Real Analysis Exchange
Vol. 29(1), 2003/2004, pp. 107–120

Jolanta Wesołowska, Department of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland. email: `jwes@math.univ.gda.pl`

ON SETS OF DISCRETE CONVERGENCE POINTS OF SEQUENCES OF REAL FUNCTIONS

Abstract

The aim of the paper is to characterize the class of sets of points at which a sequence of real functions of a distinguish family $\mathcal{F} \subset \mathbb{R}^X$ discretely converges.

1 Introduction

The notion of discrete convergence of sequences of real functions was introduced by Császár and Laczkovich in [4]. The authors describe the families of discrete limits of sequences of functions for certain classes of functions, for example continuous functions. The same problem is discussed in many recent papers of Grande (see e.g. [5]).

Our investigation of discrete convergence takes a different direction, namely we are interested in characterizing the class of sets of points at which a sequence of real functions from an established family of functions discretely converges. The first result describing sets of convergence points is due to Hahn [6] and Sierpiński [14]. It concerns pointwise convergence of sequences of continuous functions and states that a subset A of a Polish space X is of type $\mathcal{F}_{\sigma\delta}$ if and only if there exists a sequence $\{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$ of continuous functions convergent exactly at each point of A. This theorem has become the starting point for our further considerations. The problem we deal with in [16] and [17] is to find an analogous characterization of sets of convergence points for sequences of functions from some other classes, such as B_α, measurable, approximately continuous or quasi-continuous functions.

The same problem arises when we replace pointwise convergence with some other types of convergence. In [12] the convergence of transfinite sequences of

Key Words: Sequence of functions, sets of convergence points, discrete convergence.
Mathematical Reviews subject classification: 26A21, 26A03, 54C50
Received by the editors May 14, 2002
Communicated by: Udayan B. Darji

functions has been examined. In this paper we consider the discrete convergence; i.e., we give a characterization of the class of sets of discrete convergence points of sequences of functions, which are taken from a fixed class $\mathcal{F} \subset \mathbb{R}^X$. As $\mathcal{F}$ we consider Borel class α, Darboux functions, measurable functions, derivatives, approximately continuous or quasi-continuous functions. In Section 8 we also study some proper subclasses of Baire measurable functions, for example cliquish functions. We will see that for some classes of functions we get the same family of sets for pointwise or discrete convergence (e.g. for quasi-continuous functions, Corollary 7.5). But for others the considered families are different (e.g. for continuous functions, Corollary 3.4).

2 Definitions and Notation

We denote by ω and ω_1 a first infinite and uncountable ordinal number, respectively. We identify ordinal numbers with the sets of their predecessors; so ω is identified with the set $\mathbb{N} = \{0, 1, 2, \dots\}$. The letter $\mathbb{R}$ stands for the set of real numbers. For a subset A of a topological space X we denote by $\operatorname{int}(A)$, $\operatorname{cl}(A)$ and $\operatorname{fr}(A)$ the interior, closure and boundary of A, respectively. Throughout the paper the following abbreviations for some classes of subsets of a topological space X are used:

$\mathbf{\Sigma}^0_1(X)$ ($\mathbf{\Pi}^0_1(X)$) — open (closed) subsets of X;

$\mathbf{\Sigma}^0_\alpha(X)$ ($\mathbf{\Pi}^0_\alpha(X)$) — additive (multiplicative) class α of Borel subsets of X, $0 < \alpha < \omega_1$;

$\mathcal{M}(X)$ — σ-ideal of meager (first category) subsets of X;

$\mathcal{SO}(X)$ — semi-open subsets of X: $A \in \mathcal{SO}(X)$ iff $A \subset \operatorname{cl}(\operatorname{int} A)$ [11];

$\mathcal{B}aire(X)$ — the collection of subsets with the Baire property.

For $f, g\colon X \to \mathbb{R}$ let

$$[f = g] = \{x \in X : f(x) = g(x)\}.$$

Unless otherwise stated, functions considered here are real-valued functions defined on a topological space X. Instead each $\mathcal{F}(\mathbb{R}) \subset \mathbb{R}^{\mathbb{R}}$ we write $\mathcal{F}$ for short (e.g. $\mathcal{B}_1$ instead $\mathcal{B}_1(\mathbb{R})$). The same rule will be used for all classes of subsets of $\mathbb{R}$; for example we write $\mathbf{\Sigma}^0_1$ instead $\mathbf{\Sigma}^0_1(\mathbb{R})$.

A functions $f : X \to \mathbb{R}$ is called *a discrete limit* of a sequence $\{f_n\colon n \in \mathbb{N}\} \subset \mathbb{R}^X$ if for any $x \in X$ there is a number $k \in \mathbb{N}$ such that $f_n(x) = f(x)$ for each $n \geq k$ [4]. The notion of a point of discrete convergence is defined analogously.

Definition 2.1. A sequence $(a_n)_n$ of real numbers is said to be discretely convergent if there exists a number $k \in \mathbb{N}$ such that $a_n = a_k$ for each $n \geq k$. Then we say that $a = a_k$ is a discrete limit of $(a_n)_n$ and we denote it by

$$a = d - \lim_n a_n.$$

We call a sequence $\{f_n \colon n \in \mathbb{N}\} \subset \mathbb{R}^X$ *discretely convergent at* $x \in X$ if a sequence of real numbers $(f_n(x))_n$ is discretely convergent. A set of all such points we denote by

$$L^d(\{f_n \colon n \in \mathbb{N}\}) = \{x \in X \colon d - \lim_n f_n(x) \text{ exists}\}.$$

For a family of functions $\mathcal{F}(X) \subset \mathbb{R}^X$ let

$$\mathcal{L}^d(\mathcal{F}(X)) = \Big\{ L^d(\{f_n \colon n \in \mathbb{N}\}) \colon \ \{f_n : n \in \mathbb{N}\} \subset \mathcal{F}(X) \Big\}.$$

Remark 2.2. $\mathcal{L}^d(\mathcal{F}_1(X)) \subset \mathcal{L}^d(\mathcal{F}_2(X))$ for any $\mathcal{F}_1(X) \subset \mathcal{F}_2(X)$.

Remark 2.3. For any $\{f_n \colon n \in \mathbb{N}\} \subset \mathbb{R}^X$ we have

$$L^d(\{f_n \colon n \in \mathbb{N}\}) = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} [f_n = f_k] = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} [f_k = f_{k+1}].$$

Consequently, for any $\mathcal{F}(X) \subset \mathbb{R}^X$

$$\mathcal{L}^d(\mathcal{F}(X)) \subset \Big\{ \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k : A_k = [f = g] \text{ for some } f, g \in \mathcal{F}(X) \Big\}.$$

Lemma 2.4. *If $\mathcal{F}(X)$ is an additive subgroup of $\mathbb{R}^X$, then*

$$\mathcal{L}^d(\mathcal{F}(X)) = \Big\{ \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k : A_k = [f = g] \text{ for some } f, g \in \mathcal{F}(X) \Big\}.$$

Proof. The inclusion " $\subset$ " follows by Remark 2.3. To see " $\supset$ " take $A = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k$, where $A_k = [f_k = g_k]$ for some $f_k, g_k \in \mathcal{F}(X)$. Then $h_k = f_k - g_k \in \mathcal{F}(X)$ and $A = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} [h_k = 0]$. Consider a bijection $\varphi : \omega \to 2 \times \omega$, $\varphi = (\varphi_0, \varphi_1)$ and put $p_n = \varphi_0(n) \cdot h_{\varphi_1(n)}$ for $n \in \mathbb{N}$. Then $p_n \in \mathcal{F}(X)$ for every $n \in \mathbb{N}$ and $A = L^d(\{p_n : n \in \mathbb{N}\})$. $\square$

In order to describe the connections between pointwise and discrete convergence let us consider also the set $L(\{f_n \colon n \in \mathbb{N}\})$ of all convergence points of a sequence $\{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$; i.e., $x \in L(\{f_n : n \in \mathbb{N}\})$ iff $\lim_n f_n(x)$ exists and it is finite. For a family of functions $\mathcal{F}(X) \subset \mathbb{R}^X$ let

$$\mathcal{L}(\mathcal{F}(X)) = \left\{ L(\{f_n : n \in \mathbb{N}\}) : \ \{f_n : n \in \mathbb{N}\} \subset \mathcal{F}(X) \right\}.$$

It will be shown later that $\mathcal{L}(\mathcal{F}) \setminus \mathcal{L}^d(\mathcal{F}) \neq \emptyset$ for some $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ (cf. Corollary 3.4). There exist also $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ such that $\mathcal{L}^d(\mathcal{F}) \setminus \mathcal{L}(\mathcal{F}) \neq \emptyset$ as the following example shows.

Example 2.5. For every $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{n+2} & \text{for } x \in \{0, 1, \ldots, n\} \cup \{-n-1\}, \\ 1 & \text{otherwise .} \end{cases}$$

Let $\mathcal{F}$ be a family of all such functions. Then $\mathbb{R} \setminus \mathbb{N} \in \mathcal{L}^d(\mathcal{F})$, because $\mathbb{R} \setminus \mathbb{N} = L^d(\{f_n : n \in \mathbb{N}\})$. But $\mathbb{R} \setminus \mathbb{N} \notin \mathcal{L}(\mathcal{F})$. Suppose to the contrary that there exists $\{g_i : i \in \mathbb{N}\} \subset \mathcal{F}$ such that $\mathbb{R} \setminus \mathbb{N} = L(\{g_i : i \in \mathbb{N}\})$.

First suppose that $\{g_i : i \in \mathbb{N}\}$ contains a constant subsequence $\{g_{i_k} : k \in \mathbb{N}\}$ (i.e., for some $n \in \mathbb{N}$ and any $k \in \mathbb{N}$ we have $g_{i_k} = f_n$). Let us consider two possible cases.

- There is $m \in \mathbb{N}$ such that $g_i = f_n$ for all $i \geq m$. But then $L(\{g_i : i \in \mathbb{N}\}) = \mathbb{R}$.
- For every $m \in \mathbb{N}$ there exists $i_m \geq m$ such that $g_{i_m} \neq f_n$. Then $\lim_m g_{i_m}(-n-1) = 1$ and $\lim_k g_{i_k}(-n-1) = \frac{1}{n+2}$. It follows that $-n-1 \notin L(\{g_i : i \in \mathbb{N}\})$.

Both of them contradict our assumption.

Now suppose $\{g_i : i \in \mathbb{N}\}$ contains no constant subsequence. Therefore $\lim_i g_i(n) = 0$ for all $n \in \mathbb{N}$, which means that $\mathbb{N} \subset L(\{g_i : i \in \mathbb{N}\})$, a contradiction. $\square$

3 $\mathcal{B}_\alpha$ class of functions

For $\alpha < \omega_1$ denote by $\mathcal{B}_\alpha(X)$ the *Borel class* α of functions defined on a metric space X; i.e., $f \in \mathcal{B}_\alpha(X)$ iff $f^{-1}(U) \in \mathbf{\Sigma}^0_{\alpha+1}(X)$ for every open $U \subset \mathbb{R}$. In particular, $\mathcal{B}_0(X)$ denotes the family of all continuous functions.

Lemma 3.1. *Let (X, d) be an arbitrary metric space with a metric d. For any ordinal number $\alpha < \omega_1$ and for any set $A \in \mathbf{\Pi}^0_{\alpha+1}(X)$ there exists a function $h \in \mathcal{B}_\alpha(X)$ such that $A = [h = 0]$.*

Proof. First, consider $\alpha = 0$. Then a closed set $A \subset X$ takes a form $A = [h = 0]$ for a continuous function $h\colon X \to \mathbb{R}$, which is given by a formula $h(x) = \operatorname{dist}(x, A) = \inf\{d(y, a)\colon a \in A\}$. For $\alpha \geq 1$ consider a set $X \setminus A \in \mathbf{\Sigma}^0_{\alpha+1}(X)$. It is easy to see that $X \setminus A = \bigcup_{n\in\mathbb{N}} A_n$ for some sequence $\{A_n\colon n \in \mathbb{N}\} \subset \mathbf{\Pi}^0_{\alpha+1}(X) \cap \mathbf{\Sigma}^0_{\alpha+1}(X)$. Moreover, we can assume $A_k \cap A_m = \emptyset$ for $k \neq m$. Putting

$$h(x) = \begin{cases} \frac{1}{n+1} & \text{if } x \in A_n \text{ and } n \in \mathbb{N} \\ 0 & \text{if } x \in A \end{cases}$$

we get the required function. □

Theorem 3.2. *Let X be an arbitrary metric space. For any ordinal number $\alpha < \omega_1$, we have $\mathcal{L}^d(\mathcal{B}_\alpha(X)) = \mathbf{\Sigma}^0_{\alpha+2}(X)$. In particular, for the class $\mathcal{B}_0(X)$ of continuous functions we have $\mathcal{L}^d(\mathcal{B}_0(X)) = \mathbf{\Sigma}^0_2(X)$.*

Proof. Fix $\alpha < \omega_1$. For any $f, g \in B_\alpha(X)$ we have $[f = g] \in \mathbf{\Pi}^0_{\alpha+1}$. Therefore and by Lemma 2.4,

$$\begin{aligned} \mathcal{L}^d(\mathcal{B}_\alpha(X)) &= \left\{ \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} [f_k = g_k]\colon f_k, g_k \in \mathcal{B}_\alpha(X) \right\} \\ &\subset \left\{ \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k\colon A_k \in \mathbf{\Pi}^0_{\alpha+1}(X) \right\} = \mathbf{\Sigma}^0_{\alpha+2}(X). \end{aligned}$$

To get an equality fix $A = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k$ where $A_k \in \mathbf{\Pi}^0_{\alpha+1}(X)$. By Lemma 3.1, for every $k \in \mathbb{N}$ there is a function $h_k \in \mathcal{B}_\alpha(X)$ such that $A_k = [h_k = 0]$. In a consequence, $A = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} [h_k = 0] \in \mathcal{L}^d(\mathcal{B}_\alpha(X))$. □

It turns out that the assumption on X to be metric is essential.

Example 3.3. There exists a topological space X such that a family $\mathcal{B}_0(X)$ of continuous functions does not satisfy an assertion of Theorem 3.2. To prove this, consider a space X of real numbers with a topology $\mathcal{T}$ of cocountable subsets of $\mathbb{R}$; i.e.,

$$\mathcal{T} = \{A \subset \mathbb{R}\colon \mathbb{R} \setminus A \text{ is countable}\} \cup \{\emptyset\}.$$

Then $\mathcal{B}_0(X)$ consists of all constants; so $\mathcal{L}^d(\mathcal{B}_0(X)) = \{\emptyset, \mathbb{R}\}$, while $\mathbf{\Sigma}^0_2(X)$ contains all countable subsets of $\mathbb{R}$.

It it known that for an uncountable Polish space X the different Borel classes of subsets of X are not equal to each other (see [8, Theorem 22.4]). It was shown in [17] that $\mathcal{L}(\mathcal{B}_\alpha) = \mathbf{\Pi}^0_{\alpha+3}$ and it is easy to see, that also $\mathcal{L}(\mathcal{B}_\alpha(X)) = \mathbf{\Pi}^0_{\alpha+3}(X)$ for any metric space X. By Theorem 3.2 we get the following.

Corollary 3.4. *For any uncountable Polish space X we have*

$$\mathcal{L}(\mathcal{B}_\alpha(X)) \setminus \mathcal{L}^d(\mathcal{B}_\alpha(X)) \neq \emptyset.$$

4 Approximately Continuous Functions

We say that $f : \mathbb{R} \to \mathbb{R}$ is *approximately continuous* iff for every open $U \subset \mathbb{R}$ a set $f^{-1}(U)$ is open in the density topology $\mathcal{T}_d$. (Recall that $\mathcal{T}_d$ consists of all Lebesgue measurable subsets of $\mathbb{R}$ having density 1 at each of its points; see e.g. [3].) Let $\mathcal{A}$ denote the class of all such functions.

Theorem 4.1. $\mathcal{L}^d(\mathcal{A}) = \mathbf{\Sigma}^0_3$.

PROOF. It is known that $\mathcal{A} \subset \mathcal{B}_1$ ([3, Theorem 5.5, p. 21]); so if $A = [f = g]$ for some $f, g \in \mathcal{A}$, then $A \in \mathbf{\Pi}^0_2$ and $\mathbb{R} \setminus A \in \mathcal{T}_d$. Moreover, a family $\mathcal{A}$ satisfies all assumptions of Lemma 2.4, so we have

$$\begin{aligned}\mathcal{L}^d(\mathcal{A}) &= \left\{ \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k; A_k = [f = g] \ \text{ for some } \ f, g \in \mathcal{A} \right\} \\ &\subset \left\{ \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k : A_k \in \mathbf{\Pi}^0_2 \text{ and } \mathbb{R} \setminus A_k \in \mathcal{T}_d \right\} \subset \mathbf{\Sigma}^0_3.\end{aligned}$$

First we will show that the last inclusion can be replaced with an equality. Fix $A \in \mathbf{\Sigma}^0_3$. Then $A = \bigcup_{n\in\mathbb{N}} G_n$, where $G_n \in \mathbf{\Pi}^0_2$. Since every $\mathbf{\Pi}^0_2$ set is a countable union of $\mathbf{\Pi}^0_2$ sets, closed in the density topology (cf. [10, Lemma 5]), we have $A = \bigcup_{n\in\mathbb{N}} B_n$, where $B_n \in \mathbf{\Pi}^0_2$ and $\mathbb{R} \setminus B_n \in \mathcal{T}_d$. Put $A_k = \bigcup_{n\leq k} B_n$. Then $A_k \in \mathbf{\Pi}^0_2$, $\mathbb{R} \setminus A_k \in \mathcal{T}_d$ and $A = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k$. Now, it is enough to notice that every set $A_k \in \mathbf{\Pi}^0_2$ such that $\mathbb{R} \setminus A_k \in \mathcal{T}_d$ takes a form $A_k = [h_k = 0]$ for some approximately continuous function $h_k : \mathbb{R} \to [0, 1]$ (Zahorski [18, Lemma 11]). □

Denote by $b\mathcal{A}$ the class of all bounded approximately continuous functions and by Δ the class of derivatives.

Theorem 4.2. $\mathcal{L}^d(\Delta) = \mathbf{\Sigma}^0_3$.

PROOF. Note that in Theorem 4.1 we have actually proved that $\mathcal{L}^d(b\mathcal{A}) = \mathbf{\Sigma}^0_3$. Since $b\mathcal{A} \subset \Delta \subset \mathcal{B}_1$ (see [3, Theorem 5.5, p. 21]), the assertion follows by Theorem 3.2 and Remark 2.2. □

It was shown in [17], that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\Delta) = \mathbf{\Pi}^0_4$. By Theorems 4.1 and 4.2 we have the following.

Corollary 4.3. $\mathcal{L}^d(\mathcal{A}) \neq \mathcal{L}(\mathcal{A})$ *and* $\mathcal{L}^d(\Delta) \neq \mathcal{L}(\Delta)$.

5 Darboux Functions

We say that a function $f : \mathbb{R} \to \mathbb{R}$ has *the Darboux property* iff $f(A)$ is connected for every connected $A \subset \mathbb{R}$. Denote by $\mathcal{D}$ the family of all such functions. It is well known that every function $f\colon \mathbb{R} \to \mathbb{R}$ is a sum of two Darboux functions (see e.g. [3]). The next theorem is a simple consequence of this fact.

Theorem 5.1. *A family $\mathcal{L}^d(\mathcal{D})$ consists of all subsets of $\mathbb{R}$.*

Proof. Fix $A \subset \mathbb{R}$ and consider a function

$$p(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in \mathbb{R} \setminus A \end{cases}$$

and Darboux functions $g, h\colon \mathbb{R} \to \mathbb{R}$ such that $p = g - h$. Define

$$f_n = \varphi_0(n) \cdot g + (1 - \varphi_0(n)) \cdot h,$$

where $\varphi\colon \omega \to 2 \times \omega$, $\varphi = (\varphi_0, \varphi_1)$ is an arbitrary bijection. Then f_n are Darboux functions and $A = L^d(\{f_n : n \in \mathbb{N}\}) \in \mathcal{L}^d(\mathcal{D})$. $\square$

For $0 < \alpha < \omega_1$ denote by $\mathcal{DB}_\alpha$ the class of all $\mathcal{B}_\alpha$ functions having Darboux property. As a consequence of Theorem 3.2 we get the following.

Theorem 5.2. *For any ordinal number $0 < \alpha < \omega_1$ we have $\mathcal{L}^d(\mathcal{DB}_\alpha) = \mathbf{\Sigma}^0_{\alpha+2}$.*

Proof. For $\alpha = 1$ the assertion follows by inclusions $\mathcal{A} \subset \mathcal{DB}_1 \subset \mathcal{B}_1$ ([3, Theorem 5.5, p. 21]) and by Theorems 4.1, 3.2 and Remark 2.2. Now, fix $\alpha > 1$ and $A \in \mathbf{\Sigma}^0_{\alpha+2}$. By Theorem 3.2, there is a sequence $\{g_n\colon n \in \mathbb{N}\} \subset \mathcal{B}_\alpha$ such that $L^d(\{g_n\colon n \in \mathbb{N}\}) = A$. To prove $A \in \mathcal{L}^d(\mathcal{DB}_\alpha)$ we will use $\{g_n\colon n \in \mathbb{N}\}$ to construct a new sequence $\{f_n\colon n \in \mathbb{N}\} \subset \mathcal{DB}_\alpha$ such that $A = L^d(\{f_n\colon n \in \mathbb{N}\})$.

Let $\{I_m\colon m \in \mathbb{N}\}$ be an enumeration of all open intervals with rational endpoints. By induction we can choose a sequence $\{C_{m,n}\colon n, m \in \mathbb{N}\}$ of nonempty nowhere dense perfect sets satisfying the following properties:

(i) $C_{m,n} \subset I_m$ for $m, n \in \mathbb{N}$;

(ii) $C_{m,n} \cap C_{i,j} = \emptyset$ for $(m, n) \neq (i, j)$.

For fixed $n \in \mathbb{N}$ let $C_n = \bigcup_{m\in\mathbb{N}} C_{m,n}$. For every $m \in \mathbb{N}$ choose a continuous surjection $h_n^m\colon C_{m,n} \to [-m; m]$ and define $h_n\colon C_n \to \mathbb{R}$ by

$$h_n(x) = h_n^m(x) \text{ for } x \in C_{m,n} \text{ and } m \in \mathbb{N}.$$

It is easy to verify that h_n are of $\mathcal{B}_1$ class on C_n. The required sequence $\{f_n : n \in \mathbb{N}\} \subset \mathcal{DB}_\alpha$ is defined by

$$f_n(x) = \begin{cases} h_n(x) & \text{if } x \in C_n \\ g_n(x) & \text{if } x \notin C_n. \end{cases}$$

Indeed,

- $f_n(P) = \mathbb{R}$ for every nonempty interval $P \subset \mathbb{R}$; so $f_n \in \mathcal{D}$,
- $f_n^{-1}(U) \in \mathbf{\Sigma}^0_{\alpha+1}$ for every open $U \subset \mathbb{R}$ and $\alpha \geq 2$; so $f_n \in \mathcal{B}_\alpha$,
- for every $x \in \mathbb{R}$ there is at most one $n \in \mathbb{N}$ such that $x \in C_n$; so $f_k(x) = g_k(x)$ for every $k > n$. It follows that $L^d(\{f_n : n \in \mathbb{N}\}) = L^d(\{g_n : n \in \mathbb{N}\}) = A$. □

Remark 5.3. Since $\mathcal{L}(\mathcal{B}_\alpha) = \mathbf{\Pi}^0_{\alpha+3}$ (see [17, Theorem 3]), we can apply the same arguments as above to obtain $\mathcal{L}(\mathcal{DB}_\alpha) = \mathbf{\Pi}^0_{\alpha+3}$.

6 Measurable Functions

The next simple observation gives a characterization of $\mathcal{L}^d$ for measurable functions, for example for Lebesgue measurable, Borel or Baire measurable functions.

Theorem 6.1. *Let $\mathcal{A}(X)$ be a σ-algebra of subsets of X and $\mathcal{M}_\mathcal{A}(X) \subset \mathbb{R}^X$ denote a family of $\mathcal{A}(X)$-measurable real functions. Then $\mathcal{L}^d(\mathcal{M}_\mathcal{A}(X)) = \mathcal{A}(X)$.*

PROOF. The inclusion " $\subset$ " is follows from Remark 2.3. To prove " $\supset$ " note that each set $A \in \mathcal{A}(X)$ takes the form $A = L^d(\{f_n : n \in \mathbb{N}\})$ for a sequence $\{f_n : n \in \mathbb{N}\} \subset \mathcal{M}_\mathcal{A}(X)$ of functions defined by $f_n = \varphi_0(n) \cdot \chi_A + (1 - \varphi_0(n)) \cdot h$, where $\varphi\colon \omega \to 2 \times \omega$, $\varphi = (\varphi_0, \varphi_1)$ is an arbitrary bijection, χ_A denotes the characteristic function of A and $h\colon X \to \mathbb{R}$ is equal to 1 for every $x \in X$. □

7 Quasi-Continuous Functions

In this section we examine the class $\mathcal{QC}(X)$ of *quasi-continuous* functions defined on a topological space X. A function $f \in \mathcal{QC}(X)$ iff for every $p \in X$ and for every open sets $U \subset X$, $W \subset \mathbb{R}$ such that $p \in U$ and $f(p) \in W$ there exists a nonempty open set $G \subset U$ such that $f(G) \subset W$ (or, equivalently $f^{-1}(V) \in \mathcal{SO}(X)$ for every open $V \subset \mathbb{R}$, see e.g. [13, Theorem 1.1]). Denote also by $\mathcal{C}liq(X)$ the collection of all *cliquish* functions; i.e., $f \in \mathcal{C}liq(X)$ iff for every $p \in X$, its open neighborhood U and $\varepsilon > 0$ there exists a nonempty open set $G \subset U$ such that $|f(x) - f(x')| < \varepsilon$ for every $x, x' \in G$ [15].

The following lemma is a key to a construction of quasi-continuous functions.

Lemma 7.1. (cf. Borsík [1, Lemma 1]) *Let X be an arbitrary metric space. For every nowhere dense closed set $F \subset X$ satisfying $F \subset \mathrm{cl}\,(G)$ for some nonempty open set $G \subset X$ there exists a collection $\{K_{n,m} \colon n \in \mathbb{N}, m \leq n\}$ of nonempty open sets such that:*

(i) $\mathrm{cl}\,(K_{n,m}) \subset G \setminus F$ *for all $n \in \mathbb{N}$ and $m \leq n$,*

(ii) $\mathrm{cl}\,(K_{n,m}) \cap \mathrm{cl}\,(K_{i,j}) = \emptyset$ *for $(n,m) \neq (i,j)$,*

(iii) *for every $x \notin F$ there exists an open neighborhood V of x such that the set $\{(n,m) \colon \mathrm{cl}\,(K_{n,m}) \cap V \neq \emptyset\}$ has at most one element,*

(iv) *for every $x \in F$, every open neighborhood V of x and every number $m \in \mathbb{N}$, there exists $n \geq m$ such that $K_{n,m} \cap V \neq \emptyset$.*

Consequently, $F \subset \mathrm{cl}\,(\bigcup_{n \geq m} K_{n,m})$ for each $m \in \mathbb{N}$ and both $F \cup \bigcup_{n \geq m} \mathrm{cl}\,(K_{n,m})$ and $F \cup \bigcup_{n \in \mathbb{N}, m \leq n} \mathrm{cl}\,(K_{n,m})$ are closed in G. □

Corollary 7.2. *Let X be an arbitrary metric space. For every nowhere dense closed set $F \subset X$ satisfying $F \subset \mathrm{cl}\,(G)$ for some nonempty open set $G \subset X$ there exist disjoint semi-open sets $S_0, S_1 \subset G \setminus F$ such that $G \setminus F = S_0 \cup S_1$ and $F \subset \mathrm{cl}\,(S_0) \cap \mathrm{cl}\,(S_1)$.*

Proof. By Lemma 7.1, there is a collection $\{K_{n,m} \colon n \in \mathbb{N}, m \leq n\}$ of sets satisfying (i)-(iv). It is enough to take $S_0 = \bigcup_{n \in \mathbb{N}} \mathrm{cl}\,(K_{n,0})$ and $S_1 = G \setminus (F \cup \bigcup_{n \in \mathbb{N}} \mathrm{cl}\,(K_{n,0}))$.

Lemma 7.3. (cf. Borsík [2]) *Let X be an arbitrary metric space. For every cliquish function $g \colon X \to \mathbb{R}$ there are quasi-continuous functions $s, t : X \to \mathbb{R}$ such that $g = s + t$.* □

Theorem 7.4. *Let X be an arbitrary metric space. Then $\mathcal{L}^d(\mathcal{QC}(X)) = \mathcal{B}aire(X)$.*

Proof. Since $\mathcal{SO}(X) \subset \mathcal{B}aire(X)$, every quasi-continuous function has the Baire property and an inclusion "$\subset$" is a consequence of Theorem 6.1 and Remark 2.2.

To prove "$\supset$", fix $A \in \mathcal{B}aire(X)$. Then $A = (G \setminus P) \cup Q$, where G is a regular open set (i.e. $G = \operatorname{int}(\operatorname{cl}(G))$), $P, Q \in \mathcal{M}(X)$, $P \subset G$ and $Q \cap G = \emptyset$. Let $P = \bigcup_{i \in \mathbb{N}} F_i$ for an increasing sequence $\{F_i \colon i \in \mathbb{N}\}$ of nowhere dense subsets of G. We need to find $\{f_n : n \in \mathbb{N}\} \subset \mathcal{QC}(X)$ such that $A = L^d(\{f_n : n \in \mathbb{N}\})$.

First consider a function $s\colon G \to \mathbb{R}$ given by

$$s(x) = \begin{cases} 1 & \text{if } x \in F_0 \\ \frac{1}{i} & \text{if } x \in F_i \setminus F_{i-1} \text{ and } i \geq 1 \\ 0 & \text{otherwise .} \end{cases}$$

It is easy to see that $s \in \mathcal{C}liq(G)$. Moreover, $[s = 0] = G \setminus P$. By Lemma 7.3 there are quasi-continuous functions $s_0, s_1 \colon G \to \mathbb{R}$ such that $s = s_0 - s_1$. Take a bijection $\varphi\colon \omega \to 2 \times \omega$, $\varphi = (\varphi_0, \varphi_1)$ and for each $n \in \mathbb{N}$ put

$$f_n|_G = \varphi_0(n) \cdot s_0 + (1 - \varphi_0(n)) \cdot s_1.$$

Then $\{f_n|_G : n \in \mathbb{N}\} \subset \mathcal{QC}(G)$ and $L^d(\{f_n|_G : n \in \mathbb{N}\}) = G \setminus P$.

Now, take $X \setminus G$. Since G is a regular open set, $X \setminus G \in \mathcal{SO}(X)$. It follows that $X \setminus G = \operatorname{cl}(\operatorname{int}(X \setminus G)) = \operatorname{int}(X \setminus G) \cup \operatorname{fr}(G)$. Consider the first category set $Q \subset X \setminus G$. Then $Q \subset \bigcup_{i \in \mathbb{N}} N_i$ for an increasing sequence $\{N_i \colon i \in \mathbb{N}\}$ of nowhere dense closed subsets of $X \setminus G$. Fix $i \in \mathbb{N}$. Let $E_i = N_i \cup \operatorname{fr}(G)$. Then $E_i \subset \operatorname{cl}(\operatorname{int}(X \setminus G))$ and by Corollary 7.2 there are semi-open sets S_0^i, S_1^i such that:

(i) $S_0^i \cap S_1^i = \emptyset$,

(ii) $\operatorname{int}(X \setminus G) \setminus E_i = S_0^i \cup S_1^i$,

(iii) $E_i \subset \operatorname{cl}(S_0^i) \cap \operatorname{cl}(S_1^i)$.

Define $g_i, h_i \in \mathcal{QC}(X \setminus G)$ by

$$g_i(x) = \begin{cases} 0 & \text{if } x \in S_0^i \cup (E_i \cap Q) \\ 1 & \text{if } x \in S_1^i \cup (E_i \setminus Q) \end{cases}$$

and

$$h_i(x) = \begin{cases} 1 & \text{if } x \in S_0^i \\ 0 & \text{if } x \in S_1^i \cup E_i. \end{cases}$$

It is easy to check that for $f_n|_{X \setminus G} = \varphi_0(n) \cdot g_{\varphi_1(n)} + (1 - \varphi_0(n)) \cdot h_{\varphi_1(n)}$ we have $\{f_n|_{X \setminus G} : n \in \mathbb{N}\} \subset \mathcal{QC}(X \setminus G)$ and $L^d(\{f_n|_{X \setminus G} : n \in \mathbb{N}\}) = Q$. The required sequence consists of the functions $f_n = f_n|_G \cup f_n|_{X \setminus G}$. $\square$

By [16, Theorem 2] we have the following.

Corollary 7.5. $\mathcal{L}^d(\mathcal{QC}(X)) = \mathcal{L}(\mathcal{QC}(X))$.

Remark 7.6. There is a topological space X such that $\mathcal{L}^d(\mathcal{QC}(X)) \neq \mathcal{B}aire(X)$. It is enough to consider a space X of real numbers with a topology of cocountable sets (cf. Example 3.3). Then $\mathcal{L}^d(\mathcal{QC}(X)) = \{\emptyset, \mathbb{R}\}$, while $\mathcal{B}aire(X)$ contains all countable subsets of $\mathbb{R}$. □

8 Some Subclasses of Baire Measurable Functions

Let $\mathcal{M}_{\mathcal{B}aire}(X)$ be the family of Baire measurable functions. ($f \in \mathcal{M}_{\mathcal{B}aire}(X)$ iff $f^{-1}(U) \in \mathcal{B}aire(X)$ for every open $U \subset \mathbb{R}$.) For $f\colon X \to \mathbb{R}$ let $C(f)$ be the set of all continuity points of f and let $D(f) = X \setminus C(f)$. In the next theorem a class of functions $f\colon X \to \mathbb{R}$ with a dense and open set $C(f)$ is examined. This class we denote by $\mathcal{S}(X)$. Obviously, $\mathcal{S}(X) \subset \mathcal{M}_{\mathcal{B}aire}(X)$.

Theorem 8.1. *For any topological space X we have $\mathcal{L}^d(\mathcal{S}(X)) = \mathcal{B}aire(X)$.*

PROOF. Since $\mathcal{S}(X) \subset \mathcal{M}_{\mathcal{B}aire}(X)$, we have $\mathcal{L}^d(\mathcal{S}(X)) \subset \mathcal{B}aire(X)$, by Theorem 6.1 and Remark 2.2. Now, fix $A_0 \in \mathcal{B}aire(X)$ and let $A_1 = X \setminus A_0$. Of course, $A_1 \in \mathcal{B}aire(X)$.

First suppose X is a Baire space. Then for every $s \in \{0,1\}$ we have $A_s = (G_s \setminus P_s) \cup Q_s$, where G_s is a regular open set, $P_s, Q_s \in \mathcal{M}(X)$, $P_s \subset G_s$ and $Q_s \cap G_s = \emptyset$. Since X is a Baire space, $X = G_0 \cup G_1 \cup Q_0 \cup Q_1 = \mathrm{cl}\,(G_0) \cup \mathrm{cl}\,(G_1)$ and $G_0 \cap G_1 = \emptyset$. Moreover, $P_s \subset \bigcup_{n\in\mathbb{N}} F_n^s$, where $\{F_n^s : n \in \mathbb{N}\}$ is an increasing sequence of closed nowhere dense subsets of $\mathrm{cl}\,(G_s)$.

Fix $n \in \mathbb{N}$ and define $\{f_n : n \in \mathbb{N}\}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in (G_0 \setminus (F_n^0 \cap P_0)) \cup (F_n^1 \cap P_1) \\ (-1)^n & \text{if } x \in (G_1 \setminus (F_n^1 \cap P_1)) \cup (F_n^0 \cap P_0) \\ 0 & \text{if } x \in Q_0 \setminus G_1 \\ (-1)^n & \text{if } x \in Q_1 \setminus G_0. \end{cases}$$

Fix $x \in A_0 = (G_0 \setminus P_0) \cup Q_0$. Then $x \in L^d\{f_n : n \in \mathbb{N}\}$, because

- if $x \in Q_0 \setminus G_1$, then $f_n(x) = 0$ for every $n \in \mathbb{N}$; so $d - \lim_n f_n(x) = 0$,
- if $x \in Q_0 \cap G_1 = Q_0 \cap P_1 \subset Q_0 \cap \bigcup_{n\in\mathbb{N}} F_n^1$, then there is $k \in \mathbb{N}$ such that $x \in P_1 \cap F_n^1$ for every $n > k$; so $f_n(x) = 0$, which means that $d - \lim_n f_n(x) = 0$,
- if $x \in (G_0 \setminus P_0) \cap \bigcup_{n\in\mathbb{N}} F_n^0$, then there is $k \in \mathbb{N}$ such that $x \in G_0 \setminus (F_n^0 \cap P_0)$ for every $n > k$, so $f_n(x) = 0$, which means that $d - \lim_n f_n(x) = 0$,

- if $x \in G_0 \setminus \bigcup_{n\in\mathbb{N}} F_n^0$, then $f_n(x) = 0$ for every $n \in \mathbb{N}$; so $d-\lim_n f_n(x) = 0$.

Hence, $A_0 \subset L^d\{f_n : n \in \mathbb{N}\}$. In the same manner we can see that $\{f_n : n \in \mathbb{N}\}$ is not discretely convergent at any $x \in A_1$. It follows that $A_0 = L^d\{f_n : n \in \mathbb{N}\}$. Moreover, since $X = \mathrm{cl}\,(G_0) \cup \mathrm{cl}\,(G_1)$ and f_n is constant on the open sets $G_0 \setminus F_n^0$ and $G_1 \setminus F_n^1$, we have $D(f_n) \subset F_n^0 \cup F_n^1 \cup \mathrm{fr}\,(G_0) \cup \mathrm{fr}\,(G_1)$ for any $n \in \mathbb{N}$; so it is nowhere dense. Consequently, $C(f_n)$ is a dense subset of X. Since f_n has a finite range, $C(f_n)$ is open; so $f_n \in \mathcal{S}(X)$. Therefore, $A \in \mathcal{L}^d(\mathcal{S}(X))$.

Now, consider an arbitrary topological space X. Let X_1 be the union of all first category open subsets of X. Then $X_2 = X \setminus \mathrm{cl}\,(X_1)$ is an open Baire subspace of X and both $A_0 \cap X_2$ and $A_1 \cap X_2$ have the Baire property in X_2 (see e.g. [7]) so we can find (as before) a sequence $\{f_n|_{X_2} \colon n \in \mathbb{N}\} \subset \mathbb{R}^{X_2}$ of functions with finite ranges such that $A_0 \cap X_2 = L^d(\{f_n|_{X_2} \colon n \in \mathbb{N}\})$ and $D(f_n|_{X_2})$ is closed and nowhere dense in X_2 for any $n \in \mathbb{N}$.

Take X_1. By the Banach Category Theorem ([7, Theorem 1.6]), X_1 is an open set of the first category in X; so $X_1 \subset \bigcup_{n\in\mathbb{N}} F_n \subset \mathrm{cl}\,(X_1)$, where F_n is an increasing sequence of nowhere dense sets closed in X. Fix $n \in \mathbb{N}$. Let $E_n = F_n \cup \mathrm{fr}\,(X_1)$. Then E_n is a nowhere dense closed subset of X and $\mathrm{cl}\,(X_1) = \bigcup_{n\in\mathbb{N}} E_n$. Define $f_n|_{\mathrm{cl}\,(X_1)} \colon \mathrm{cl}\,(X_1) \to \mathbb{R}$ by

$$f_n|_{\mathrm{cl}\,(X_1)}(x) = \begin{cases} 0 & \text{if } x \in E_n \cap A_0 \\ (-1)^n & \text{otherwise.} \end{cases}$$

Note that $f_n|_{\mathrm{cl}\,(X_1)}$ is constant on the open set $X_1 \setminus E_n \subset C(f_n|_{\mathrm{cl}\,(X_1)})$, dense in $\mathrm{cl}\,(X_1)$. Moreover, $A_0 \cap \mathrm{cl}\,(X_1) = L^d(\{f_n|_{\mathrm{cl}\,(X_1)} \colon n \in \mathbb{N}\})$. Put $f_n = f_n|_{\mathrm{cl}\,(X_1)} \cup f_n|_{X_2}$. Then $C(f_n)$ is open for any $n \in \mathbb{N}$ and $D(f_n)$ is a subset of $D(f_n|_{X_2}) \cup E_n \cup \mathrm{fr}\,(X_1)$, which is nowhere dense in X. Consequently, $\{f_n \colon n \in \mathbb{N}\}$ has all of the required properties. □

As a consequence of Theorem 8.1 and Remark 2.2 we get a characterization of $\mathcal{L}^d(\mathcal{F}(X))$ for any class $\mathcal{F}(X) \subset \mathbb{R}^X$ between $\mathcal{S}(X)$ and $\mathcal{M}_{\mathcal{B}aire}(X)$, such as *pointwise discontinuous* functions (see [9], p. 74), *simply continuous* functions (cf. [13]), or cliquish functions. In this case we get the same collection of sets for discrete or pointwise convergence (cf. [16, Remark 2]).

Corollary 8.2. *For any $\mathcal{F}(X) \subset \mathbb{R}^X$ such that $\mathcal{S}(X) \subset \mathcal{F}(X) \subset \mathcal{M}_{\mathcal{B}aire}(X)$ we have $\mathcal{L}^d(\mathcal{F}(X)) = \mathcal{L}(\mathcal{F}(X)) = \mathcal{B}aire(X)$.*

References

[1] J. Borsík, *Algebraic structures generated by real quasicontinuous functions*, Tatra Mt. Math. Publ., **8** (1996), 175–184.

[2] J. Borsík, *Sums of quasicontinuous functions defined on a pseudometrizable spaces* Real Anal. Exchange, **22** (1996–97), no. 1, 328–337.

[3] A. M. Bruckner, *Differentiation of real functions*, Springer, Berlin 1978.

[4] A. Császár, M. Laczkovich, *Discrete and equal convergence*, Studia Sci. Math. Hungar, **10** (1975), 463–472.

[5] Z. Grande, *On discrete limits of sequences of functions satisfying some special approximate quasi-continuity conditions*, Real Anal. Exchange, **25** (1999–2000), no. 2, 703–710.

[6] H. Hahn, *Über die Menge der Konvergenzpunkte einer Funktionfolge*, Archiv. der Math. und Physik, **28** (1919), 34–45.

[7] R. C. Haworth, R. A. McCoy, *Baire spaces*, Dissertationes Math., **141** (1977), 1–77.

[8] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, New York 1995.

[9] K. Kuratowski, *Topologie*, vol. 1, PWN, Warszawa 1958.

[10] M. Laczkovich, G. Petruska, *A theorem on approximately continuous functions*, Acta Math. Acad. Sci. Hungar., **24** (1973), 383–387.

[11] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.

[12] T. Natkaniec, J. Wesołowska, *On the convergence of ω_1 sequences of real functions*, Acta Math. Hungar, **91** no. 4 (2001), 333–350.

[13] A. Neubrunnová, *On certain generalizations of the notion of continuity*, Mat. Časopis, **23** (1973), 374–380.

[14] W. Sierpiński, *Sur l'ensemble des points de convergence d'une suite de fonctions continues*, Fund. Math., **2** (1921), 41–49.

[15] H. P. Thielman, *Types of functions*, Amer. Math. Monthly, **60** (1953), 156–161.

[16] J. Wesołowska, *On sets determined by sequences of quasi-continuous functions*, Journal of Appl. Annal., **7** (2001), no. 2, 271–283.

[17] J. Wesołowska, *On sets of convergence points of sequences of some real functions*, Real Anal. Exchange, **25** (1999–2000), no. 2, 937–942.

[18] Z. Zahorski, *Sur la première dérivée*, Trans. Amer. Math. Soc., **69** (1950), 1–54.