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ON THE LEVEL STRUCTURE OF BOUNDED DERIVATIVES

Abstract

We prove: In the space $\mathcal{C}$ of continuous functions on $[0,1]$ under the *sup* metric, the functions all of whose level sets (in every direction) have measure zero, form a residual subset of $\mathcal{C}$. In the space $\mathcal{D}$ of bounded derivatives of $[0,1]$, the derivatives all of whose level sets are nowhere dense sets of measure zero form a residual subset of $\mathcal{D}$. Moreover, there exists a derivative in $\mathcal{D}$ all of whose level sets have measure zero and one of whose level sets is dense in $[0,1]$.

Let $\mathcal{C}$ denote the family of continuous real valued functions on $[0,1]$, and let $\mathcal{D}$ be the family of bounded derivatives on $[0,1]$:

$$\mathcal{D} = \big\{ f : f \text{ is bounded and there exists an } F \in \mathcal{C} \text{ such that } F'(x) = f(x) \text{ for } 0 \le x \le 1 \big\}.$$

Then $\mathcal{D}$ and $\mathcal{C}$ are complete metric spaces under the *sup* metric (see [W1])

$$d(f,g) = \sup_{0 \le x \le 1} \big| f(x) - g(x) \big|.$$

We say that a set $E \subset [0,1]$ is a "level" set if $\big\{(x, f(x)) : x \in E\big\}$ is the intersection of the graph of f with a nonvertical line in the plane. The "slope" of the level set is the slope of this line. (We do not define the slopes of singleton level sets, but it won't matter.) In this paper we study the (linear) measures of the level sets of functions in certain residual subsets of $\mathcal{D}$ and of $\mathcal{C}$. In [BG], A. M. Bruckner and K. M. Garg proved the existence of a residual subset of $\mathcal{C}$ such that the level sets of each member of this residual subset consists of a perfect set or the union of a perfect set with a singleton or doubleton set. We conclude that each and every one of these perfect sets can have measure

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zero. In other words, the functions in $\mathcal{C}$, all of whose level sets have measure zero, form a residual subset of $\mathcal{C}$. Moreover we prove that the functions in $\mathcal{D}$, all of whose level sets are nowhere dense sets of measure zero, form a residual subset of $\mathcal{D}$. In [W2], Clifford Weil used a category argument on a complete subset of $\mathcal{D}$ to prove the existence of derivatives in $\mathcal{D}$ that take both positive and negative values on every subinterval of $[0,1]$. We will prove the existence of such a derivative all of whose level sets have measure zero.

In this paper, $m(X)$ will denote the measure of X. The level sets of functions in $\mathcal{C}$ or $\mathcal{D}$ are necessarily measurable. We begin with some needed lemmas.

Lemma 1. *Let (a,b) be an open subinterval of $[0,1]$, and put*

$$P=\left\{f\in\mathcal{D}: f \text{ has a level set } E \text{ such that } m\big(E\cap(a,b)\big)\geq\frac{3}{4}(b-a)\right\}.$$

Then P is a first category subset of $\mathcal{D}$.

Proof. For definiteness we assign to each $f\in P$ a level set $E(f)$ of f such that $m\big(E(f)\cap(a,b)\big)\geq\frac{3}{4}(b-a)$, and let $c(f)$ be the slope of $E(f)$. For each positive integer N, put

$$P_N=\left\{f\in P: \big|c(f)\big|\leq N\right\}.$$

It suffices to prove that P_N is a nowhere dense subset of D. Select $\epsilon>0$ and $f\in P_N$. We prove that there is a function $h\in\mathcal{D}\setminus P_N$ with $|f-h|\leq\epsilon$. To do this put $h(x)=f(x)+\epsilon x^2$. Assume to the contrary, that $h\in P_N$. Necessarily the intersection $E(f)\cap E(h)\cap(a,b)$ has a positive measure. Fix $u\in E(f)\cap E(h)\cap(a,b)$. For any x in this intersection, we have

$$\begin{aligned}h(u)-h(x)-(u-x)c(h)&=0,\\ f(u)-f(x)-(u-x)c(f)&=0,\end{aligned}$$

and by taking differences,

$$\epsilon u^2-\epsilon x^2-(u-x)\big(c(h)-c(f)\big)=0\,.$$

But u is fixed, so this polynomial equation can have at most two roots in x. This contradiction proves that P_N has a dense complement.

It remains to prove that P_N is closed in $\mathcal{D}$. Let g lie in the closure of P_N. Let (f_n) be a sequence of functions in P_N converging uniformly to g. Now

$$\bigcap_{n=1}^{\infty}\bigcup_{j\geq n}E(f_j)\cap(a,b)$$

has positive measure; fix a point v in this intersection. Then v lies in $E(f_n) \cap (a, b)$ for infinitely many n. By passing to a subsequence if necessary, we assume without loss of generality, that $v \in E(f_n) \cap (a, b)$ for all n and the sequence $c(f_n)$ converges, say to t. Observe that $|t| \leq N$.

For each positive index j we define the set

$$A_j = \Big\{x \in (a, b) : \big|g(x) - g(v) - (x - v)t\big| < 2^{-j}\Big\} \tag{1}$$

and put $A = \cap_j A_j$. For fixed j choose n so large that

$$|f_n - g| < \frac{2^{-j}}{3}, \ \big|c(f_n) - t\big| < \frac{2^{-j}}{3}.$$

Then

$$\begin{gathered} \big|f_n(x) - g(x)\big| < \frac{2^{-j}}{3}, \ \big|f_n(v) - g(v)\big| < \frac{2^{-j}}{3}, \\ \text{and } \Big|(x - v)c(f_n) - (x - v)t\Big| < \frac{2^{-j}}{3}, \end{gathered} \tag{2}$$

and for $x \in E(f_n) \cap (a, b)$,

$$f_n(x) - f_n(v) - (x - v)c(f_n) = 0\,. \tag{3}$$

Now from (2) and (3) we deduce that

$$\Big|g(x) - g(v) - (x - v)t\Big| < 2^{-j}$$

and $x \in A_j$. Thus $A_j \supset E(f_n) \cap (a, b)$ for this n, and

$$m(A_j) \geq m\big(E(f_n) \cap (a, b)\big) \geq \frac{3}{4}(b - a).$$

Clearly $A_1 \supset A_2 \supset A_3 \supset \ldots \supset A_j \supset \ldots$ and hence $m(A) \geq \frac{3}{4}(b-a)$. Moreover $g(x) - g(v) - (x - v)t = 0$ for $x \in A$ by (1). We deduce that A is a subset of a level set of g (the line here is $y = g(v) + (x - v)t$) and therefore $g \in P_N$. This proves that P_N is closed. □

Now let $f \in \mathcal{D}$ and let f vanish on a dense subset of (a, b). Because f is a Baire 1 function, $f^{-1}(0)$ is a G_δ-set and f vanishes on a residual subset of $[a, b]$.

Lemma 2. *Let (a, b) be an open subinterval of $[0, 1]$, and put*

$$P = \Big\{f \in \mathcal{D} : f \text{ has a level set } E \text{ such that } E \cap (a, b) \text{ is dense in } (a, b)\Big\}.$$

Then P is a first category subset of $\mathcal{D}$.

PROOF. We deduce from our preceding comment that if $f \in P$ then $E \cap (a,b)$ is a residual subset of $[a,b]$. The proof proceeds just like the proof of Lemma 1 where $m\big(E \cap (a,b)\big) \geq \frac{3}{4}(b-a)$ is replaced by $E \cap (a,b)$ is a dense subset of (a,b), only easier. So we leave it. $\square$

Theorem 1. *The functions in $\mathcal{D}$, all of whose level sets are nowhere dense sets of measure zero, form a residual subset of $\mathcal{D}$.*

PROOF. For any rational numbers a and b, $0 \leq a < b \leq 1$, define

$$P(a,b) = \Big\{ f \in \mathcal{D} : f \text{ has a level set } E \text{ such that either } E \cap (a,b) \text{ is a dense subset of } (a,b), \text{ or } m\big(E \cap (a,b)\big) \geq \frac{3}{4}(b-a) \Big\}.$$

Then $\cup_{a<b} P(a,b)$ is a first category subset of $\mathcal{D}$, by Lemmas 1 and 2. If $g \in \mathcal{D}$ has a level set E_g such that either $m(E_g) > 0$ or E_g is not nowhere dense, then g lies in some $P(a,b)$. The conclusion follows. $\square$

Theorem 2. *The functions in $\mathcal{C}$, all of whose level sets are nowhere dense sets of measure zero, form a residual subset of $\mathcal{C}$.*

PROOF. The argument is just like the proof of Theorem 1, only easier. Lemma 2 is not needed this time. So we leave it. $\square$

Let $\mathcal{B}$ denote the family of all bounded Baire class 1 functions, that is, pointwise limits of sequences of continuous functions on $[0,1]$. Then $\mathcal{B}$ is a vector space under the usual addition and scalar multiplication of functions, and $\mathcal{B}$ is complete under the *sup* metric (see [Gf, Theorem 1, p. 138]). It is easy to see that the proof of Theorem 1 goes through word for word when $\mathcal{B}$ replaces $\mathcal{D}$. Moreover the proof goes through when $\mathcal{D}$ is replaced by any closed vector subspace of $\mathcal{B}$ that contains a nonlinear polynomial.

Let $\mathcal{W}$ denote the family of all members of $\mathcal{D}$ that vanish on a dense subset of $[0,1]$. In [W2] it was observed that $\mathcal{W}$ is a complete vector space under the *sup* metric. It was further observed that the functions in $\mathcal{W}$ that take both positive and negative values on each subinterval of $[0,1]$ form a residual subset of $\mathcal{W}$. Here we show that the functions in $\mathcal{W}$ all of whose level sets have measure zero, form a residual subset of $\mathcal{W}$. An additional problem here is that there are no nonzero polynomials in $\mathcal{W}$.

Lemma 3. *Let (a,b) be an open subinterval of $[0,1]$ and let*

$$P = \Big\{ f \in \mathcal{W}; f \textit{ has a level set } E \textit{ such that } m\big(E \cap (a,b)\big) \geq \frac{3}{4}(b-a) \Big\}.$$

Then P is a first category subset of $\mathcal{W}$.

Proof. Choose $f \in P$ and let $\epsilon > 0$. We will construct a $g \in \mathcal{W} \setminus P$ such that $|f - g| < \epsilon$. The function $m\big(E \cap (a, x)\big)$ is a continuous nondecreasing function in x whose range is an interval containing points 0 and $\frac{3}{4}(b - a)$. Select $p \in (a, b)$ such that $m\big(E \cap (a, p)\big) = \frac{b-a}{2}$.

We deduce from [KS, Theorem, p. 351] that there is a derivative T on $[0, 1]$ such that $1 - T$ is positive on a dense subset of $[0, 1]$, such that $T = 1$ on a dense subset of $[0, 1]$ containing 0 and 1, and $0 < T \le 1$. Put $h_1 = 1 - T$. Then $0 \le h_1 \le 1$, $h_1 > 0$ on a dense subset of $[0, 1]$, and h_1 vanishes on a dense subset of $[0, 1]$ containing 0 and 1. Now h_1 is a bounded derivative and is hence the derivative of an absolutely continuous function. It follows that the set $\big\{x : h_1(x) > 0\big\}$ has positive measure; say its measure is $\delta > 0$. We extend h_1 to the real line by making h_1 periodic with period 1. Put

$$h_2(x) = \sum_{j=1}^{\infty} 2^{-j} h_1(2^j x) \ (0 \le x \le 1).$$

For any interval I of the form $\big((i - 1)2^{-j}\,;\, i2^{-j}\big)$ ($i,\ j$ positive integers) the function $h_1(2^j x)$ is positive on a subset of I with measure $\delta m(I)$. Likewise h_2 is positive on a subset of I of measure at least $\delta m(I)$. It follows that the set $\big\{x : h_2(x) = 0\big\}$ can have no point of density, and hence $h_2 > 0$ almost everywhere on $[0, 1]$. Clearly h_1 and h_2 both vanish on residual subsets of $[0, 1]$ so h_1 and h_2 lie in $\mathcal{W}$.

As in [W2, p. 389], by pushing and crushing it is not hard to use h_2 to prove the existence of a function h_3 such that $0 \le h_3 < \epsilon$, h_3 vanishes on a dense subset of $[p, b]$ containing p and b, and $h_3 > 0$ almost everywhere on $[p, b]$. Make $h_3 = 0$ for $x < p$ and $x > b$. Then h_3 lies in $\mathcal{W}$.

Let L be the line such that $\big\{(x, f(x)) : x \in E\big\} \subset L$. Then there is a real number r such that $0 < r < 1$ and the set

$$\Big\{x \in (p, b) : \big(x, f(x) + rh_3(x)\big) \in L\Big\}$$

has measure zero. Put $g(x) = f(x) + rh_3(x)$ for $0 \le x \le 1$. Let L_0 be a line in the plane different from L. It follows from the choice of p and the definition of h_3 that the measure of the set

$$\Big\{x \in (a, b) : \big(x, g(x)\big) \in L\Big\}$$

is $\frac{b-a}{2}$ and (because $L \cap L_0$ contains at most one point) the measure of the set

$$\Big\{x \in (a, b) : \big(x, g(x)\big) \in L_0\Big\}$$

can not exceed $\frac{b-a}{2}$. Therefore $g \notin P$. But $|f-g| < r\epsilon < \epsilon$. It follows that P has a dense complement in $\mathcal{W}$.

The proof that P is the union of countably many closed sets is just like the corresponding argument in the proof of Lemma 1, so we leave it. □

Theorem 3. *The functions in $\mathcal{W}$, all of whose level sets have measure zero, form a residual subset of $\mathcal{W}$.*

Proof. The argument is just like the proofs of Theorem 1 and Theorem 2 where Lemma 1 is replaced by Lemma 3. So we leave it. □

Now let f be a function in $\mathcal{W}$ whose graph meets the x-axis in a set of measure zero. Then f is discontinuous at each point where it is nonzero, so f is discontinuous almost everywhere on $[0,1]$. The family of derivatives in $\mathcal{W}$ that vanish on a set of positive measure in every subinterval of $[0,1]$ and are discontinuous almost everywhere on $[0,1]$, form a first category subset of $\mathcal{W}$. The question arises: are there any such derivatives? The answer is *yes*. We gave a constructive definition of one such derivative in [C].

For a detailed discussion of the topology of level sets of functions in $\mathcal{C}$, consult [BG].

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