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A GENERALIZED HENSTOCK INTEGRAL

Abstract

The present paper concerns a general version of the Henstock integral that includes Schwabik's generalized Perron integral and all the Stieltjes type integrals induced by measures as well.

1 Introduction

Let k be a positive integer and a, b be fixed real numbers. For $g : [a, b] \to \mathbb{R}$ and for $k+1$ distinct points $x_0, x_1, \dots, x_k$ belonging to $[a, b]$, not necessarily in linear order, the kth divided difference of g is defined as

$$Q_k(g; x_0, x_1, \dots, x_k) = \sum_{i=0}^{k} \frac{g(x_i)}{\Pi_{j=0, j\neq i}^{k}(x_i - x_j)}.$$

If $Q_k(g; x_0, x_1, \dots, x_k) \geq 0$ for all choices of $k+1$ distinct points $x_0, x_1, \dots, x_k$ in $[a, b]$, then g is called k-convex on $[a, b]$.

Let x be a given point in $[a, b]$. The kth Riemann derivative of $g : [a, b] \to \mathbb{R}$ at x (Bullen [2], p. 83) is defined by

$$\widehat{D}^k g(x) = k! \lim_{|x_k - x|\to 0} \lim_{|x_{k-1} - x|\to 0} \cdots \lim_{|x_1 - x|\to 0} \lim_{|x_0 - x|\to 0} Q_k(g; x_0, x_1, \dots, x_k),$$

if the iterated limit exists for all choices of $k+1$ distinct points $x_0, x_1, \dots, x_k$ belonging to $[a, b]$ satisfying $0 \leq |x_0 - x| < |x_1 - x| < \cdots < |x_k - x|$. The right and left kth Riemann derivatives $\widehat{D}^k_+ g(x)$, $\widehat{D}^k_- g(x)$ respectively are defined in the obvious way.

Key Words: divided difference, k-convex function, RS^*_k-, $\mathbb{R}S^*_k$-, LS_k-, HS_k-, GSH-, GR_k-integrals, Henstock Integral, Schwabik's generalized Perron integral, δ-fine tagged k-partition, GH_k integral, Saks-Henstock lemma, Cauchy Extension Formula
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If in the above definition we take $h_0 = 0$, then, in view of Definition 2 of Russell [11], we obtain the corresponding definitions of the kth Riemann*derivatives $D^k g(x)$, $D_+^k g(x)$, $D_-^k g(x)$ of g at x. It is known that if the ordinary kth derivative $g^{(k)}(x)$ exists, then the kth Riemann*derivative $D^k g(x)$ exists and equals $g^{(k)}(x)$. In general, the converse is true for $k = 1$. In view of Lemma 1(d) of Bullen [2] or Lemma 4 of Russell [11], we have

$$(x_k - x_0)Q_k(g; x_0, x_1, \ldots, x_k) = Q_{k-1}(g; x_1, \ldots, x_k) - Q_{k-1}(g; x_0, \ldots, x_{k-1}).$$

This shows that for a k-convex function g on $[a, b]$ the $(k-1)$th divided difference is non-decreasing with the increase of any of the component points x_i. It is shown in Russell [11] that for a k-convex function g on $[a, b]$, $D_+^{k-1} g(a)$, $D_-^{k-1} g(b)$ may not exist. If g is k-convex on $[a, b]$ and $D_+^{k-1} g(a)$, $D_-^{k-1} g(b)$ exist, then it is shown in the proof of Lemma 3.2 of Das and Das [3] that for $a < x < y < b$

$$D_+^{k-1} g(a) \le D_-^{k-1} g(x) \le D_+^{k-1} g(x) \le D_-^{k-1} g(y) \le D_+^{k-1} g(y) \le D_-^{k-1} g(b).$$

It follows that $D_-^{k-1} g(x)$ is non-decreasing on $(a, b]$ and $D_+^{k-1} g(x)$ is non-decreasing on $[a, b)$. The $(k-1)$th Riemann* derivative $D^{k-1} g(x)$ exists at each point of continuity of either sided derivative. Since $D^{k-2} g(x)$ is continuous at each point of $[a, b]$ (see Russell [11]), we observe, in view of Verblunsky [14], that the above $(k-1)$th Riemann* derivatives can be replaced by the corresponding $(k-1)$th ordinary derivatives. It therefore follows that if g is k-convex on $[a, b]$, then $g^{(k-1)}(x)$ exists on $[a, b]$ except possibly for a countable set of points in $[a, b]$.

Russell [12] obtains a definition of Stieltjes type integral which he calls the RS_k^* integral.

Definition 1.1 ([12], Definition 4, amended on p. 441)**.** Let $f, g : [a, b] \to \mathbb{R}$. The function f is said to be RS_k^* integrable with respect to g on $[a, b]$ if there exists a real number I such that to each $\epsilon > 0$ there corresponds a real constant $\delta > 0$ such that for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ with $\max\{(x_i - x_{i-1}), 1 \le i \le n\} < \delta$ the inequality

$$\Big|\sum_{i=0}^{n-k} f(\xi_i)[Q_{k-1}(g; x_{i+1}, \ldots, x_{i+k}) - Q_{k-1}(g; x_i, \ldots, x_{i+k-1})] - I\Big| < \epsilon$$

holds independent of the choice of points $\xi_i \in [x_i, x_{i+k}], i = 0, 1, \ldots, n-k$.

If the integral exists we write $(f, g) \in RS_k^*[a, b]$. The number I is called the RS_k^*integral and we write $I = (RS_k^*) \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}}$.

It is shown in Russell [12] that given g, k-convex on $[a,b]$ and $g_+^{k-1}(a)$, $g_-^{k-1}(b)$ existing, $(f,g) \in RS_k^*[a,b]$ if f is continuous on $[a,b]$. Bhattacharaya and Das [1] show by an example that the RS_k^* integral fails to exist if the interior of $[a,b]$ contains a common point of discontinuity of f and nonexistence of $(k-1)$th derivative of g. Ray and Das [10] overcome this difficulty of the RS_k^* integral by introducing the definition of the $\mathbb{R}S_k^*$ integral as follows.

Definition 1.2 ([10], Definition 2.2)**.** Let f be bounded and g be k-convex on $[a,b]$ and $g_+^{(k-1)}(a)$, $g_\pm^{(k-1)}(b)$ exist. For any partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of $[a,b]$, we write gk-mesh$(P) = \max\limits_{1 \le i \le n} \; [g_-^{(k-1)}(x_i) - g_+^{(k-1)}(x_{i-1})]$ and

$$\begin{aligned} S(P,f,g) = &\sum_{i=1}^{n} f(x_i)[g_+^{(k-1)}(x_i) - g_-^{(k-1)}(x_i)]/(k-1)! \\ &+ \sum_{i=1}^{n} f(\xi_i)[g_-^{(k-1)}(x_i) - g_+^{(k-1)}(x_{i-1})]/(k-1)! \end{aligned}$$

where $\xi_i \in (x_{i-1}, x_i)$, $i = 1, 2, \ldots, n$.

The $\mathbb{R}S_k^*$ integral, written as $(\mathbb{R}S_k^*)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}$, is the real number I if it exists uniquely, and if for each $\epsilon > 0$ there corresponds a real number $\delta(\epsilon) > 0$ such that for any partition P of $[a,b]$ with gk-mesh$(P) < \delta$, the inequality $|S(P,f,g) - I| < \epsilon$ is satisfied. If the integral exists, we write $(f,g) \in \mathbb{R}S_k^*[a,b]$.

It is shown in Ray and Das [10] that this integral exists if f is BV on $[a,b]$, and that $(RS_k^*) \subset (\mathbb{R}S_k^*) \subset (LS_k)$, where (I) stands for the class of I-integrable functions. (For the LS_k integral the readers are referred to [1].)

Das et al. [5] obtain the Henstock version of the $\mathbb{R}S_k^*$ integral as below.

Definition 1.3 ([5], Definition 2.1 and Remark 2.15)**.** Let f be defined on $[a,b]$ and g be k-convex on $[a,b]$ with $g_+^{(k-1)}(a)$, $g_\pm^{(k-1)}(b)$ existing. The HS_k integral of f with respect to g is the real number I if for every arbitrary $\epsilon > 0$ there is a positive function δ, called a gauge, on $[a,b]$ such that for every δ-fine partition $P = \{(\xi_j, [x_{j-1}, x_j]), 1 \le j \le q\}$ of $[a,b]$ with each $\xi_j \in [x_{j-1}, x_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$,

$$\begin{aligned} \Big| \Big\{ &\sum_{j=1}^{q} f(x_j)[g_+^{(k-1)}(x_j) - g_-^{(k-1)}(x_j)]/(k-1)! \\ &+ \sum_{j=1}^{q} f(\xi_j)[g_-^{(k-1)}(x_j) - g_+^{(k-1)}(x_{j-1})]/(k-1)! \Big\} - I \Big| < \epsilon. \end{aligned}$$

If the integral exists, we write $(f, g) \in HS_k[a, b]$ and $I = (HS_k) \int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}}$.

It is proved in ([5], Theorem 3.2) that the HS_k integral includes the LS_k integral of Bhattacharaya and Das [1], so that we obtain

$$(RS_k^*) \subset \mathbb{R}S_k^*) \subset (LS_k) \subset (HS_k).$$

Das and Sahu [7] apply the HS_k integral to obtain the existence theorem for the solutions of certain differential equations. An equivalent Denjoy type definition of the HS_k integral, the DS_k^* integral, is obtained by Das and Sahu [8].

Das and Sahu [6] further introduced the definition of a general integral, the GSH integral, in light of Schwabik's definition of the generalized Perron integral in Schwabik [13]. The GSH integral includes the HS_k integral in the same way as the Schwabik integral includes the Henstock integral. For ready references we produce the Schwabik integral [13] and the GSH integral [6].

Definition 1.4 ([13], Definition 1.2)**.** A function $U : [a, b] \times [a, b] \to \mathbb{R}$ is called integrable over $[a, b]$ if there is an $I \in \mathbb{R}$ such that given $\epsilon > 0$, there is a gauge $\delta : [a, b] \to (0, \infty)$ such that

$$|S(U, D) - I| = \Big| \sum_{j=1}^{n} [U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})] - I \Big| < \epsilon$$

for every δ-fine partition $D = \{(\tau_j, [\alpha_{j-1}, \alpha_j]),\ j = 1, 2, \ldots, n\}$ of $[a, b]$. The real number I is called the generalized Perron integral of U over the interval $[a, b]$ and will be denoted by $\int_a^b DU(\tau, t)$.

We see that for $f, g : [a, b] \to \mathbb{R}$, letting $U(\tau, t) = f(\tau)g(t)$ reduces the Schwabik integral to the Henstock Stieltjes integral $(HS) \int_a^b f\, dg$.

Definition 1.5 ([6], Definition 2.1)**.** Let U and V be two real valued functions defined on $[a, b] \times [a, b]$. The pair (U, V) is called integrable on $[a, b]$ if there is a real number I such that given $\epsilon > 0$ there is a gauge $\delta : [a, b] \to (0, \infty)$ such that

$$\Big| \sum_{j=1}^{n} \{[V(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})] + [U(\alpha_j, \alpha_j) - V(\alpha_j, \alpha_j)]\} - I \Big| < \epsilon$$

for every δ-fine partition $P = \{(\tau_j, [\alpha_{j-1}, \alpha_j]),\ j = 1, 2, \ldots, n\}$ of $[a, b]$. The real number I is called the generalized Schwabik-Henstock integral of the pair (U, V) on the interval $[a, b]$ and will be denoted by $(GSH) \int_a^b (U, V)$.

If $f, g : [a,b] \to \mathbb{R}$ and g is k-convex with $g_+^{k-1}(a)$, $g_\pm^{k-1}(b)$ existing, then letting $U(\tau,t) = f(\tau)g_+^{k-1}(t)$ and $V(\tau,t) = f(\tau)g_-^{k-1}(t)$ in Definition 1.5 yields the HS_k integral of Das et al. [5].

The main clue in introducing the $\mathbb{R}S_k^*$-, LS_k-, HS_k-, DS_k-, GSH- integrals of A. G. Das and his collaborators is based on the concept of gk-measure of an open interval $J = (u,v)$ defined by $|J|_{gk} = g_-^{(k-1)}(v) - g_+^{(k-1)}(u)$ and the gk-saltus at $x \in (a,b)$ defined by $g_+^{(k-1)}(x) - g_-^{(k-1)}(x)$ (see Bhattacharaya and Das [1]). The HS_k integral is a Henstock type generalization of the $\mathbb{R}S_k^*$ integral of Ray and Das [10] that uses gk-measure and gk-saltus instead of the RS_k^* integral of Russell [12] that uses divided differences of the integrator function. Obviously the HS_k integral includes the RS_k^* integral too. It is expected that a Henstock type integral be introduced that directly uses the *divided differences* of the integrator function as in the RS_k^* integral.

Pal, Ganguly and Lee [9] make an attempt in this regard. They offer a concept of δ^k-fine division of $[a,b]$ and obtain a general version of a Henstock type integral. We produce below certain definitions and results from Pal et al. [9] for ready references.

Given a positive function $\delta : [a,b] \to (0,\infty)$, there always exists (see [9], p. 854) a δ^k-fine division $D = \{([x_i, x_{i+k}], \xi_i),\ i = 0,1,\dots,n-k\}$ of $[a,b]$ given by $a = x_0 < x_1 < \cdots < x_n = b$ with associated points $\{\xi_0, \xi_1, \dots, \xi_{n-k}\}$ satisfying

$$\xi_i \in [x_i, x_{i+k}] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)),\ i = 0,1,\dots,n-k.$$

Definition 1.6 ([9], page 854)**.** Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b]^{k+1} \to \mathbb{R}$. The function f is said to be GR_k integrable with respect to g to real number I on $[a,b]$ if for every $\epsilon > 0$ there is a positive function δ on $[a,b]$ such that for any δ^k-fine division $D = \{([x_i, x_{i+k}], \xi_i),\ i = 0,1,\dots,n-k\}$ of $[a,b]$ we have

$$\Big|\sum_{i=0}^{n-k} f(\xi_i)g(x_i, x_{i+1}, \dots, x_{i+k}) - I\Big| < \epsilon.$$

The expression $\sum_{i=0}^{n-k} f(\xi_i)g(x_i, x_{i+1}, \dots, x_{i+k})$ is often denoted by $s(f,g;D)$.

For $k = 1$, $g : [a,b] \times [a,b] \to \mathbb{R}$ and $\alpha : [a,b] \to \mathbb{R}$, letting $g(x_i, x_{i+1}) = \alpha(x_{i+1}) - \alpha(x_i)$, provides the classical Henstock-Stieltjes integral $\int_a^b f\, d\alpha$.

Pal et al. [9] claim that for $g : [a,b]^{k+1} \to \mathbb{R}$ and $\alpha : [a,b] \to \mathbb{R}$, letting

$$\begin{aligned} g(x_i, x_{i+1}, \dots, x_{i+k}) &= (x_{i+k} - x_i)Q_k(\alpha; x_i, x_{i+1}, \dots, x_{i+k}) \\ &= Q_{k-1}(\alpha; x_{i+1}, \dots, x_{i+k}) - Q_{k-1}(\alpha; x_i, \dots, x_{i+k}) \end{aligned} \tag{1}$$

reduces the GR_k integral to the Henstock type generalization of the RS_k^* integral of Russell [12].

For further development of the theory of the GR_k integral Pal et al. [9] need the following concept of *jump*.
The *jump* of $g : [a, b]^{k+1} \to \mathbb{R}$ at x, denoted by $J(g; x)$, is defined by

$$J(g; x) = \lim_{x_0 \to x, x_k \to x} g(x_0, x_1, \ldots, x_k)$$

where $x \in [x_0, x_k]$ and $x_0 < x_1 < \cdots < x_k$. We note here that for $k = 1$ and $\alpha : [a, b] \to \mathbb{R}$, letting $g(x_0, x_1) = \alpha(x_1) - \alpha(x_0)$, yields $J(g; x) = \alpha(x+) - \alpha(x-)$ or $\alpha(x+) - \alpha(x)$ or $\alpha(x) - \alpha(x-)$ according as $x_0 < x < x_1$ or $x_0 = x < x_1$ or $x_0 < x = x_1$. In any case $J(g; x) = 0$ if α is continuous at x. For $k > 1$ and g as in (1), $J(g; x)$ exists and equals 0 provided α has finite kth divided differences or the $(k-1)$th Riemann derivative (see Bullen [2], p. 83) of α exists at x.

Prior to producing the Saks-Henstock lemma analog, Pal et al. [9] introduce a concept of partial division of $[a, b]$ as follows.
Let $[a_i, b_i]$, $i = 1, 2, \ldots, p$ be pairwise non-overlapping intervals in $[a, b]$ such that $\cup_{i=1}^{p} [a_i, b_i] \subset [a, b]$. The $\{D_i\}_{i=1,2,\ldots,p}$ is said to be a δ^k-fine partial division of $[a, b]$ if each D_i is a δ^k-fine division of $[a_i, b_i]$. Its corresponding partial Riemann type sum is given by $\sum_{i=1}^{p} s(f, g; D_i)$.

Theorem 1.7 ([9], Theorem 2.5). *If $(f, g) \in GR_k[a, b]$ and $J(g, c)$ exists for all $c \in (a, b)$, then for every $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that for any δ^k-fine division D of $[a, b]$ and for any δ^k-fine partial division $\{D_i\}_{i=1,2,\ldots,p}$ of $[a, b]$*

$$\|s(f, g; D) - F(a, b)| < \epsilon \text{ and } \left| \sum_{i=1}^{p} [s(f, g; D_i) - F(a_i, b_i)] \right| < (k+1)\epsilon \quad (2)$$

where D_i is a δ^k-fine division of $[a_i, b_i]$ and $F(u, v)$ denotes the GR_k integral on $[u, v] \subseteq [a, b]$.

The process of the proof of the above theorem in [9] requires refinements of gauge functions arising out of the overlap of the point-interval pairs. As such from a given δ^k-fine division D satisfying the first inequality in (2), one cannot choose a finite number of point interval pairs $\{([x_i, x_{i+k}], \xi_i)\}$ as in D. This results in the non-additive behavior of the GR_k integrals over subintervals of $[a, b]$.

To overcome this unpleasant behavior of the GR_k integral the authors of the present paper introduce the definition of a general integral that extends the idea of the generalized Perron integral of Schwabik [13] in the sense of higher dimension. It is worth mentioning that the new integral presented here accommodates Saks-Henstock lemma analog (see Theorem 3.1 and Corollary 3.2 below) without any *jump* concept which is essential for the similar result in the GR_k integral of Pal et al. [9]. Cauchy Extension analogs have also been obtained. It is shown that for a particular choice of the defining function the new integral is the GR_kintegral of Pal et al. [9]. The definition of the integral here does not involve repetition of division points and as such the approach seems to be simpler.

2 Definitions and Elementary properties

Let a, b be fixed real numbers such that $a < b$, and let k be a fixed positive integer. Let $x_{1,0} < x_{1,1} < \cdots < x_{1,k} \leq x_{2,0} < x_{2,1} < \cdots < x_{2,k} \leq \cdots \leq x_{n,0} < x_{n,1} < \cdots < x_{n,k}$ be any system of points in $[a,b]$. We say that the intervals $[x_{i,0}, x_{i,k}], i = 1, 2, \ldots, n$ form an elementary system

$$\{(x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

in $[a,b]$. If each interval $[x_{i,0}, x_{i,k}]$ along with the interior points $x_{i,1} < x_{i,2} < \cdots < x_{i,k-1}$ is tagged with $\xi_i \in [x_{i,0}, x_{i,k}]$ we call the system a *tagged elementary k-system* and denote it by

$$\{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}.$$

A tagged elementary k-system

$$\{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

is a *tagged k-partition* of $[a,b]$ if $\cup_{i=1}^{n} [x_{i,0}, x_{i,k}] = [a,b]$.

Given a positive function $\delta : [a,b] \to (0, \infty)$, a tagged elementary k-system and in particular a tagged k-partition

$$\{(\xi_i; x_{i,1}, x_{i,2}, \ldots ; x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

of $[a,b]$ is said to be *δ-fine* if $\xi_i \in [x_{i,0}, x_{i,k}] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, 2, \ldots, n$. We shall often call such a positive function δ a gauge on $[a,b]$. We note that a δ-fine tagged k-partition

$$\{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

of $[a,b]$ exists because it is simply a usual δ-fine tagged partition

$$\{\xi_i; [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

of $[a,b]$ along with a set of $(k-1)$ points $x_{i,1} < x_{i,2} < \cdots < x_{i,k-1}$ in $(x_{i,0}, x_{i,k}), i = 1,2,\ldots,n$. Clearly for $k=1$, a δ-fine tagged k-partition is a δ-fine tagged partition.

Definition 2.1. A function $U : [a,b]^{k+1} \to \mathbb{R}$ is called GH_k integrable on $[a,b]$ if there is an $I \in \mathbb{R}$ such that given $\epsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\Big|\sum_{i=1}^{n}[U(\xi_i; x_{i,1},\ldots,x_{i,k}) - U(\xi_i; x_{i,0},\ldots,x_{i,k-1})] - I\Big| < \epsilon$$

for every δ-fine tagged k-partition

$$P = \{(\xi_i; x_{i,1}, x_{i,2},\ldots,x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1,2,\ldots,n\}$$

of $[a,b]$. The real number I is called the GH_k integral of U on $[a,b]$ and we write $(GH_k)\int_a^b U = I$.

If $(GH_k)\int_a^b U$ exists, we often write $U \in GH_k[a,b]$. We use the notation $S(U,P)$ for the Riemann type sum

$$\sum_{i=1}^{n}[U(\xi_i; x_{i,1},\ldots,x_{i,k}) - U(\xi_i; x_{i,0},\ldots,x_{i,k-1})]$$

corresponding to the function U and the k-partition

$$P = \{(\xi_i; x_{i,1}, x_{i,2},\ldots,x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1,2,\ldots,n\}.$$

Remark 2.2. For $k=1$, the GH_k integral is the generalized Perron integral of Schwabik [13]. As such the GH_k integral includes the Henstock, the generalized Perron and all the Stieltjes type integrals on $[a,b]$ induced by measure.

For $k>1$, we set

$$U(\tau; t_1, t_2, \ldots, t_k) = f(\tau)\alpha(t_1, t_2, \ldots, t_k)$$

for $f : [a,b] \to \mathbb{R}$, $\alpha : [a,b]^k \to \mathbb{R}$ so as to obtain a kth Riemann Stieltjes type sum

$$\sum_{i=1}^{n} f(\xi_i)[\alpha(x_{i,1},\ldots,x_{i,k}) - \alpha(x_{i,0},\ldots,x_{i,k-1})].$$

If the integral exists, we often write $(f,\alpha) \in GH_k[a,b]$ and the integral will be denoted by $(GH_k)\int_a^b f d\alpha$. In particular, for $f : [a,b] \to \mathbb{R}$, and $h : [a,b] \to \mathbb{R}$, letting

$$U(\tau; t_1, t_2, \ldots, t_k) = f(\tau)Q_{k-1}(h; t_1, t_2, \ldots, t_k)$$

where $Q_{k-1}(h; t_1, t_2, \ldots, t_k)$ is the $(k-1)$th divided difference of h, leads to the kth Riemann-Stieltjes sum

$$s(f, h; P) = \sum_{i=1}^{n} f(\xi_i)[Q_{k-1}(h; x_{i,1}, \ldots, x_{i,k}) - Q_{k-1}(h; x_{i,0}, \ldots, x_{i,k-1})]$$

corresponding to the k-partition

$$P = \{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, 2, \ldots, n\}$$

of $[a, b]$ (cf. Russell [12], Das and Lahiri [4]). We note further that the resulting integrals do not involve the repeated terms as in [12], [9] and in some others.

Definition 2.3. A function $U : [a, b]^{k+1} \to \mathbb{R}^n$ is called GH_k integrable on $[a, b]$ if there exists an $I \in \mathbb{R}^n$ such that given $\epsilon > 0$, there exists a gauge δ on $[a, b]$ such that

$$\|S(U, P) - I\| = \left\| \sum_{i=1}^{n} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1})] - I \right\| < \epsilon$$

for any δ-fine tagged k-partition

$$P = \{(\xi_i; x_{i,1}, x_{i,2}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, 2, \ldots, n\}.$$

of $[a, b]$. The integral $I \in \mathbb{R}^n$ is called the GH_k integral of U on $[a, b]$ and we write $(GH_k) \int_a^b U = I$.

If the integral exists, we often write $U \in GH_k[a, b]$. Here the norm $\|.\|$ is any norm in $\mathbb{R}^n$, for example, the Euclidean one.

Note 2.4. Following Schwabik[13] it is not difficult to show that an $\mathbb{R}^n$-valued function $U : [a, b]^{k+1} \to \mathbb{R}^n$, $U = (U_1, U_2, \ldots, U_n)$, is GH_k integrable if and only if every component $U_m, m = 1, 2, \ldots, n$, is GH_k integrable in the sense of Definition 1.1.

Theorem 2.5. *The function $U : [a, b]^{k+1} \to \mathbb{R}^n$ is GH_k integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there is a gauge δ on $[a, b]$ such that*

$$\|S(U, P_1) - S(U, P_2)\| < \epsilon$$

for any δ-fine tagged k-partitions P_1, P_2 of $[a, b]$.

PROOF. By Note 2.4, it is sufficient to prove the theorem for a real valued function $U : [a, b]^{k+1} \to \mathbb{R}$. We prove the sufficient part. The necessary part is immediate. Denote by M the set of all s in $\mathbb{R}$ such that for every δ-fine tagged k-partition P of $[a, b]$ we have $s \leq S(U, P)$. Assume that P_0 is an arbitrary δ-fine tagged k-partition of $[a, b]$. By the hypothesis, for every δ-fine tagged k-partition P of $[a, b]$ we have

$$S(U, P_0) - \epsilon < S(U, P) < S(U, P_0) + \epsilon.$$

Therefore $(-\infty, S(U, P_0) - \epsilon) \subset M \subset (-\infty, S(U, P_0) + \epsilon)$ and so the set M is non-empty and bounded above. Consequently $\sup M$ exists and

$$S(U, P_0) - \epsilon < \sup M < S(U, P_0) + \epsilon,$$

so that for every δ-fine tagged k-partition P of $[a, b]$

$$|S(U, P) - \sup M| \leq |S(U, P) - S(U, P_0)| + |S(U, P_0) - \sup M| < 2\epsilon$$

holds. Hence $U \in GH_k[a, b]$ and $(GH_k) \int_a^b U = \sup M$. □

For any k-partition $P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}); [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$ of $[a, b]$ and for arbitrary $c_1, c_2 \in \mathbb{R}$ we evidently have

$$S(c_1U + c_2V, P) = c_1S(U, P) + c_2S(V, P)$$

for the Riemann sums of the functions $U : [a, b]^{k+1} \to \mathbb{R}^n$, $V : [a, b]^{k+1} \to \mathbb{R}^n$. We immediately obtain the following linear property.

Theorem 2.6 (Linear Property). *If $U, V \in GH_k[a, b]$ and c_1, $c_2 \in \mathbb{R}$, then $c_1U + c_2V \in GH_k[a, b]$ and*

$$(GH_k) \int_a^b (c_1U + c_2V) = c_1(GH_k) \int_a^b U + c_2(GH_k) \int_a^b V.$$

Theorem 2.7. *If $U \in GH_k[a, b]$, then for every $[c, d] \subset [a, b]$, $U \in GH_k[c, d]$.*

PROOF. Consider any two δ-fine tagged k-partitions P_1, P_2 of $[c, d]$, namely

$$P_j = \{(\xi_i^j; x_{i,1}^j, \ldots, x_{i,k-1}^j) : [x_{i,0}^j, x_{i,k}^j], i = 1, 2, \ldots, q_j\}, x_{1,0}^j = c, x_{q_j,k}^j = d$$

for $j = 1, 2$. Assume that $a < c < d < b$. Let

$$P_3 = \{(\xi_i^3; x_{i,1}^3, \ldots, x_{i,k-1}^3) : [x_{i,0}^3, x_{i,k}^3], i = 1, 2, \ldots, p\}, x_{1,0}^3 = a, x_{p,k}^3 = c$$

be a δ-fine tagged k-partition of $[a, c]$ and let

$$P_4 = \{(\xi_i^4; x_{i,1}^4, \ldots, x_{i,k-1}^4) : [x_{i,0}^4, x_{i,k}^4], i = 1, 2, \ldots, r\}, x_{1,0}^4 = d, x_{r,k}^4 = b$$

be a δ-fine tagged k-partition of $[d, b]$. Clearly the union $P_3 \cup P_1 \cup P_4$ constitute a δ-fine tagged k-partition P_1' of $[a, b]$. Similarly the union $P_3 \cup P_2 \cup P_4$ constitute a δ-fine tagged k-partition P_2' of $[a, b]$. So using the necessary part of Theorem 2.5, we have $\|S(U, P_1') - S(U, P_2')\| < \epsilon$. Then

$$\|S(U, P_1) - S(U, P_2)\| = \|S(U, P_1') - S(U, P_2')\| < \epsilon.$$

Therefore by the sufficient part of Theorem 2.5, $U \in GH_k[c, d]$. □

Corollary 2.8. *If $U \in GH_k[a, b]$ and if $a < c < b$, then $U \in GH_k[a, c]$, $U \in GH_k[c, b]$ and $(GH_k) \int_a^b U = (GH_k) \int_a^c U + (GH_k) \int_c^b U$.*

Proof. By Theorem 2.7, $U \in GH_k[a, c]$ and $U \in GH_k[c, b]$. For arbitrary $\epsilon > 0$ there exists a positive function $\delta : [a, b] \to (0, \infty)$ such that for every δ-fine tagged k-partition P_1 of $[a, c]$ and P_2 of $[c, b]$ and consequently $P = P_1 \cup P_2$ of $[a, b]$, we have

$$\begin{aligned} \|S(U, P_1) - (GH_k) \int_a^c U\| &< \epsilon/3; \\ \|S(U, P_2) - (GH_k) \int_c^b U\| &< \epsilon/3; \\ \|S(U, P) - (GH_k) \int_a^b U\| &< \epsilon/3. \end{aligned}$$

Then

$$\begin{aligned} &\|(GH_k) \int_a^c U + (GH_k) \int_c^b U - (GH_k) \int_a^b U\| \\ \le &\|(GH_k) \int_a^c U - S(U, P_1)\| + \|(GH_k) \int_c^b U - S(U, P_2)\| \\ &+ \|(GH_k) \int_a^b U - S(U, P)\| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

As $\epsilon > 0$ is arbitrary, we have $(GH_k) \int_a^b U = (GH_k) \int_a^c U + (GH_k) \int_c^b U$. □

Prior to the next result we produce a definition.

Definition 2.9. Let $U : [a,b]^{k+1} \to \mathbb{R}$ and let $c \in [a,b]$. By $J(U,c)$ we mean the iterated limit

$$\lim_{|t_k - c| \to 0} \lim_{|t_{k-1} - c| \to 0} \cdots \lim_{|t_1 - c| \to 0} U(c; t_1, t_2, \ldots, t_k),$$

if it exists for all choices of $t_1, t_2, \ldots, t_k$ in $[a,b]$ satisfying $0 \leq |t_1 - c| < |t_2 - c| < \cdots < |t_k - c|$.

Equivalently, $J(U,c)$ exists if for every $\epsilon > 0$ there is a positive number $\delta(c)$ such that $|U(c; t_1, t_2, \ldots, t_k) - J(U,c)| < \epsilon$ for all $t_1, t_2, \ldots, t_k \in [a,b]$ satisfying $0 \leq |t_1 - c| < |t_2 - c| < \cdots < |t_k - c| < \delta(c)$.

For $c \leq t_1 < t_2 < \cdots < t_k \leq b$, we define $J^+(U,c)$ and similarly $J^-(U,c)$ for $c \geq t_1 > t_2 > \cdots > t_k \geq a$.

In particular, if we set $U(\tau, t) = f(\tau)g(t)$, then $J(U,\tau) = f(\tau) \lim_{|t-\tau| \to 0} g(t)$. If $U(\tau; t_1, t_2, \ldots, t_k) = f(\tau) Q_{k-1}(g; t_1, t_2, \ldots, t_k)$, then

$$J(U,\tau) = f(\tau) \widehat{D}^{k-1} g(\tau)/(k-1)!,$$

provided the $(k-1)$th Riemann derivative, $\widehat{D}^{k-1} g(\tau)$, exists. We note that if any t_i coincides with τ for some i, we have $J(U,\tau) = f(\tau) D^{k-1} g(\tau)/(k-1)!$, where $D^{k-1} g(\tau)$ is the $(k-1)$th Riemann* derivative (see Russell [11], p. 548). We further note that $\widehat{D}^{k-1} g(\tau) = D^{k-1} g(\tau)$ whenever $\widehat{D}^{k-1} g(\tau)$ exists.

For $f : [a,b] \to \mathbb{R}$, $\alpha : [a,b]^k \to \mathbb{R}$ and $U(\tau; t_1, \ldots, t_k) = f(\tau)\alpha(t_1, \ldots, t_k)$, we shall sometimes use the notation $J(f,\alpha;c)$ for $J(U,c)$ when $a < c < b$, $J^+(f,\alpha;c)$ for $J^+(U,c)$ when $a \leq c < b$ and $J^-(f,\alpha;c)$ for $J^-(U,c)$ when $a < c \leq b$.

Theorem 2.10. *Let $c \in (a,b)$ and let $J(U,c)$ exist. If $U \in GH_k[a,c]$ and $U \in GH_k[c,b]$, then $U \in GH_k[a,b]$ and $(GH_k) \int_a^b U = (GH_k) \int_a^c U + (GH_k) \int_c^b U$.*

Proof. Let $\epsilon > 0$ be arbitrary. There exist $\delta_1 : [a,c] \to (0,\infty)$ and $\delta_2 : [c,b] \to (0,\infty)$ such that for any δ_1-fine tagged k-partition P_1 of $[a,c]$ and for any δ_2-fine tagged k-partition P_2 of $[c,b]$, we have

$$\|S(U,P_1) - (GH_k) \int_a^c U\| < \epsilon, \; \|S(U,P_2) - (GH_k) \int_c^b U\| < \epsilon.$$

Since $J(U,c)$ exists, there is $\eta > 0$ such that for $\max_{1 \leq j \leq k} |t_j - c| < \eta$,

$$\|U(c; t_1, t_2, \ldots, t_k) - J(U,c)\| < \epsilon/6. \tag{3}$$

We define a positive function δ on $[a,b]$ by

$$\delta(x)=\begin{cases}\min\{\delta_1(x),c-x\} & \text{if } a\le x<c\\ \min\{\delta_2(x),x-c\} & \text{if } c<x\le b\\ \min\{\delta_1(c),\delta_2(c),\eta\} & \text{if } x=c.\end{cases}$$

Consider any δ-fine tagged k-partition

$$P=\{(\xi_i;x_{i,1},\ldots,x_{i,k-1}):[x_{i,0},x_{i,k}],\ i=1,2,\ldots,n\}$$

of $[a,b]$. Clearly c is a tag for some interval $[x_{m,0},x_{m,k}]$, $1\le m\le n$. No other interval except $[x_{m,0},x_{m,k}]$ can include c. If $S(U,P)$ denotes the approximating sum of U corresponding to the δ-fine tagged k-partition P, we have

$$\begin{aligned}S(U,P)=&\sum_{i=1}^{m-1}[U(\xi_i;x_{i,1},\ldots,x_{i,k})-U(\xi_i;x_{i,0},\ldots,x_{i,k-1})]\\&+[U(c;x_{m,1},\ldots;x_{m,k})-U(c;x_{m,0},\ldots,x_{m,k-1})]\\&+\sum_{i=m+1}^{n}[U(\xi_i;x_{i,1},\ldots,x_{i,k})-U(\xi_i;x_{i,0},\ldots,x_{i,k-1})].\end{aligned}$$

Consider the sets of k-points

$$c-\delta(c)<x_{m-1,k}=y_{m,0}<y_{m,1}<\cdots<y_{m,k}=c,$$

$$c=z_{m,0}<z_{m,1}<\cdots<z_{m,k}=x_{m+1,0}<c+\delta(c).$$

Then the parts of the partition P for $i=1,2,\ldots,m-1$ together with the single system $\{(c;y_{m,1},\ldots,y_{m,k-1}):[y_{m,0},c]\}$ constitute a δ_1-fine tagged k-partition P_1 of $[a,c]$. Also the parts of the partition P for $i=m+1,\ldots,n$ together with $\{(c;z_{m,1},\ldots,z_{m,k-1}):[c,z_{m,k}]\}$ constitute a δ_2-fine tagged k-partition P_2 of $[c,b]$. We have then using (3)

$$\begin{aligned}\|S(U,P)-&S(U,P_1)-S(U,P_2)\|\\&=\|[U(c;x_{m,1},\ldots,x_{m,k})-U(c;x_{m,0},\ldots,x_{m,k-1})]\\&\quad-[U(c;y_{m,1},\ldots,y_{m,k}=c)-U(c;y_{m,0},\ldots,y_{m,k-1})]\\&\quad-[U(c;z_{m,1},\ldots,z_{m,k})-U(c;z_{m,0}=c,\ldots,z_{m,k-1})]\|\ <\epsilon.\end{aligned}$$

So

$$\begin{aligned}
&\|S(U,P) - (GH_k)\int_a^c U - (GH_k)\int_c^b U\| \\
&\quad \leq \|S(U,P_1) - (GH_k)\int_a^c U\| + \|S(U,P_2) - (GH_k)\int_c^b U\| \\
&\quad\quad + \|S(U,P) - S(U,P_1) - S(U,P_2)\| < 3\epsilon.
\end{aligned}$$

Since P is arbitrary δ-fine tagged k-partition of $[a,b]$, it follows that $U \in GH_k[a,b]$, and $(GH_k)\int_a^b U = (GH_k)\int_a^c U + (GH_k)\int_c^b U$. □

Remark 2.11. Apparently there seems to be a contrast between Corollary 2.8 and Theorem 2.10. In Corollary 2.8, the equality holds without the limit concept on U at c whereas it is essential in establishing the equality in Theorem 2.10. In fact, if U is given to be GH_k integrable on $[a,b]$, we are free to consider any point $c \in (a,b)$ as a partition point $x_{m,0}, m < n$, of a δ-fine tagged k-partition

$$P = \{(\xi_i; x_{i,1}, \dots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, 2, \dots, n\}$$

of $[a,b]$. For $k = 1$, Theorem 2.10 analog does not require the existence of $J(U,c)$ (see Schwabik [13]).

3 Some Fundamental Results

Theorem 3.1 (Saks-Henstock analog). *Let $U : [a,b]^{k+1} \to \mathbb{R}^n$ be GH_k integrable on $[a,b]$. Given $\epsilon > 0$ assume that the gauge δ on $[a,b]$, $\delta : [a,b] \to (0,\infty)$ is such that for every δ-fine tagged k-partition*

$$P = \{(\xi_i; x_{i,1}, \dots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \dots, n\}$$

of $[a,b]$

$$\Big\| \sum_{i=1}^{n} [U(\xi_i; x_{i,1}, \dots, x_{i,k}) - U(\xi_i; x_{i,0}, \dots, x_{i,k-1})] - (GH_k)\int_a^b U \Big\| < \epsilon.$$

If $\{(\eta_i; y_{i,1}, \dots, y_{i,k-1}) : [y_{i,0}; y_{i,k}],\ i = 1, 2, \dots, m\}$ where $a \leq y_{1,0}$, $y_{i-1,k} \leq y_{i,0} (i = 2, \dots, m)$, $y_{m,k} \leq b$, represents a δ-fine elementary k-system in $[a,b]$, then

$$\Big\| \sum_{i=1}^{m} [U(\eta_i; y_{i,1}, \dots, y_{i,k}) - U(\eta_i; y_{i,0}, \dots, y_{i,k-1}) - (GH_k)\int_{y_{i,0}}^{y_{i,k}} U] \Big\| < 2\epsilon.$$

Proof. If $y_{i,k} < y_{i+1,0}$ for some $i = 1, 2, \ldots, m$; $y_{m+1,0} = b$, then by Theorem 2.7, $U \in GH_k[y_{i,k}, y_{i+1,0}]$. Given $\epsilon > 0$ there exists a gauge δ_i on $[y_{i,k}, y_{i+1,0}]$ such that $\delta_i(x) < \delta(x)$ for all $x \in [y_{i,k}, y_{i+1,0}]$ and for every δ_i-fine tagged k-partition P^i of $[y_{i,k}, y_{i+1,0}]$ we have

$$\|S(U, P^i) - (GH_k)\int_{y_{i,k}}^{y_{i+1,0}} U\| < \frac{\epsilon}{m+1}. \tag{4}$$

If $y_{i,k} = y_{i+1,0}$, we consider $S(U, P^i) = 0$. The expression

$$\sum_{i=1}^{m}[U(\eta_i; y_{i,1}, \ldots, y_{i,k}) - U(\eta_i; y_{i,0}, \ldots, y_{i,k-1})] + \sum_i S(U, P^i)$$

represents a GH_k integral sum which corresponds to a certain δ-fine tagged k-partition of $[a, b]$ and consequently

$$\begin{aligned}\Big\| \sum_{i=1}^{m}&[U(\eta_i; y_{i,1}, \ldots, y_{i,k}) - U(\eta_i; y_{i,0}, \ldots, y_{i,k-1})] \\ &+ \sum_i S(U, P^i) - (GH_k)\int_a^b U\Big\| < \epsilon.\end{aligned} \tag{5}$$

Hence in view of Corollary 2.8 and the inequalities (4), (5)

$$\begin{aligned}\Big\| \sum_{i=1}^{m}&\big[U(\eta_i; y_{i,1}, \ldots, y_{i,k}) - U(\eta_i; y_{i,0}, \ldots, y_{i,k-1}) - (GH_k)\int_{y_{i,0}}^{y_{i,k}} U\big]\Big\| \\ \le&\Big\| \sum_{i=1}^{m}[U(\eta_i; y_{i,1}, \ldots, y_{i,k}) - U(\eta_i; y_{i,0}, \ldots, y_{i,k-1})] \\ &+ \sum_i S(U, P^i) - (GH_k)\int_a^b U\Big\| + \sum_i \Big\|S(U, P^i) - (GH_k)\int_{y_{i,k}}^{y_{i+1,0}} U\Big\| \\ <& \epsilon + \frac{m}{m+1}\epsilon < 2\epsilon.\end{aligned}$$

□

Corollary 3.2. *Let $U : [a, b]^{k+1} \to \mathbb{R}^n$ be GH_k integrable on $[a, b]$. Then to each $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that for every δ-fine tagged k-partition*

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n\}$$

of $[a, b]$

$$\sum_{i=1}^{n} \Big\|[U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1}) - (GH_k)\int_{x_{i,0}}^{x_{i,k}} U]\Big\| < \epsilon.$$

Proof. We prove the result for real valued function $U : [a,b]^{k+1} \to \mathbb{R}$. The general case for $\mathbb{R}^n$ follows from Note 2.4. Since $U \in GH_k[a,b]$, for every $\epsilon > 0$ there is a positive function δ on $[a,b]$ such that for every δ-fine tagged k-partition

$$P = \{(\xi_i; x_{i,1}, \dots, x_{i,k-1}) : [x_{i.0}, x_{i,k}],\ i = 1, 2, \dots, n\}$$

of $[a,b]$ we have, in view of Corollary 2.8

$$\Big|\sum_{i=1}^{n}[U(\xi_i; x_{i,1}, \dots, x_{i,k}) - U(\xi_i; x_{i,0}, \dots, x_{i,k-1}) - (GH_k)\int_{x_{i,0}}^{x_{i,k}} U]\Big| < \frac{\epsilon}{4}.$$

Let Σ^+ denote that part of the above sum for which

$$[U(\xi_i; x_{i,1}, \dots, x_{i,k}) - U(\xi_i; x_{i,0}, \dots, x_{i,k-1}) - (GH_k)\int_{x_{i,0}}^{x_{i,k}} U] \geq 0,$$

and let Σ^- denote that part for which the above left expression is less than 0. Then utilizing Theorem 3.1, we obtain

$$\begin{aligned}&\textstyle\sum_{i=1}^{n}\Big|[U(\xi_i; x_{i,1}, \dots, x_{i,k}) - U(\xi_i; x_{i,0}, \dots, x_{i,k-1}) - (GH_k)\displaystyle\int_{x_{i,0}}^{x_{i,k}} U]\Big| \\ &\quad = \Sigma^+ - \Sigma^- = |\Sigma^+| + |\Sigma^-| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□

Theorem 3.3 (Cauchy Extension analog). *Let $U : [a,b]^{k+1} \to \mathbb{R}^n$, $k > 1$, be such that $U \in GH_k[a,c]$ for every $c \in [a,b)$ and let $\lim_{c\to b}(GH_k)\int_a^c U = I$ exist finitely. If $J^-(U,b)$ exists finitely, then $U \in GH_k[a,b]$ and $(GH_k)\int_a^b U = I$.*

Proof. Let $\epsilon > 0$ be arbitrary. There is a number $\eta_1 > 0$ such that for every $c \in (b - \eta_1, b)$

$$\Big\|(GH_k)\int_a^c U - I\Big\| < \epsilon. \tag{6}$$

Let $\{c_p\}_{p=0}^{\infty}$ be an increasing sequence in $[a,b), c_0 = a$ with $c_p \to b$ so that $U \in GH_k[a, c_p]$ for every $p = 1, 2, \dots$. So for every $p = 1, 2, \dots$, there exists a gauge $\delta_p : [a, c_p] \to (0, \infty)$ such that for any δ_p-fine tagged k-partition P_p of $[a, c_p]$ we have

$$\Big\|S(U, P_p) - (GH_k)\int_a^{c_p} U\Big\| < \epsilon/2^{p+1}, p = 1, 2, \dots. \tag{7}$$

For any $\xi \in [a,b)$ there is exactly one $p(\xi) = 1, 2, \ldots$ for which $\xi \in [c_{p(\xi)-1}, c_{p(\xi)}]$. Given $\xi \in [a,b)$ choose $\hat{\delta}(\xi) > 0$ such that $\hat{\delta}(\xi) \le \delta_{p(\xi)}(\xi)$ and

$$(\xi - \hat{\delta}(\xi), \xi + \hat{\delta}(\xi)) \cap [a,b) \subset [a, c_{p(\xi)}).$$

Assume that c is given in $[a,b)$ and that

$$\hat{P} = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}); [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n-1\}$$

is a $\hat{\delta}$-fine tagged k-partition of $[a,c]$. If $p(\xi_i) = p$, then

$$[x_{i,0}, x_{i,k}] \subset (\xi_i - \hat{\delta}(\xi_i), \xi_i + \hat{\delta}(\xi_i)) \subset [a, c_p].$$

Also we have $[x_{i,0}, x_{i,k}] \subset (\xi_i - \delta_p(\xi_i), \xi_i + \delta_p(\xi_i))$. Let

$$\sum_{\substack{i=1 \\ p(\xi_i)=p}}^{n-1} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1}) - (GH_k)\int_{x_{i,0}}^{x_{i,k}} U]$$

be the sum of those terms in

$$\sum_{i=1}^{n-1} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1}) - (GH_k)\int_{x_{i,0}}^{x_{i,k}} U]$$

for which the tag ξ_i satisfies the relation $\xi_i \in [c_{p-1}, c_p]$. Since (7) holds we obtain, by Saks-Henstock analog (Theorem 3.1),

$$\Big\| \sum_{\substack{i=1 \\ p(\xi_i)=p}}^{n-1} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1}) - (GH_k)\int_{x_{i,0}}^{x_{i,k}} U]\Big\| < \frac{\epsilon}{2^p}. \tag{8}$$

Since $U \in GH_k[a,c]$ for every $c \in [a,b)$, we have by Corollary 2.8

$$(GH_k)\int_a^c U = \sum_{i=1}^{n-1} (GH_k)\int_{x_{i,0}}^{x_{i,k}} U.$$

Therefore, using (8)

$$
\begin{aligned}
&\Big\| \sum_{i=1}^{n-1} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1})] - (GH_k) \int_a^c U \Big\| \\
=&\| \sum_{i=1}^{n-1} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1}) - (GH_k) \int_{x_{i,0}}^{x_{i,k}} U] \| \qquad (9) \\
\leq& \sum_{p=1}^{\infty} \| \sum_{\substack{i=1 \\ p(\xi_i)=p}}^{n-1} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1}) - (GH_k) \int_{x_{i,0}}^{x_{i,k}} U] \| \\
<& \sum_{p=1}^{\infty} \frac{\epsilon}{2^p} = \epsilon.
\end{aligned}
$$

If $J^-(U, b)$ exists, there is $\eta_2 > 0$ such that for every $b - \eta_2 < t_1 < t_2 < \cdots < t_k < b$ we have

$$\|U(b; t_2, \ldots, t_k, b) - U(b; t_1, t_2, \ldots, t_k)\| < \epsilon. \qquad (10)$$

Let $\eta = \min(\eta_1, \eta_2)$. Define a gauge δ on $[a, b]$ such that

$$\delta(\xi) = \min(\hat{\delta}(\xi), b - \xi) \text{ if } \xi \in [a, b), \ \delta(b) < \eta.$$

Let $P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, 2, \ldots, n\}$ be an arbitrary δ-fine tagged k-partition of $[a, b]$. Clearly $\xi_n = x_{n,k} = b$ and $x_{n-1,k} = x_{n,0} \in (b - \eta, b) \subset (b - \eta_1, b)$. Utilizing (6) and (10)

$$
\begin{aligned}
&\|S(U, P) - I\| \\
=&\Big\| \sum_{i=1}^{n-1} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1})] \\
&+ [U(b; x_{n,1}, \ldots, b) - U(b; x_{n,0}, \ldots, x_{n,k-1})] - I \Big\| \\
\leq&\Big\| \sum_{i=1}^{n-1} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1})] - (GH_k) \int_a^{x_{n-1,k}} U \Big\| \\
&+ \|(GH_k) \int_a^{x_{n-1,k}} U - I\| + \|U(b; x_{n,1}, \ldots, b) - U(b; x_{n,0}, \ldots, x_{n,k-1})\| \\
<&\Big\| \sum_{i=1}^{n-1} [U(\xi_i; x_{i,1}, \ldots, x_{i,k}) - U(\xi_i; x_{i,0}, \ldots, x_{i,k-1})] - (GH_k) \int_a^{x_{n-1,k}} U \Big\| + 2\epsilon.
\end{aligned}
$$

Since $x_{n-1,k} < b$ and $\hat{P} = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}],\ i = 1, 2, \ldots, n-1\}$ is a $\hat{\delta}$-fine tagged k-partition of $[a, x_{n-1,k}]$, using (9) the first term on the right hand side of the last inequality is less than ϵ. Hence we obtain

$$\|S(U, P) - I\| < 3\epsilon.$$

This yields $U \in GH_k[a, b]$ and that $(GH_k) \int_a^b U = I$. □

Remark 3.4. For $k = 1$ see Schwabik ([13], Theorem 1.14). In fact, there the existence of $J^-(U, b)$ doesn't necessarily imply $\lim\limits_{c\to b-} [U(b, b) - U(b, c)] = 0$. For example, let $U(\tau, t) = f(\tau)g(t)$ so that $\lim\limits_{c\to b-} [U(b, b) - U(b, c)] = f(b)[g(b) - g(b-)]$, when $g(b-)$ exists. However, if $U(\tau, t) = f(\tau).t$, then $\lim\limits_{c\to b-} [U(b, b) - U(b, c)] = 0$.

The left analog of Theorem 3.3 can similarly be obtained.

Theorem 3.5. *Let $U : [a, b]^{k+1} \to \mathbb{R}^n$, $k > 1$, be such that $U \in GH_k[\alpha, b]$ for every $\alpha \in (a, b]$ and let $\lim\limits_{\alpha\to a} (GH_k) \int_\alpha^b U = I$ exist finitely. If $J^+(U, a)$ exists finitely, then $U \in GH_k[a, b]$ and $(GH_k) \int_a^b U = I$.*

Theorem 3.6. *Let $f : [a, b] \to \mathbb{R}$, $\alpha : [a, b]^k \to \mathbb{R}$, $g : [a, b]^{k+1} \to \mathbb{R}$ and let*

$$g(t_0, t_1, \ldots, t_k) = \alpha(t_1, \ldots, t_k) - \alpha(t_0, \ldots, t_{k-1})$$

for $t_0, t_1, \ldots, t_k \in [a, b]$. If $(f, \alpha) \in GH_k[a, b]$ and $J^+(f, \alpha; a)$, $J^-(f, \alpha; b)$ exist, then $(f, g) \in GR_k[a, b]$ and $(GR_k) \int_a^b f\, dg = k(GH_k) \int_a^b f\, d\alpha$.

PROOF. Let $\epsilon > 0$ be arbitrary. There exists a positive function $\delta : [a, b] \to (0, \infty)$ such that for every δ-fine tagged k-partition

$$P = \{(\xi_i; x_{i,1}, \ldots, x_{i,k-1}) : [x_{i,0}, x_{i,k}], i = 1, 2, \ldots, n\}$$

of $[a, b]$ and for every δ-fine tagged k-system $\{(\zeta_i; y_{i,1}, \ldots, y_{i,k-1}) : [y_{i,0}, y_{i,k}]\}$ in $[a, b]$ we have from Definition 2.1 and Theorem 3.1

$$\Big| \sum_{i=0}^{n} f(\xi_i)[\alpha(x_{i,1}, \ldots, x_{i,k}) - \alpha(x_{i,0}, \ldots, x_{i,k-1})] - (GH_k)\int_a^b f\, d\alpha \Big| < \epsilon, \quad (11)$$

$$\Big| \sum_{i} \{f(\zeta_i)[\alpha(y_{i,1}, \ldots, y_{i,k}) - \alpha(y_{i,0}, \ldots, y_{i,k-1})] - (GH_k)\int_{y_{i,0}}^{y_{i,k}} f\, d\alpha\} \Big| < 2\epsilon \quad (12)$$

where we take $U(\tau; t_1, t_2, \ldots, t_k) = f(\tau)\alpha(t_1, t_2, \ldots, t_k)$, $\tau, t_1, t_2, \ldots, t_k \in [a, b]$.

Take an arbitrary δ^k-fine division $D = \{([x_i, x_{i+k}], \xi_i),\ i = 0, 1, \ldots, n-k\}$ of $[a, b]$. The points $x_1, \ldots, x_{k-1}$ and $x_{n-k+1}, \ldots, x_{n-1}$ are accommodated satisfying the *jump* effects at a, b respectively. For each $j = 0, 1, \ldots, k-1$ we may consider

$$\{(\xi_{i+j}; x_{i+j+1}, \ldots, x_{i+j+k-1}) : [x_{i+j}, x_{i+j+k})],$$

$$i \in A_{pk} = \{pk, p = 0, 1, \ldots, x_{i+j+k} \leq b\}\}$$

as a δ-fine tagged k-system in $[a, b]$. So utilizing Cauchy Extension analogs (Theorems 3.3, 3.5) and using inequalities (11), (12) above, we obtain

$$\Big|\sum_{i \in A_{pk}} f(\xi_{i+j})[\alpha(x_{i+j+1}, \ldots, x_{i+j+k}) - \alpha(x_{i+j}, \ldots, x_{i+j+k-1})] - (GH_k)\int_a^b f\, d\alpha\Big| < 2\epsilon$$

for each $j = 0, 1, \ldots, k-1$. Using the definition of g and the notation of Pal et al. [9], we observe that

$$s(f, g; D) = \sum_{j=0}^{k-1} \sum_{i \in A_{pk}} f(\xi_{i+j})[\alpha(x_{i+j+1}, \ldots, x_{i+j+k}) - \alpha(x_{i+j}, \ldots, x_{i+j+k-1})].$$

We therefore obtain $|s(f, g; D) - k(GH_k) \int_a^b f d\alpha| < 2k\epsilon$. It follows that $(f, g) \in GR_k[a, b]$, and $(GR_k) \int_a^b f dg = k(GH_k) \int_a^b f d\alpha$. $\square$

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