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COVERING THE CIRCLE WITH RANDOM OPEN SETS

Abstract

The Dvoretzky covering problem is to cover the circle with random intervals. We consider the covering of the circle with random open sets. We find a necessary and sufficient condition for the circle to be covered almost surely when each open set is composed of a finite number of intervals which are separated by a positive distance.

1 Introduction

The classical Dvoretzky problem is as follows([D]). Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle. We consider a decreasing sequence of positive numbers $\{l_n\}_{n\geq 1}$ with $0 < l_n < 1$ and an i.i.d. sequence of random variables $\{\omega_n\}_{n\geq 1}$ of uniform distribution (Lebesgue measure). We let $I_n = \omega_n + (0, l_n)$. The Dvoretzky covering problem is to find conditions on the length sequence $\{l_n\}_{n\geq 1}$ of the random intervals $\{I_n\}$ in order to cover the whole circle $\mathbb{T}$ almost surely (a.s. for short); i.e., $\mathbb{T} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} I_n$ a.s. After several contributions due to P. Levy, J. P. Kahane, P. Erdŏs, P. Billard (see[K1]), L. Shepp [S1, S2] gave the following necessary and sufficient condition for covering.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{(l_1+\cdots+l_n)} = \infty. \tag{1.1}$$

The reader can see the survey papers [K2, K3] for more information on the subject and related topics.

What about covering the circle by random translates of open sets instead of random intervals I_n? This problem was considered by M. Wschebor [W1].

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In his paper, he pursued the extremal character of intervals among open sets but we shall study this problem in a quite different way in our paper.

Let $\{O_n\}_{n\geq 1}$ be a sequence of open sets in $\mathbb{T}$. (O_n will play the role of the interval $(0, l_n)$.) Let $\mathcal{O}_n = O_n + \omega_n$ be the translation of O_n by ω_n. As in the Dvoretzky model, we assume that the $\omega_n' s$ are i.i.d. random variables with Lebesgue distribution. We say that $\mathbb{T}$ is covered if $\mathbb{T} = \limsup_{n\to\infty} \mathcal{O}_n$ a.s.

We denote by l_n the Lebesgue measure of O_n. Clearly $\sum_{n=1}^{\infty} l_n = +\infty$ is necessary for $\mathbb{T}$ to be covered. So, in the following, we always assume that $\sum_{n=1}^{\infty} l_n = +\infty$. Furthermore, we assume that $\sum_{n=1}^{\infty} l_n^2 < +\infty$.

Denote by χ_n the characteristic function of the open set $\mathcal{O}_n$. Let

$$\Phi(t) = \sum_{n=1}^{\infty} \xi_n(t) \text{ with } \xi_n(t) = \chi_n * \chi_n(t). \tag{1.2}$$

If we consider the $\mathbb{T}$-martingale,

$$\prod_{n=1}^{N} \frac{1 - \chi_n(t - \omega_n)}{1 - l_n},$$

in the same way as in [K2], we can get that

$$\int_0^1 \exp(\Phi(t))\, dt < \infty \Longleftrightarrow \int_0^1 \exp(\Phi(t)) \frac{d\Phi'(t)}{(\Phi'(t))^2} < \infty$$

and

$$\int_0^1 \exp(\Phi(t)) \frac{d\Phi'(t)}{(\Phi'(t))^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{(l_1 + \cdots + l_n - n l_n)}.$$

Combining (1.1) and Proposition 4 in Chapter 11 of [K1], it's easy to see that

$$\int_0^1 \exp \Phi(t)\, dt = \infty \tag{1.3}$$

is a necessary condition for $\mathbb{T}$ to be covered.

In this paper, we will prove that this necessary condition is also sufficient, when some supplement separation conditions are satisfied. Suppose that, for any $n \geq 1$, the open set O_n is composed of $t_{n,k}$ open intervals of length

$l_{n,k}(k = 1, 2, \ldots, m_n)$. Let t_n be the number of the component intervals of O_n. We have $t_n = t_{n,1}+t_{n,2}+\cdots+t_{n,m_n}$. Without loss of generality, we can assume $l_{n,1} > l_{n,2} > \cdots > l_{n,m_n}$. Consider the set of lengths $\{l_{n,k} : n \geq 1, 1 \leq k \leq m_n\}$ and reorder them by $x_1 > x_2 > \cdots > x_n > \ldots$. Let

$$p_j = \mathrm{Card}\{l_{n,k} : l_{n,k} = x_j, n \geq 1, 1 \leq k \leq m_n\}.$$

Assume that O_n is composed of open intervals $I_{n,1}, I_{n,2}, \ldots, I_{n,t_n}$. Throughout this paper, we make the following separation hypothesis

$$d := \inf_{n\geq 1} \inf_{\substack{1\leq j,k\leq t_n \\ j\neq k}} d(I_{n,j}, I_{n,k}) > 0. \tag{1.4}$$

where $d(I, I')$ denotes the distance between the two sets I and I'. The main result of this paper is the following assertion.

Theorem. *Under the separation hypothesis (1.4), we have*

$$\mathbb{T} = \limsup_{n\to\infty} \mathcal{O}_n \ \ a.s. \iff \sum_{n=1}^{\infty} \frac{p_{n+1}}{(p_1 + \cdots + p_n)^2} e^{\Phi(x_{n+1})} = \infty.$$

A very special case is that O_n is composed of two disjoint intervals of length $p\frac{\alpha}{n}$ and $(1-p)\frac{\alpha}{n}$ with $0 < p < 1$ and $\alpha > 0$. If we assume that the separation condition is satisfied, then as a consequence of the theorem, we can conclude that $\mathbb{T}$ is covered iff $\alpha \geq 1$. Note that the covering condition is independent of $0 < p < 1$.

2 The Proof of Theorem

We can get our theorem after proving a series of lemmas.

Let $U_n = \mathcal{O}_1 \bigcup \mathcal{O}_2 \bigcup \cdots \bigcup \mathcal{O}_n$, $F = \mathbb{T} \setminus U_n$. If some of the sets of $\{O_n : n \geq 1\}$ are composed of one interval exactly, we denote by h_1 the length of the longest one of those sets.

For any given n and interval $[\alpha, \beta]$, denote the Lebesgue measure of $F_n \bigcap [\alpha, \beta]$ by $\mu_n(\alpha, \beta)$ and $\mu_n(0, \varepsilon)$ by $\mu_n(\varepsilon)$.

Let $A_n = \{\omega : F_n \bigcap [0, \varepsilon] \neq \emptyset\}$.

Let $\mathbf{E}$ be the expectation operator. With this notation the first lemma can be stated.

Lemma 1. $\mathbf{E}(\mu_n(2\varepsilon)) \geq P(A_n)\mathbf{E}(\mu_n(\varepsilon) \mid 0 \in F_n)$.

Proof. Let $A_{n,N} = \{\omega : F_n \bigcap [0, \varepsilon]$ contains an interval of length $\frac{1}{N}\}$. Obviously, $P(A_n) = \lim_{N\to\infty} P(A_{n,N})$. Choose an appropriate ε such that

$0 < 2\varepsilon < \min\{d, 1-h_1\}$. (Note that if $t_n \geq 2$ for all n, then we only need to choose $0 < 2\varepsilon < d$.) If the event $A_{n,N}$ occurs, then at least one more point of $\{\frac{j}{N} : j = 0, 1, \ldots, [N\varepsilon]\}$ belong to F_n. Write

$$\begin{aligned} A_{n,N:0} =& \{\omega : 0 \in F_n\} \\ A_{n,N:j} =& \{\omega : 0 \in U_n, \frac{1}{N} \in U_n, \ldots, \frac{j-1}{N} \in U_n, \frac{j}{N} \in F_n\} \ (j = 1, 2, \ldots, [N\varepsilon]). \end{aligned}$$

Clearly, $P(A_{n,N}) \leq \sum_{j=0}^{[N\varepsilon]} P(A_{n,N:j})$.

In order to prove this lemma, we only need to prove

$$\begin{aligned} \mathbf{E}(\mu_n(2\varepsilon) I_{A_{n,N:j}}) \geq& \mathbf{E}(\mu_n(\frac{j}{N}, \frac{j}{N} + \varepsilon) I_{A_{n,N:j}}) \\ \geq& P(A_{n,N:j}) \mathbf{E}(\mu_n(\varepsilon) \mid 0 \in F_n) \ (j = 0, 1, \ldots, [N\varepsilon]). \end{aligned}$$

The first inequality of the above expression is obvious and the second one can be rewritten as

$$\mathbf{E}(\mu_n(\frac{j}{N}, \frac{j}{N} + \varepsilon) \mid A_{n,N:j}) \geq \mathbf{E}(\mu_n([\frac{j}{N}, \frac{j}{N} + \varepsilon]) \mid \frac{j}{n} \in F_n). \tag{2.1}$$

Therefore, if

$$P(x \in F_n \mid A_j) \geq P(x \in F_n \mid \frac{j}{N} \in F_n) \tag{2.2}$$

holds for all $x \in (\frac{j}{N}, \frac{j}{N} + \varepsilon)$, we can easily get this lemma.

We rewrite inequality (2.2) as $P(x \in U_n \mid A_j) \leq P(x \in U_n \mid \frac{j}{N} \in F_n)$, which is equivalent to

$$\begin{aligned} &P(0 \in U_n, \ldots, \frac{j-1}{N} \in U_n \mid \frac{j}{N} \in F_n, x \in U_n) \\ \leq& P(0 \in U_n, \ldots, \frac{j-1}{N} \in U_n \mid \frac{j}{N} \in F_n). \end{aligned} \tag{2.3}$$

Thus if

$$\begin{aligned} &P(0 \in U_n, \ldots, \frac{j-1}{N} \in U_n \mid \frac{j}{N} \in F_n, x \in U_n) \\ \leq& P(0 \in U_n, \ldots, \frac{j-1}{N} \in U_n \mid \frac{j}{N} \in F_n, x \in F_n), \end{aligned} \tag{2.4}$$

holds, then inequality (2.3) follows immediately.

Now we proceed to prove that for $\forall\, k = 1, 2, \ldots, n$

$$\begin{aligned}
&P(0 \in U_n, \ldots, \frac{j-1}{N} \in U_n \mid \frac{j}{N} \in F_n, x \notin \mathcal{O}_1 \bigcup \cdots \bigcup \mathcal{O}_{k-1}, x \in \mathcal{O}_k) \\
\leq &P(0 \in U_n, \ldots, \frac{j-1}{N} \in U_n \mid \frac{j}{N} \in F_n, x \in F_n).
\end{aligned}$$

In fact, if $0 < 2\varepsilon < \min\{d, 1 - h_1\}$ and $\mathcal{O}_k$ contains x but does not contains $\frac{j}{N}$, then under the separation hypothesis (1.4), $\mathcal{O}_k \bigcap [0, \frac{j-1}{N}] = \emptyset$. Moreover, $\mathcal{O}_n$ $(n \geq 1)$ are i.i.d. Thus

$$\begin{aligned}
&P(0 \in U_n, \ldots, \frac{j-1}{N} \in U_n \mid \frac{j}{N} \in F_n, x \notin \mathcal{O}_1 \bigcup \cdots \bigcup \mathcal{O}_{k-1}, x \in \mathcal{O}_k) \\
=&P(0 \in U_{k-1}, \ldots, \frac{j-1}{N} \in U_{k-1}, 0 \in \mathcal{O}_{k+1} \bigcup \cdots \bigcup \mathcal{O}_n, \ldots, \\
&\frac{j-1}{N} \in \mathcal{O}_{k+1} \bigcup \cdots \bigcup \mathcal{O}_n \mid \frac{j}{N} \in F_{k-1}, \frac{j}{N} \notin \mathcal{O}_k, \frac{j}{N} \notin \mathcal{O}_{k+1} \bigcup \cdots \bigcup \\
&\mathcal{O}_n, x \notin \mathcal{O}_1 \bigcup \cdots \bigcup \mathcal{O}_{k-1}, x \in \mathcal{O}_k) \\
\leq &P(0 \in U_{k-1}, \ldots, \frac{j-1}{N} \in U_{k-1} \mid \frac{j}{N} \in F_{k-1}, x \in F_{k-1})
\end{aligned}$$

and

$$\begin{aligned}
&P(0 \in U_n, \ldots, \frac{j-1}{N} \in U_n \mid \frac{j}{N} \in F_n, x \in F_n) \\
\geq &P(0 \in U_{n-1}, \ldots, \frac{j}{N} \in U_{n-1} \mid \frac{j}{N} \in F_{n-1}, \frac{j}{N} \notin \mathcal{O}_n, x \notin \mathcal{O}_1 \bigcup \cdots \\
&\quad \bigcup \mathcal{O}_{n-1}, x \notin \mathcal{O}_n) \\
= &P(0 \in U_{n-1}, \ldots, \frac{j}{N} \in U_{n-1} \mid \frac{j}{N} \in F_{n-1}, x \notin \mathcal{O}_1 \bigcup \cdots \bigcup \mathcal{O}_{n-1}) \\
\geq &P(0 \in U_{k-1}, \ldots, \frac{j}{N} \in U_{k-1} \mid \frac{j}{N} \in F_{k-1}, x \notin \mathcal{O}_1 \bigcup \cdots \bigcup \mathcal{O}_{k-1})
\end{aligned}$$

for all $k \geq 1$, which implies that the inequality (2.3) holds. $\square$

Lemma 2. *Under the separation hypothesis (1.4), if* $\int\limits_0^{\varepsilon} \exp(\Phi(t))\, dt = \infty$ $(0 < \varepsilon < d)$, *then* $\mathbb{T} = \limsup\limits_{n \to \infty} \mathcal{O}_n$ *a.s.*

PROOF. Firstly, we will show that $P(A_n) \to 0 (n \to \infty)$. In fact

$$\begin{aligned}\mathbf{E}\mu_n(2\varepsilon) =& \int_0^{2\varepsilon} \mathbf{E} \prod_{j=1}^{n} (1 - \chi_j(t - \omega_j))\, dt \\ =& \int_0^{2\varepsilon} \prod_{j=1}^{n} \mathbf{E}(1 - \chi_j(t - \omega_j))\, dt = 2\varepsilon \prod_{j=1}^{n} (1 - l_j).\end{aligned}$$

$$\begin{aligned}\mathbf{E}(\mu_n(\varepsilon)|0 \in F_n) =& \int_0^{\varepsilon} \mathbf{E} \prod_{j=1}^{n} (1 - \chi_j(t - \omega_j)) \times \frac{\prod\limits_{j=1}^{n} (1 - \chi_j(-\omega_j))}{\prod\limits_{j=1}^{n} (1 - l_j)}\, dt \\ =& (\prod_{j=1}^{n} (1 - l_j))^{-1} \int_0^{\varepsilon} \prod_{j=1}^{n} (1 - 2l_j + \xi_j(t))\, dt.\end{aligned}$$

Yy applying lemma 1 we can get

$$\begin{aligned}2\varepsilon \geq& P(A_n)(\prod_{j=1}^{n} (1 - l_j))^{-2} \int_0^{\varepsilon} \prod_{j=1}^{n} (1 - 2l_j + \xi_j(t))\, dt \\ =& P(A_n) \int_0^{\varepsilon} \prod_{j=1}^{n} (1 + \frac{\xi_j(t) - l_j^2}{(1 - l_j)^2})\, dt.\end{aligned} \tag{2.5}$$

Under the separation hypothesis, $0 \leq \xi_j(t) \leq l_j$, combining with the assumption that $\sum\limits_{n=1}^{\infty} l_n^2 < \infty$, we have

$$\sum_{j=1}^{n} \frac{\xi_j(t) - l_j^2}{(1 - l_j)^2} = \sum_{j=1}^{n} \xi_j(t) + O(1) \text{ and } \sum_{j=1}^{n} (\frac{\xi_j(t) - l_j^2}{(1 - l_j)^2})^2 = O(1).$$

Hence from (2.5) we can get $P(A_n) \int\limits_0^{\varepsilon} \exp \sum\limits_{j=1}^{n} \xi_j(t)\, dt \leq C\varepsilon$, where C depends only on $l_1, l_2, \ldots, l_n$. Obviously, if $\int\limits_0^{\varepsilon} \exp(\Phi(t))\, dt = \infty$ $(0 < \varepsilon < d)$, then $P(A_n) \to 0 (n \to \infty)$.

By substituting any interval $[\alpha, \beta]$ for $[0, \varepsilon]$, it's easy to get the this lemma. $\square$

Lemma 3. *Under the separation hypothesis (1.4), if* $\int_0^\varepsilon \exp(\Phi(t))\,dt < \infty$ $(0 < \varepsilon < d)$, *then* $\mathbb{T} \neq \limsup_{n\to\infty} \mathcal{O}_n$ *a.s.*

The proof of this lemma is very similar to that of proposition 3 in chapter 11 of [K1]. Consequently we omit it here.

Note that $\int_0^\varepsilon \exp(\Phi(t))\,dt < \infty$ $(0 < \varepsilon < d)$ is a necessary and sufficient condition for $\mathbb{T}$ to be covered under the separation hypothesis (1.4). So now we proceed to give a concrete expression of $\int_0^\varepsilon \exp(\Phi(t))\,dt$. A useful lemma must be inserted here.

Lemma 4. [K1] *If* $\Phi(t)$ *is convex and decreasing on* $(0,\varepsilon)$, *then*

$$\int_0^\varepsilon \exp(\Phi(t))\,dt < \infty \Longleftrightarrow \int_0^\varepsilon \exp(\Phi(t))\frac{d\Phi'(t)}{(\Phi'(t))^2} < \infty.$$

We will use Lemma 4 to calculate $\int_0^\varepsilon \exp(\Phi(t))\,dt$. First of all, there exist $\delta > 0$ such that $\Phi(t)$ is convex and decreasing on $(0,\delta)$. In fact, for any $n \geq 1$ and $k \geq 1$, there exists $j_k(n) \geq 0$ such that

$$l_{k,1}, \ldots, l_{k,j_k(n)} \geq x_n, \quad l_{k,j_k(n)+1}, \ldots, l_{k,m_k} \leq x_{n+1}$$

and $\forall\, t \in (0,d)$ we have

$$\xi_n(t) = \sum_{i=1}^{m_n} t_{n,i} \cdot \sup\{0, l_{n,i} - t\}.$$

Write $n_0 = \inf\{n : x_n \leq \frac{\min\{d,1-h_1\}}{2}\}$. Then for any $n \geq n_0$ and $t \in [x_{n+1}, x_n)$ as well as $k \geq 1$, $\xi_k(t) = \sum_{i=1}^{j_k(n)} t_{k,i}(l_{k,i} - t)$. Thus $\forall\, n \geq n_0$ and $\forall\, t \in [x_{n+1}, x_n)$ we have $\Phi(t) = \sum_{k=1}^{\infty}\sum_{i=1}^{j_k(n)} t_{k,i}(l_{k,i} - t)$, where $\sum_{k=1}^{\infty}\sum_{i=1}^{j_k(n)} t_{k,i} = p_1 + \cdots + p_n$. Furthermore, we can get that

$$\Phi(x_{n+1}) = \sum_{k=1}^{\infty}\sum_{i=1}^{j_k(n)} t_{k,i}(l_{k,i} - x_{n+1}) = \sum_{k=1}^{\infty}\sum_{i=1}^{j_k(n)} t_{k,i} l_{k,i} - (p_1 + \cdots + p_n)x_{n+1},$$

$$\Phi'(t) = -\sum_{k=1}^{\infty}\sum_{i=1}^{j_k(n)} t^*_{k,i} = -(p_1 + \cdots + p_n),\ \forall\, t \in [x_{n+1}, x_n).$$

Similarly, for any $n \geq n_0$ and $\forall t \in [x_{n+2}, x_{n+1})$ we have

$$\Phi(t) = \sum_{k=1}^{\infty} \sum_{i=1}^{j_k(n+1)} t_{k,i}(l_{k,i} - t).$$

However, for any $k \geq 1$ and $n \geq 1$

$$j_k(n+1) = \begin{cases} j_k(n), & \text{if } l_{k,j_k(n)+1} < x_{n+1} \\ j_k(n)+1, & \text{if } l_{k,j_k(n)+1} = x_{n+1}; \end{cases}$$

so it's easy to prove that $\Phi(t) \to \Phi(x_{n+1})$ $(t \to x_{n+1})$.

Take $\delta = x_{n_0}$. Then the above facts show that $\Phi(t)$ is convex and decreasing on $(0, \delta]$. For any $n \leq n_0$ define

$$\begin{aligned} \Phi(x_1) =& 0 \\ \Phi(x_n) =& \sum_{k=1}^{\infty} \sum_{i=1}^{j_k(n-1)} t_{k,i}(l_{k,i} - x_n)(n_0 \geq n \geq 2). \end{aligned}$$

Note that $d\Phi'(x_{n+1}) = p_{n+1}$ and $\Phi'(t) = -(p_1 + \cdots + p_n)$ for any $t \in [x_{n+1}, x_n)$. Then it's easy to check that

$$\int_0^{\delta} \exp(\Phi(t)) \frac{d\Phi'(t)}{(\Phi'(t))^2} = \sum_{n=n_0+1}^{\infty} \frac{\exp[\Phi(x_n)] p_n}{(p_1 + \cdots + p_{n-1})^2}.$$

Combining all the above conclusions, we can get our Theorem. □

Notation. If $t_n = 1$ $(n \geq 1)$, then our covering problem becomes the classical Dvoretzky covering problem and in this case, $x_n = l_n$, $p_n = 1$ and $\Phi(x_n) = l_1 + l_2 + \cdots + l_n - n l_n$.

3 Examples

Example 1. Suppose O_n is composed of two disjoint intervals of lengths $p\frac{\alpha}{n}$ and $(1-p)\frac{\alpha}{n}$ respectively, where $0 < p < 1$ and $\alpha > 0$. We assume that the separation condition is satisfied. Without loss of generality, we suppose $p < \frac{1}{2}$.

Corollary. *For this special case,* $\mathbb{T}$ *is covered iff* $\alpha \geq 1$. *The covering is independent of* $0 < p \leq 1$.

PROOF. We need to sort the lengths of all intervals into $x_1, \ldots, x_n, \ldots$ by their sizes and to calculate $\Phi(t)$.

For any positive number y, denote by $[y]$ the integer part of y. Write $z_p = [\frac{1-p}{p}]$ and $y_p = \frac{1-p}{p} - z_p$.

(**I**) p is irrational. Then $\frac{p}{m} \neq \frac{1-p}{n}$ for all n and m, which implies that the lengths of all intervals are different and hence $p_n = 1 (n \geq 1)$.

Note that for any $\frac{p}{n}$, $\frac{1-p}{k} > \frac{p}{n}$ for $k \leq [n \cdot \frac{1-p}{p}]$ and $\frac{1-p}{k} < \frac{p}{n}$ for $k \geq [n \cdot \frac{1-p}{p}] + 1$. Let $k_m = [m \cdot \frac{1-p}{p}] (m \geq 1)$. Then

$$k_{m+1} = \begin{cases} k_m + z_p & \text{if } [my_p + y_p] = [my_p] \\ k_m + z_p + 1 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned}
x_1 &= (1-p)\alpha, \\
x_2 &= \frac{1-p}{2}\alpha, \ldots, x_{k_1} = \frac{1-p}{k_1}\alpha, \\
x_{k_1+1} &= p\alpha, \\
x_{k_1+2} &= \frac{1-p}{k_1+1}\alpha, x_{k_1+3} = \frac{1-p}{k_1+2}\alpha, \ldots, x_{k_2+1} = \frac{1-p}{k_2}\alpha, \\
x_{k_2+2} &= \frac{p}{2}\alpha, \\
x_{k_2+3} &= \frac{1-p}{k_2+1}\alpha, x_{k_2+4} = \frac{1-p}{k_2+2}\alpha, \ldots, x_{k_3+2} = \frac{1-p}{k_3}\alpha, \\
x_{k_3+3} &= \frac{p}{3}\alpha, \\
&\vdots \\
x_{k_n+n-1} &= \frac{1-p}{k_n}\alpha, \\
x_{k_n+n} &= \frac{p}{n}\alpha, \\
x_{k_n+n+1} &= \frac{1-p}{k_n+1}\alpha, x_{k_n+n+2} = \frac{1-p}{k_n+2}\alpha, \ldots, x_{k_{n+1}+n} = \frac{1-p}{k_{n+1}}\alpha, \\
&\vdots
\end{aligned}$$

Now we want to compute $\Phi(x_n)$.

(1) If $n = k_m + m$ for some m, then $x_n = p\alpha\frac{1}{m}$ and

$$k_{m+1} = \begin{cases} (m+1)z_p & \text{if } [my_p + y_p] = [my_p] \\ (m+1)z_p + m & \text{otherwise} \end{cases}$$

$$n = \begin{cases} (z_p+1)m & \text{if } [my_p + y_p] = [my_p] \\ (z_p+2)m - 1 & \text{otherwise.} \end{cases}$$

Which leads to

$$k_m = \begin{cases} \frac{z_p}{z_p+1} \cdot n & \text{if } [my_p + y_p] = [my_p] \\ \frac{z_p+1}{z_p+2}(n+1) - 1 & \text{otherwise} \end{cases}$$

$$m = \begin{cases} \frac{n}{z_p+1} & \text{if } [my_p + y_p] = [my_p] \\ \frac{n+1}{z_p+2} & \text{otherwise.} \end{cases}$$

In addition, we have

$$\begin{aligned} \Phi(x_n) =& (1 + \frac{1}{2} + \cdots + \frac{1}{m})p\alpha + (1 + \frac{1}{2} + \cdots + \frac{1}{k_m})(1-p)\alpha - n \cdot \frac{1}{m}p\alpha \\ =& p\alpha \ln m + (1-p)\alpha \ln k_m - \frac{n}{m}p\alpha - a_1, \end{aligned}$$

where a_1 is independent of n.

(2) If $k_m + m < n < k_{m+1} + m + 1$, that means that $n = k_m + m + j$ for some $1 \leq j \leq k_{m+1} - k_m \leq z_p + 1$. In this case, $x_n = \frac{1-p}{k_m+j}\alpha$ and

$$k_m = \begin{cases} \frac{z_p}{z_p+1} \cdot (n-j) & \text{if } [my_p + y_p] = [my_p] \\ \frac{z_p+1}{z_p+2}(n+1-j) - 1 & \text{otherwise} \end{cases}$$

$$m = \begin{cases} \frac{n-j}{z_p+1} & \text{if } [my_p + y_p] = [my_p] \\ \frac{n+1-j}{z_p+2} & \text{otherwise.} \end{cases}$$

Meanwhile, we have

$$\begin{aligned} \Phi(x_n) =& (1 + \frac{1}{2} + \cdots + \frac{1}{m})p\alpha + (1 + \frac{1}{2} + \cdots + \frac{1}{k_m + j})(1-p)\alpha - n \cdot \frac{1-p}{k_m + j}\alpha \\ =& p\alpha \ln m + (1-p)\alpha \ln(k_m + j) - \frac{n}{k_m + j}(1-p)\alpha - a_2, \end{aligned}$$

where a_2 is independent of n.

According to our theorem, it's not difficult to check if $\alpha \geq 1$, then $\mathbb{T} = \limsup_{n\to\infty} \mathcal{O}_n$ a.s. otherwise, if $\alpha < 1$ then $\mathbb{T} \neq \limsup_{n\to\infty} \mathcal{O}_n$ a.s.

(**II**) p is rational.

(1) If $\frac{1-p}{p} = z_p$,;i.e., $\frac{1-p}{p}$ is an integer, then for $[n\frac{1-p}{p}] = n \cdot z_p\ \forall\ n$ and $\frac{1-p}{n \cdot z_p}\alpha = \frac{p}{n}\alpha$. In this case, $x_n = \frac{1-p}{n}\alpha$. If $n = m \cdot z_p$ for some m, then $x_n = \frac{1-p}{mz_p}\alpha$ and

$$\begin{aligned}\Phi(x_n) =&(1 + \cdots + \frac{1}{mz_p})(1-p)\alpha + (1 + \cdots + \frac{1}{m})p\alpha - (n+m)x_n \\ =&(1-p)\alpha \lg n + p\alpha \lg m - \frac{n+m}{n}(1-p)\alpha - a_1 \\ =&(1-p)\alpha \lg n + p\alpha \lg \frac{n}{z_p} - \frac{n+m}{n}(1-p) - a_1.\end{aligned}$$

Otherwise, if $n = mz_p + j$ for some m and $j \leq z_p - 1$, then

$$\begin{aligned}\Phi(x_n) =&(1 + \cdots + \frac{1}{n})(1-p)\alpha + (1 + \cdots + \frac{1}{m})p\alpha - (n+m)x_n \\ =&(1-p)\alpha \lg n + p\alpha \lg m - \frac{n+m}{n}(1-p)\alpha - a_1 \\ =&(1-p)\alpha \lg n + p\alpha \lg \frac{n-j}{z_p} - \frac{n+m}{n}(1-p) - a_1.\end{aligned}$$

It's clear that $\mathbb{T} = \limsup_{n\to\infty} \mathcal{O}_n$ a.s iff $\alpha \geq 1$.

(2) If $\frac{1-p}{p}$ isn't an integer, let $\frac{1-p}{p} = \frac{y_1}{z_1}$, where y_1 and z_1 are irreducible. In this case, only when $m = k \cdot z_1 (k \geq 1)$ and $n = \frac{1-p}{p}m = ky_1$, we have $\frac{1-p}{n} = \frac{p}{m}$. Repeating the above procedure, we can get the same conclusion as before. □

Example 2. Suppose O_n is divided into m disjoint intervals of the same length $\frac{\alpha}{n \cdot m}$ and assume that the separation condition is satisfied. Then by the way Corollary 3 was proved, we can also verify the fact that $\mathbb{T} = \limsup_{n\to\infty} \mathcal{O}_n$ a.s. iff $\alpha \geq 1$.

4 Remark

If we remove the separation hypothesis (1.4), we can get a sufficient condition for $\mathbb{T}$ to be covered a.s. Let $s_{n,k} (k = 1, \ldots, t_n)$ be the smallest integers z satisfied $l_{n,k} \geq \frac{l_n}{z}$ and s_n be the biggest one of $\{s_{m,k_m} : m = 1, \ldots, n, k_m = 1, \ldots, t_m\}$. Obviously, $s_{n,k} \geq 1$ and $\sum_{k=1}^{t_n} \frac{1}{s_{n,k}} \leq 1$.

Proposition. *If* $\limsup_{n\to\infty} \frac{1}{ns_n} \exp(l_1 + l_2 + \cdots + l_n) = \infty$*, then* $\mathbb{T} = \limsup_{n\to\infty} \mathcal{O}_n$ *a.s. Specially, if* $\sum_{n=1}^{\infty} l_n^2 = \infty$ *and* $s_n = O(n^\alpha)$ *for some* $\alpha > 0$*, then* $\mathbb{T} = \limsup_{n\to\infty} \mathcal{O}_n$ *a.s.*

PROOF. Write $u_n = \frac{1}{n}\exp(l_1 + \cdots + l_n)$. If $u_n \geq \sup_{m<n} u_m$, then we say $n \in \Lambda$. From the condition of this proposition, we know Λ is infinite and $\lim_{n\to\infty} u_n = \infty$. For any $n \in \Lambda$, we have $u_n \geq u_{n-1}$ and $l_n \geq \lg \frac{n}{n-1} \geq \frac{1}{2n}$, which implies that for any given $n \in \Lambda$ and $m \leq n$, we have $l_m \geq \frac{1}{2n}$.

For convenience, denote the interval of center x and radius l by $I(x, l)$. Let $m \in \Lambda$ be a given but arbitrary. For any $n \leq m$, let $\tilde{\mathcal{I}}_{n,k}(k = 1, \ldots, t_n)$ be the intervals with the same centers as $\mathcal{I}_{n,k}$ and the length of $l_{n,k} - \frac{1}{2n\cdot s_{n,k}}$. Write $\tilde{\mathcal{O}}_n = \bigcup_{k=1}^{t_n} \tilde{\mathcal{I}}_{n,k}$. Let $x_j = \frac{j}{2m\cdot s_m}(j = 0, \ldots, 2m \cdot s_m)$ be those points on $\mathbb{T}$ which divide $\mathbb{T}$ into $2ms_m$ parts; that is, $\mathbb{T}$ is covered by $\bigcup_{j=0}^{2m\cdot s_m} I(x_j, \frac{1}{4ms_m})$; so it's easy to get that

$$\begin{aligned}\{\omega : \mathbb{T} \neq U_m\} &\subseteq \bigcup_{j=0}^{2ms_m} \{\omega : I(x_j, \frac{1}{4ms_m}) \not\subset U_m \\ &\subseteq \bigcup_{j=0}^{2ms_m} \{\omega : x_j \notin \bigcup_{n=1}^{m} \tilde{\mathcal{O}}_n\} = \bigcup_{j=0}^{2ms_m} \{\omega : x_j \notin \bigcup_{n=1}^{m} \bigcup_{k=1}^{t_n} \tilde{I}_{n,k}\}.\end{aligned}$$

Otherwise, if $\exists\ j$ such that $x_j \in \bigcup_{n=1}^{m} \tilde{\mathcal{O}}_n$, then $\exists\ n_0$ and $k_0 \in \{1, \ldots, t_{n_0}\}$ such that $x_j \in \tilde{\mathcal{I}}_{n_0,k_0}$. Then

$$I(x_j, \frac{1}{4ms_m}) \subset I(x_j, \frac{1}{4ms_{n_0,k_0}}) \subset \mathcal{I}_{n_0,k_0} \subset \mathcal{O}_{n_0} \subset U_m.$$

However, since $\tilde{\mathcal{O}}_n = \bigcup_{k=1}^{t_n} \tilde{\mathcal{I}}_{n,k}$, we have

$$P(x_j \notin \bigcup_{n=1}^{m} \tilde{\mathcal{O}}_n) = \prod_{n=1}^{m} P(x_j \notin \tilde{\mathcal{O}}_n) = \prod_{n=1}^{m} [1 - \sum_{k=1}^{t_n} (l_{n,k} - \frac{1}{2ms_{n,k}})].$$

It follows that

$$\begin{aligned} P(T \neq U_m) \leq & 2ms_m \prod_{n=1}^{m}[1-\sum_{k=1}^{t_n}(l_{n,k}-\frac{1}{2ms_{n,k}})] \\ = & 2ms_m \prod_{n=1}^{m}[1-l_n+\frac{1}{2m}\sum_{k=1}^{t_n}\frac{1}{s_{n,k}}] \\ \leq & 2ms_m \exp[-\sum_{n=1}^{m}(l_n-\frac{1}{2m}\sum_{k=1}^{t_n}\frac{1}{s_{n,k}}] \\ \leq & 2ms_m \exp[-\sum_{n=1}^{m}(l_n-\frac{1}{2m})]. \end{aligned}$$

For the next to the last inequality, we have use the fact $\sum_{k=1}^{t_n}\frac{1}{s_{n,k}} \leq 1$. Let $m \to \infty$ $(m \in \Lambda)$. We get $P(T \neq \sum_{n=1}^{\infty}\mathcal{O}_n) = 0$. If $\sum_{n=1}^{\infty} l_n^2 = \infty$, then there exists infinite l_n such that $l_n > n^{-\frac{2}{3}}$, this implies that $l_1 + \cdots + l_n > n^{\frac{1}{3}}$ when n is large enough. □

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