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CONDITIONS WHICH IMPLY CONTINUITY

Abstract

In this paper we look at various generalizations of continuity for a function and determine necessary additional conditions which result in continuity in the ordinary sense.

1 Introduction and Definitions

There are many notions in the literature which are generalizations of continuity. For each of them, there are examples showing that a function can have this property without being continuous in the ordinary sense. The natural question to follow then is, "What conditions can be added to this form of continuity to make a function continuous in the usual sense?" That is precisely the question we wish to look at in this paper. We start by defining some of the generalizations of continuity we shall deal with.

Definition 1. Let X, Y, and Z be topological spaces and $f : X \times Y \to Z$. We say that

1. f is quasi-continuous at (x, y) if for open sets $U \subset X$ and $V \subset Y$ with $(x, y) \in U \times V$ and open set $W \subset Z$ where $f(x, y) \in W$, there is a nonempty open set $U' \subset U$ and a nonempty open set $V' \subset V$ such that $f(U' \times V') \subset W$.

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2. f is quasi-continuous at (x, y) with respect to x (alternatively y) if we also insist $x \in U'$ $(y \in V')$.

3. f is symmetrically quasi-continuous at (x, y) if it is quasi-continuous with respect to x and with respect to y.

An example to show how all of these differ from ordinary continuity is the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) \begin{cases} 1 & \text{if } x = y,\ \ (x, y) \neq (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

A related, but stronger, idea is separate continuity. Here a function defined on a product space is refined by holding one of the arguments constant.

Definition 2. Let X, Y, and Z be topological spaces and let $f : X \times Y \to Z$. For every fixed $x \in X$, the function $f_x : Y \to Z$ defined by $f_x(y) = f(x, y)$ is called an x-section of f. The y-section is similarly defined. We say $f : X \times Y \to Z$ is separately continuous if each x-section and each y-section is a continuous function.

The standard example of a separately continuous function which is not continuous at the origin is

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

The relationships between these various notions are summarized in the following diagram where C represents the continuous functions, SC the separately continuous functions, QC the quasi-continuous functions, SQC the separately quasi-continuous functions (for all (x, y) each x-section and y-section is a quasi-continuous function), and $SymQC$ the symmetrically quasi-continuous functions.

$$\begin{array}{ccccc} & & C & & \\ & \swarrow & \downarrow & \searrow & \\ SymQC & \leftarrow & SC & \rightarrow & QC \\ & \searrow & \downarrow & \nearrow & \\ & & SQC & & \end{array}$$

There are an abundance of examples to show that none of these arrows may be reversed.

For this paper we shall restrict ourselves to $X = Y = Z = \mathbb{R}$. Under this circumstance we can add a fourth to our list of quasi-continuities. We begin with the single variable definition found in Grande and Natkaniec [7].

Definition 3. A function $f : \mathbb{R} \to \mathbb{R}$ is bilaterally quasi-continuous at a point x if for every positive η there are non-empty open sets $V \subset (x - \eta, x)$ and $W \subset (x, x + \eta)$ such that

$$f(V \cup W) \subset (f(x) - \eta, f(x) + \eta).$$

While the idea of bilateral does not easily generalize into all topological spaces we can expand this idea to functions whose domain is $\mathbb{R}^n$.

Definition 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x = (x_1, x_2, \ldots, x_n)$. We say f is quasi-continuous from the right with respect to x_i if for every positive η there are non-empty open sets U_j in $\mathbb{R}$ where U_j contains x_j if $j \neq i$ and $U_i = (a, b) \subseteq (x_i, x_i + \eta)$ and both

$$\prod_{j=1}^{n} U_j \subseteq B(x, \eta)$$

and

$$f\left(\prod_{j=1}^{n} U_j\right) \subset (f(x) - \eta, f(x) + \eta).$$

Similarly, there is quasi-continuous from the left ($b < x_i$) with respect to x_i. Finally, f is bilaterally quasi-continuous with respect to x_i, if it is quasi-continuous from both directions at x_i and f is bilaterally quasi-continuous at x if it is bilaterally quasi-continuous at every x_i.

Bilaterally quasi-continuous functions fit between continuous functions and symmetrically quasi-continuous functions in the chart.

Our last type of function to be defined is the nearly continuous function. The concept of nearly continuous at a point was introduced by V. Pták in [12].

Definition 5. Let X and Y be topological spaces. A function $f : X \to Y$ is nearly continuous at $x \in X$ if, for each neighborhood V of $f(x)$, $\text{int}(\text{cl}(f^{-1}(V)))$ is a neighborhood of x.

In our situation, where we'll have $X = \mathbb{R}^2$ and $Y = \mathbb{R}$. It can be shown that (see [8]) a function $f : \mathbb{R}^2 \to \mathbb{R}$ is nearly continuous at a point x if there exists D, a set dense in a neighborhood of x, such that the restriction, $f\mid_D$, is continuous at x.

The classic salt and pepper function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is a nearly continuous function which is not continuous at any point.

2 Results

The motivation for this paper were the following results. C E. Burgess in [1] gives a condition on locally bounded functions which implies ordinary continuity.

Lemma 6. *If $f : \mathbb{R} \to \mathbb{R}$ is locally bounded and the graph of f is closed in $\mathbb{R}^2$, then f is continuous.*

In addition, Z. Piotrowski and E. Wingler in [11] showed the following.

Theorem 7. *Let X and Y be topological spaces with Y locally connected. Let Z be locally compact and suppose $f : X \times Y \to Z$ has continuous y-sections and connected x-sections. If f has a closed graph, then f is continuous.*

The authors have been working with quasi-continuous functions and decided, in a similar vein, to look for a condition on quasi-continuity which would imply ordinary continuity. That result is the following theorem.

Theorem 8. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is bilaterally quasi-continuous and has a closed graph, then f is continuous.*

PROOF. Assume not. Say there exists a point (x_0, y_0) where f is not continuous. So there exists an $\varepsilon > 0$ and (x_n, y_n) converging to (x_0, y_0) such that

$$|f(x_n, y_n) - f(x_0, y_0)| > \varepsilon$$

for all n. Let B be the open ball about (x_0, y_0) of radius one. Look at $f(B)$. From Corollary 1 in [9] $f(B)$ is connected, so there exists a point at least ε away from the triple $(x_0, y_0, f(x_0, y_0))$ on the line parallel to the z-axis through this triple which is a limit point of the graph. This contradicts the graph being closed. □

Let us give some examples regarding this result. The first shows that we cannot replace bilaterally quasi-continuous with the weaker symmetrically quasi-continuous. The second shows that this is different than the result of Burgess by exhibiting a bilaterally quasi-continuous function which is not locally bounded. Lastly, we show that being bilaterally quasi-continuous everywhere is necessary to have $f(B)$ be connected.

Example 9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \frac{1}{x-y} & \text{if } x < y \\ 0 & \text{if } x \geq y. \end{cases}$$

This function is symmetrically quasi-continuous and has a closed graph, but is not continuous.

Example 10. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$g(x,y) = \begin{cases} \frac{1}{\sqrt{x^2+y^2}} \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

This g is bilaterally quasi-continuous, but not bounded at the origin.

Example 11. Let A be the set of points (x,y) in the interior of the first quadrant such that $y < x/5$ and let $L_1 = \{(x,y) : x > 0 \text{ and } y = x/5\}$. Let B be the set of points (x,y) in the interior of the fourth quadrant such that $y > -x/5$ and let $L_2 = \{(x,y) : x > 0 \text{ and } y = -x/5\}$. Let $C = \{(x,0) : x > 0\}$. Let $\operatorname{dist}((x,y), A)$ denote the distance from a point (x,y) to the set A. Finally we define $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(x) = \begin{cases} 1/\operatorname{dist}((x,y), L_1) & \text{if } (x,y) \in A \\ 1/\operatorname{dist}((x,y), L_2) & \text{if } (x,y) \in B \\ 1/\operatorname{dist}((x,y), L_1) & \text{if } (x,y) \in C \\ 0 & \text{otherwise.} \end{cases}$$

Now h has closed graph and is bilaterally quasi-continuous at the origin, but $h(B)$, the image of the unit ball, it not connected.

Let $\mathcal{C}$ be a class of functions from a space X into a space Y. We say that the class $\mathcal{C}$ has the unique determination property on dense sets, for f, g from $\mathcal{C}$, if f agrees with g on any dense subset of X, then f and g agree throughout X.

It is well-known that the class of all continuous functions $f : X \to Y$ from any topological space X into any Hausdorff space Y does have the unique determination property on dense sets (see [3]). W. Sierpinski showed that every separately continuous function $f : \mathbb{R}^n \to \mathbb{R}$ has this unique determination property. From now on, if a class $\mathcal{C}$ of functions from a given space X into a given space Y has the unique determination property, we will simply say that $\mathcal{C}$ has the Sierpinski property. We will (informally) say f has the Sierpinski property if X, Y, and $\mathcal{C}$ were specified.

Our result deals with nearly continuous functions and the Sierpinski property.

Theorem 12. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is nearly continuous and has the Sierpinski property, then f is continuous.*

PROOF. Suppose that f is not continuous at some (x_0, y_0). So there exists an $\varepsilon > 0$ and (x_n, y_n) converging to (x_0, y_0) such that

$$|f(x_n, y_n) - f(x_0, y_0)| > \varepsilon$$

for all n. Since f is nearly continuous at (x_0, y_0), there exists a dense neighborhood D such that $f|_D$ is continuous at (x_0, y_0). Choose an $(x_n, y_n) \in \mathrm{cl}(D)$, the closure of D. Since f is nearly continuous at (x_n, y_n), there exists a dense neighborhood $\widetilde{D}$ of (x_n, y_n) such that $f|_{\widetilde{D}}$ is continuous at (x_n, y_n).

Now pick a region where D and $\widetilde{D}$ are co-dense. Define a new function $\widehat{f}$ by

$$\widehat{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in \widetilde{D} \\ f(x, y) + 1 & \text{elsewhere.} \end{cases}$$

Then $f = \widehat{f}$ on $\widetilde{D}$, $f \neq \widehat{f}$ which contradicts the Sierpinski property. □

Since all separately continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ have the Sierpinski property, we have the following Corollary. It should be noted that C. Goffman and C. J. Neugebauer (in [5]) have given a separately continuous function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ for which the Sierpinski property fails.

Corollary 13. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is nearly continuous and separately continuous, then f is continuous.*

This is related to results by T. Neubrunn and J. Ewert. In [10] Neubrunn showed that if $f : \mathbb{R}^2 \to \mathbb{R}$ is quasi-continuous and nearly continuous, then f is continuous. Because the separately continuous functions are a proper subset of the quasi-continuous functions Neubrunn's paper contains the previous corollary. Our result differs from Neubrunn's because quasi-continuous functions do not have the Sierpinski property as the following example shows.

Example 14. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \sin \frac{1}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$g(x, y) = \begin{cases} \sin \frac{1}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0). \end{cases}$$

Obviously, f and g agree on a dense set, but are not equal.

In [4] Ewert has the following corollary where (X, T_X) is a topological space and (Z, q) is a metric space.

Corollary 15. *If $f : X \to Z$ is a cliquish and almost continuous (in the sense of Husain) function, then f is a continuous function.*

Let us close our article with an open problem that is closely related to our findings. The question is attributed to Prof. Cz. Ryll-Nardzewski.

Problem 16. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ has a closed and connected graph. Must f be continuous?

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