n \prod_{j=1}^{p} E_{k_{j}},$$

with $\text{diam}(f(D_{i_{k}(n+1,1)})) \leq \varepsilon$ ($1 \leq k \leq n+1$); in other words $B_{j} = \prod_{l=1}^{\infty} E_{l}$ with $E_l \in \mathcal{D}$ and $\text{diam}(f(E_l)) \leq \varepsilon$, $l \in \mathbb{N}^{*}$. Let $l \in \mathbb{N}^{*}$; $E_l$ corresponds to a $D_{i_{l}(1,n_{l})}$ (which can be $\emptyset$), for a $n_l \in \mathbb{N}^{*}$; if $E_l = D_{i_{l}(1,n_{l})} \neq \emptyset$, let $\alpha_l = f(\tilde{x}_l)$ for $\tilde{x}_l \in E_l$; if $E_l = \emptyset$, let $\alpha_l = 0$.

Note that $\|f \cdot 1_{E_l} - \alpha_l \cdot 1_{E_l}\|_{\infty} \leq \text{diam}(f(E_l)) \leq \varepsilon$. Let $f_3 = \sum_{l=1}^{\infty} \alpha_l \cdot 1_{E_l}$; so, we have $\|f_2 - f_3\|_{\infty} = \left\| \sum_{l=1}^{\infty} f \cdot 1_{E_l} - \sum_{l=1}^{\infty} \alpha_l \cdot 1_{E_l} \right\|_{\infty} \leq \varepsilon$. Given $S \subset \mathbb{N}^{*}$, $S \neq \emptyset$, let $B_S = \prod_{l \in S} E_{l}$. Prove that $B_S \in \mathcal{B}$. If $S$ is finite, then $B_S \in \mathcal{B}$ (because $\mathcal{B}$ is an algebra containing $\mathcal{D}$ and $E_s \in \mathcal{D}$, $s \in S$). If $S$ is infinite, write $S = \{s_1, s_2, \ldots\}$ with $s_i < s_j$ if $i < j$.

Let $\eta > 0$; there exists $n_0 = n_0(\eta) \in \mathbb{N}^{*}$ such that for every $n \geq n_0$, we have $\mu(A_n(\varepsilon)) < \eta$. Now, for each $p \in \mathbb{N}^{*}$, $E_{s_p} = D_{i_{p}(s_p),n_{s_p}}$ for a $n_{s_p} \in \mathbb{N}^{*}$. Observe that from the “construction” of the $E_l$, if $p_1 < p_2$, then $n_{s_{p_1}} \leq n_{s_{p_2}}$. Consider $n_1 \geq \max\{n_0(\eta), n_{s_1}\}$ and let $p_0 = \min\{p \in \mathbb{N}^{*} : n_{s_p} > n_1\}$. So, we have $n_{s_{p_0}} > n_1$, $p_0 \geq 2$ (because $n_{s_1} \leq n_1$), and $n_{s_{p_0-1}} \leq n_1$. Moreover, $B_S = \prod_{p=1}^{p_0-1} E_{s_p} \prod_{p=p_0}^{\infty} E_{s_p}$.
But, for every \( p \geq p_0 \geq 2 \), we can write
\[
E_{sp} = D_{ij(\varepsilon)}(n_{sp},n_{sp}) \subset \left( A_{n_{sp}-1}(\varepsilon) \setminus A_{n_{sp}}(\varepsilon) \right) \subset A_{n_{sp}-1}(\varepsilon) \\
\subset A_{n_{sp}}(\varepsilon) \subset A_{n_1}(\varepsilon).
\]
It follows \( B_S \subset \bigcap_{p=1}^{p_0-1} E_{sp} \bigcap_{j=1}^{\infty} A_{n_1}(\varepsilon) =: U \).

Note that we really have a disjoint union, because \( E_{sp} = D_{ij(\varepsilon)}(n_{sp},n_{sp}) \subset (A_{n_{sp}}(\varepsilon))^c \). Now, for every \( 1 \leq p \leq p_0 - 1 \), we have \( n_{sp} \leq n_1 \), and therefore \( A_{n_1}(\varepsilon) \subset A_{n_{sp}}(\varepsilon) \); so that
\[
E_{sp} \bigcap_{j=1}^{n_1} A_{n_j}(\varepsilon) \subset \left( (A_{n_{sp}}(\varepsilon))^c \bigcap A_{n_1}(\varepsilon) \right) \subset \left( (A_{n_1}(\varepsilon))^c \bigcap A_{n_1}(\varepsilon) \right) = \emptyset.
\]
As \( A_{n_1}(\varepsilon) = \bigcap_{j=1}^{n_1} D_{j,n_1} \), it follows that \( U = \bigcap_{p=1}^{p_0-1} E_{sp} \bigcap_{j=1}^{n_1} D_{j,n_1} \).

If \( U = \emptyset \), then \( B_S = \emptyset \in \mathcal{B} \). Suppose \( U \neq \emptyset \). From Remark 1.7, there exists \( \pi = \{ C_1, \ldots, C_r \} \in \Pi_X \) such that the non-empty elements of \( D \) which constitute \( U \) appear among the \( C_j \).

Suppose that \( C_j \bigcap B_S \neq \emptyset \) and \( C_j \bigcap (B_S)^c \neq \emptyset \); then, \( C_j \) cannot be one of the \( E_{sp} \), \( 1 \leq p \leq p_0 - 1 \), because \( E_{sp} \subset B_S \). As \( C_j \) cannot be in \( U^c \), the only possibility is that \( C_j \) is one of the \( D_{j,n_1} \) for a \( j \in J_{n_1}(\varepsilon) \). Hence, \( \bigcap_{j \in J_{n_1}(\varepsilon)} C_j \subset A_{n_1}(\varepsilon) \), and so \( 0 \leq \sum_{j \in J_{n_1}(\varepsilon)} \mu(C_j) \leq \mu(A_{n_1}(\varepsilon)) < \eta \). It follows that \( B_S \in \mathcal{B} \) for every \( S \in \mathbb{N}^*, S \neq \emptyset \).

Recall that \( f_3 = \sum_{l=1}^{q} \alpha_l \cdot 1_{E_l} \) and \( f(X) \subset \bigcup_{j=1}^{q} V_j \), where \( V_j \neq \emptyset \) and \( ||y-z|| < 2\varepsilon \) if \( y,z \in V_j \), for \( 1 \leq j \leq q \). We observe that \( f_3(X) \subset \bigcup_{j=1}^{q} V_j \) (because \( 0 \in V_{j_0} \) and if \( f_3(x) \neq 0 \), then \( f_3(x) = \alpha_l(x) = f(\hat{x}_l(x)) \in \prod_{j=1}^{q} V_j \)). For every \( 1 \leq j \leq q \), let \( \hat{B}_j = f_3^{-1}(V_j) \). Then \( \hat{B}_j \in \mathcal{B}, 1 \leq j \leq q \). This is true because \( \emptyset \in \mathcal{B} \), and if \( j \neq j_0 \) with \( f_3^{-1}(V_j) \neq \emptyset \), then \( f_3^{-1}(V_j) = \bigcap_{l: \alpha_l \in V_j} E_l \in \mathcal{B} \) (from what precedes), and finally, we have \( f_3^{-1}(V_{j_0}) = \bigcap_{l: \alpha_l \in V_{j_0}} E_l \bigcap_{l=1}^{\infty} E_l \in \mathcal{B} \).

Moreover, we have \( X = \bigcup_{j=1}^{q} \hat{B}_j \). For each \( 1 \leq j \leq q \), let \( \hat{b}_j \in V_j \) and consider \( \psi_\varepsilon = \sum_{j=1}^{q} \hat{b}_j \cdot 1_{\hat{B}_j} \in \mathcal{S}_F(\mathcal{B}) \). So, we have \( ||f_3 - \psi_\varepsilon||_\infty \leq 2\varepsilon \). Let
\(\xi_\varepsilon = \varphi_\varepsilon + \psi_\varepsilon \in S_F(\mathcal{B})\): we can write
\[
\|f - \xi_\varepsilon\|_\infty = \|f - \varphi_\varepsilon - \psi_\varepsilon\|_\infty = \|f_1 + f_2 - \varphi_\varepsilon - \psi_\varepsilon\|_\infty \\
\leq \|f_1 - \varphi_\varepsilon\|_\infty + \|f_2 - f_3\|_\infty + \|f_3 - \psi_\varepsilon\|_\infty \leq 5\varepsilon.
\]
As \(\varepsilon > 0\) is arbitrary, we deduce that \(f\) is the uniform limit of functions of \(S_F(\mathcal{B})\).

### 2.6 Remarks

1. If \(C\) is an algebra of subsets of \(X\) such that for every \(C \in C\), the function \(\varphi = a \cdot 1_C\) is \(\mathcal{D}\)-\(F\)-integrable for an \(a \in F \setminus \{0\}\) (\(F\) is supposed to be non-reduced to \(\{0\}\)), then \(C \subset \mathcal{B}\). (Indeed, for every \(\varepsilon > 0\), there exists \(\pi_\varepsilon = \{D_1, \ldots, D_n\}\) a finite \(\mathcal{D}\)-partition of \(X\) verifying
\[
\varepsilon \cdot \|a\| > \sum_{i=1}^{n} \text{diam}(\varphi(D_i)) \cdot \mu(D_i) = \sum_{i \in \Delta_{\pi_\varepsilon,C}} \|a\| \cdot \mu(D_i),
\]
therefore
\[
\sum_{i \in \Delta_{\pi_\varepsilon,C}} \mu(D_i) < \varepsilon.
\]
It follows that \(C \in \mathcal{B}\), and then we have the assertion.)

2. Note that \(\mathcal{B}\) is independent of \(F\). As a matter of fact, \(\mathcal{B}\) depends only on \((X, \mathcal{D}, \mu)\).

The following corollary corresponds to exercise 116 from §7 of [2] adapted to the case of a basic space.

### 2.7 Corollary

Let \((X, \mathcal{D}, \mu)\) be a basic space, \(F\) a Banach space of finite dimension and \(f : X \mapsto F\) a (bounded) function. Then, \(f\) is \(\mathcal{D}\)-\(F\)-integrable if and only if \(\|f\|_{S_F(\mathcal{B})} = 0\).

**Proof.** As \(\overline{f(X)}\) is compact, the result follows from Theorem 2.5.

### 2.8 Examples

1. Consider \(X = [a, b]\) with
\[
\mathcal{D} = \{ [a, \beta] : a \leq \beta \leq b \} \cup \{ [\alpha, \beta] : a \leq \alpha \leq \beta \leq b \},
\]
\[
\mu([a, \beta]) = (\beta - a) , \mu([\alpha, \beta]) = (\beta - \alpha),
\]
where \(X\) is provided with the usual topology \(\Theta\), and let \(F = \mathbb{R}\).

With reference to [6], let \(\mathcal{B} = \mathcal{A}\), the algebra of subsets \(A\) of \([a, b]\) such that \(\Lambda(\text{Fr}(A)) = 0\), where \(\Lambda\) denotes the Lebesgue measure, which is complete.
Let $B \in B$. For every $\varepsilon > 0$, there exists $\pi_\varepsilon = \{D_1, \ldots, D_l\} \in \Pi_X$ such that $\sum_{i \in \Delta_{\pi_\varepsilon, B}} \mu(D_i) < \varepsilon$. Considering if necessary $\{a\}$ and $[a, \beta]$, we can suppose that $D_1 = \{a\}$ and for $2 \leq i \leq l$, $D_i = \{\alpha_i = \beta_{i-1}, \beta_i\}$ with $\beta_{i-1} < \beta_i$ and $\alpha_2 = a$. Suppose that $x \in \text{Fr}(B)$. There exists $1 \leq i_x \leq l$ such that $x \in D_{i_x}$. If $x \in [\alpha_{i_x}, \beta_{i_x}]$, then $i_x \in \Delta_{\pi_\varepsilon, B}$, so that $\text{Fr}(B) \subset \bigcup_{i \in \Delta_{\pi_\varepsilon, B}} [a; \beta_i : 2 \leq i \leq l]$. We deduce $0 \leq \Lambda(\text{Fr}(B)) \leq \sum_{i \in \Delta_{\pi_\varepsilon, B}} \mu(D_i) + \Lambda([a; \beta_i : 2 \leq i \leq l]) < \varepsilon + 0 = \varepsilon$, for every $\varepsilon > 0$. Consequently, $\Lambda(\text{Fr}(B)) = 0$. We conclude $B \in A$. It follows $B \subset A$.

But, for every $A \in A$, the function $f = 1_A : [a, b] \mapsto \mathbb{R}$ is Riemann-integrable by the article [6]; therefore, $f$ is $D$-$\mathbb{R}$-integrable, as we have seen in Application 1.20. From the remark (1) of 2.6, we obtain $A \in B$. It follows $A \subset B$, and finally $B = A$.

(2) Consider $X = \mathbb{N}$, $D = A(D) = \{D \subset X : D$ or $D^c$ is finite\}, $\mu(D) = 0$ if $D$ is finite, and $\mu(D) = 1$ if $D^c$ is finite. Let $E \subset X$ such that $E$ and $E^c$ are infinite. Then, given $D \in D$ with $D^c$ finite, we deduce that $D \cap E \neq \emptyset$ and $D \cap E^c \neq \emptyset$. So, as $\mu(D) = 1$, $E$ cannot be an element of $B$. We conclude from the definition of $D$ that $B = D$.

### 3 Darboux-Integrability and Semi-Norm $\| \cdot \|_\mu$

In this paragraph, we only cite some results which are related to the Darboux-integrability and a semi-norm defined on $B(X, F)$. This semi-norm allows, especially, to consider the sequences of $D$-$F$-integrable functions, and also to characterize the $D$-$F$-integrable functions by the $D$-simple functions.

#### 3.1 Definition

Given a basic space $(X, D, \mu)$ and $F$ a Banach space, for every function $f \in B(X; F)$, let

$$
\|f\|_\mu = \inf_{\gamma \in \mathcal{S}_\mu(D) \text{ and } \gamma \geq \|f\|} I_D(\gamma),
$$

where $I_D(\gamma) = \sum_{i=1}^n r_i \cdot \mu(D_i)$ (if $\gamma = \sum_{i=1}^n r_i \cdot 1_{D_i}$, and $\gamma \geq \|f\|$ means $\gamma(x) \geq \|f(x)\|$, $x \in X$.

We note that this definition extends to the case of a basic space a notion (of superior Riemann-integral) introduced in [5].
3.2 Lemma

(a) $|| \cdot ||_{\mu}$ is a semi-norm on $B(X; F)$.

Moreover, $||f||_{\mu} \leq ||f||_{\infty} \cdot \mu(X)$ for every $f \in B(X; F)$.

(b) For every $f \in S_F(D)$, $||f||_{\mu} = I_D(||f||)$.

(c) Let $f : X \mapsto F$ be a bounded function such that $||f|| : X \mapsto \mathbb{R}$ is $D$-$\mathbb{R}$-integrable. Then, $||f||_{\mu} = I_D(||f||)$.

3.3 Proposition

Let $(X, D, \mu)$ be a basic space and $F$ a Banach space. Consider $(f_n)_{n \geq 1}$ a sequence of $D$-$F$-integrable functions, $f_n : X \mapsto F$, and let $f \in B(X; F)$.

Suppose that $\lim_{n \to \infty} ||f - f_n||_{\mu} = 0$. Then, $f$ is $D$-$F$-integrable and $I_D(f) = \lim_{n \to \infty} I_D(f_n)$.

3.4 Corollary

Let $f_n : X \mapsto F$ $(n \geq 1)$ be a sequence of $D$-$F$-integrable functions and $f \in B(X; F)$ such that $f_n \longrightarrow_f f$ uniformly. Then, $f$ is $D$-$F$-integrable and $I_D(f) = \lim_{n \to \infty} I_D(f_n)$.

3.5 Proposition

Let $(X, D, \mu)$ be a basic space, $F$ a Banach space, and $f \in B(X; F)$. Then,

$$f \text{ is } D\text{-}F\text{-integrable if and only if } f \in \overline{S_F(D)},$$

that is, there exists a sequence $(f_n)_{n \geq 1}$ of functions of $S_F(D)$ with $\lim_{n \to \infty} ||f - f_n||_{\mu} = 0$.

Remark: From Proposition 3.3, we have $I_D(f) = \lim_{n \to \infty} I_D(f_n)$.

3.6 Remarks

(1) Precise however that, even if the notations and the approach used are different, the essential of Proposition 3.5 is in exercise 99 of §7 of [2].

(2) Note that if the function $f$ is defined on $D$ instead of $X$, it is possible to consider an interesting type of integral (similar, but different, to those presented in [1]) as it is suggested by the article [3], where the author establishes, under the continuum hypothesis, an integral representation of the second dual of $C([0, 1])$. Add that in [4], the author extends the integral representation in a more general context and in relation with the axioms of the set theory.
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