Real Analysis Exchange
Vol. 29(2), 2003/2004, pp. 821–835

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PSEUDO-CHARACTERISTIC FUNCTIONS FOR CONVEX POLYHEDRA

Abstract

An algorithm is given for constructing polynomials that determine approximately whether a point p is inside or outside a given polyhedron C_n in Euclidean n-dimensional space. The polynomials are of degree $2r$, where r is a positive integer and the order of the approximation can be made arbitrarily small by taking r sufficiently large. For $n = 2$, the square, triangle, trapezoid, and pentagon are used as examples. For $n = 3$ and $n = 4$, the tetrahedron and equilateral simplex are used as examples. We conjecture that the center of mass of the region determined by the approximating polynomial is the same for all values of r, and hence coincides with the center of the polyhedra.

1 Introduction

It has been noted that the function

$$f_r(x, y) = x^{2r} + y^{2r}$$

describes, in the limit $r \to \infty$, a square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$: all points inside the square have $f_\infty(x, y) = 0$, points on the boundary have $f_\infty(x, y) = 1$ or $f_\infty(x, y) = 2$, and points outside the square have $f_\infty(x, y) \to \infty$. We term such a function a *pseudo-characteristic function*, and generalize this idea to arbitrary convex polyhedra in $\mathbb{R}^n$. Whereas we focus

Key Words: Convex Polyhedra, Euclidean N-space, simplex, characteristic functions, psuedo-characteristic functions
Mathematical Reviews subject classification: 52A20, 52B11, 51M15
Received by the editors October 5, 2003
Communicated by: R. Daniel Mauldin

on polynomials, similar results involving exponential functions can be found in [4].

The problem may be stated as follows. Given a convex polyhedron $P \subset \mathbb{R}^n$ defined by a finite set of J linear inequalities with real $a_{i,j}$ and c_i (not all zero):

$$\sum_{j=1}^{n} a_{i,j} x_j + c_i \leq 0, \; i = 1, 2, \ldots, J$$

and a point p in Euclidean n-space E_n, find a polynomial $f(x_1, x_2, \ldots, x_n)$ whose value $f(p)$ determines if p is inside or outside P. (For a discussion of convex polyhedra, see [5] or [10].) The polynomial we construct is really a set of polynomials $F_r(x_1, x_2, \ldots, x_n)$ whose limiting values, 0 or ∞, as $r \to \infty$ determine if the point p is inside or outside of the polyhedron P.

2 Algorithm for Two Dimensions

For clarity, we first discuss the case $n = 2$. Let P be a convex polygon in two dimensions with k edges e_i and k vertices $\mathbf{v}_i$, where $i = 1, 2, \ldots, k$, written in order about the polygon. Let $l_i : a_i x + b_i y + c_i = 0$ be the line containing the pair of vertices $(\mathbf{v}_i, \mathbf{v}_{i+1})$ for $i = 1, 2, \ldots, k-1$ and l_k be the line containing the pair $(\mathbf{v}_k, \mathbf{v}_1)$. Following the usual terminology, we say that l_i supports P because all of P is on one side of l_i and l_i contains at least one point of P.

Consider the pairs of vertices $(\mathbf{v}_i, \mathbf{v}_{i+1})$ for $i < k$ and $(\mathbf{v}_k, \mathbf{v}_1)$. For $i = 1, 2, \ldots, k$ let $\mathbf{v}_i^* = (x_i^*, y_i^*)$ be one of the possibly several vertices P farthest from l_i. Let $l_i^* : a_i x + b_i y + c_i^* = 0$ be the line through v_i^* parallel to l_i. Then l_i^* supports P.

Put

$$f_i(x, y) = \gamma(a_i x + b_i y + c_i) + 1. \tag{1}$$

Thus, $f_i(x, y)$ has the value 1 at any point on l_i, independently of γ. Now choose γ so that at $v_i^* = (x_i^*, y_i^*)$ and therefore on l_i^*, $f_i(x, y)$ has the value -1:

$$f_i(x_i^*, y_i^*) = \gamma(a_i x_i^* + b_i y_i^* + c_i) + 1 = -1, \; i = 1, 2, \ldots, k.$$

That is, $\gamma = \dfrac{-2}{a_i x_i^* + b_i y_i^* + c_i}$, so that

$$f_i(x, y) = 1 - \frac{2(a_i x + b_i y + c_i)}{a_i x_i^* + b_i y_i^* + c_i}, \; i = 1, 2, \ldots, k.$$

Note that the denominator cannot be zero.

In the strip between l_i and l_i^*, $-1 < f_i(x, y) < 1$ because f_i is linear in x and y and f_i has the value +1 at l_i and -1 at l_i^*. Furthermore, $f_i(x, y) > 1$ for (x, y) on the other side of l_i from l_i^* and $f_i(x, y) < -1$ for (x, y) on the other side of l_i^* from l_i.

We define the "pseudo-characteristic function" $F_r(x, y)$ as

$$F_r(x, y) = \sum_{i=1}^{k} [f_i(x, y)]^{2r}, \tag{2}$$

where the sum is over all k vertices and r is a positive integer. The point (x, y) is inside the polygon if and only if $-1 \leq f_i(x, y) \leq 1$ for all i. Therefore the point (x, y) is in P if and only if all the terms in the sum are no more than 1. If any term in the sum is greater than 1, the point is not in P.

A more qualitative method is to take r large. If the sum is small, the point is inside P. If the sum is large, the point is outside P. These qualitative statements can be made more precise. For example, if the value of F_r is moderate, increase r until a decision can be made. Of course, one could test all k inequalities. If any fail, the point is outside P. Otherwise the point is inside P.

3 Examples of Application of the Algorithm in 2 Dimensions

Example A: **Square**. Let P be defined as the intersection of the four half-planes:

$$\begin{aligned} &H_1 : x - 1 \leq 0, \\ &H_2 : y - 1 \leq 0, \\ &H_3 : -x - 1 \leq 0, \\ &H_4 : -y - 1 \leq 0. \end{aligned}$$

These four closed half-planes describe a 2×2 square centered at the origin, with sides parallel to the coordinate axes. The four vertices are: $v_1 : (1, -1)$, $v_2 : (1, 1)$, $v_3 : (-1, 1)$, $v_4 : (-1, -1)$. For the line through v_1 and v_2, we have $l_1 : x - 1 = 0$. A farthest vertex from l_1 is $v_3 : (-1, 1)$. So $x_1^* = -1$, $y_1^* = 1$, and $c_1 = -1$. For $i = 1$, (1) becomes

$$f_1(x, y) = 1 - \frac{2(x-1)}{-1-1} = 1 + (x - 1) = x$$

Likewise $l_2 : y - 1 = 0$ and $f_2(x, y) = y$. Lines l_3 and l_4 yield $f_3 = -x$ and $f_4 = -y$. Thus we have $F_r(x, y) = x^{2r} + y^{2r}$, where we have omitted an unnecessary factor of 2.

Example B: **Triangle**. Consider the triangle defined by the inequalities:

$$H_1 : x \geq 0,$$
$$H_2 : y \geq 0,$$
$$H_3 : x + y - 1 \leq 0.$$

Application of the algorithm gives the pseudo-characteristic function

$$F_r(x, y) = [2(x + y) - 1]^{2r} + [1 - 2x]^{2r} + [1 - 2y]^{2r}.$$

Example C: **Trapezoid**. Consider the trapezoid defined by the inequalities:

$$H_1 : x + y - 2 \leq 0,$$
$$H_2 : -x - y + 1 \leq 0,$$
$$H_3 : y \geq 0,$$
$$H_4 : x \geq 0.$$

Application of the algorithm gives

$$F_r(x, y) = 2[2(x + y) - 3]^{2r} + [1 - x]^{2r} + [1 - y]^{2r}.$$

For $r = 1$ we have $F_1 = 8(x^2 + xy + y^2 - x - y) + 3$.

Example D: **Equilateral Pentagon**. It is convenient to discuss the closed pentagon in terms of its bounding lines in place of half planes. These five bounding lines are:

$$l_1 : y = 0$$
$$l_2 : -y + \left(x - \frac{1}{2}\right) \sec\left(\frac{2\pi}{5}\right) = 0,$$
$$l_3 : -y - \left(x + \frac{1}{2}\right) \sec\left(\frac{2\pi}{5}\right) = 0,$$
$$l_4 : y - \frac{1}{2}\left[\csc\left(\frac{\pi}{5}\right) + \cot\left(\frac{\pi}{5}\right)\right] + x \tan\left(\frac{\pi}{5}\right) = 0,$$
$$l_5 : y - \frac{1}{2}\left[\csc\left(\frac{\pi}{5}\right) + \cot\left(\frac{\pi}{5}\right)\right] - x \tan\left(\frac{\pi}{5}\right) = 0.$$

These five lines describe an equilateral pentagon of unit edge being the x axis from -.5 to .5, with the pentagon bisected by the y axis. The vertex v_i^* farthest from l_i is $v_i^* = \{x_i^*, y_i^*\}$, where

$$v_1^* = \left\{0,\ \frac{1}{2}\left[\csc\left(\frac{\pi}{5}\right) + \cot\left(\frac{\pi}{5}\right)\right]\right\} \simeq \{0,\ 1.53884\ldots\},$$

$$\begin{aligned} v_2^* = \Bigg\{ & -\frac{1}{2}\left[\frac{\cot\left(\frac{\pi}{5}\right) + \csc\left(\frac{\pi}{5}\right) + \sec\left(\frac{2\pi}{5}\right)}{\sec\left(\frac{2\pi}{5}\right) + \tan\left(\frac{\pi}{5}\right)}\right], \\ & \frac{1}{2}\sec\left(\frac{2\pi}{5}\right)\left[\frac{\cot\left(\frac{\pi}{5}\right) + \csc\left(\frac{\pi}{5}\right) + \sec\left(\frac{2\pi}{5}\right)}{\sec\left(\frac{2\pi}{5}\right) + \tan\left(\frac{\pi}{5}\right)} - 1\right]\Bigg\} \\ & \simeq \{-.796666\ldots,\ .96003\ldots\}, \end{aligned}$$

$$\begin{aligned} v_3^* = \Bigg\{ & \frac{1}{2}\left[\frac{\cot\left(\frac{\pi}{5}\right) + \csc\left(\frac{\pi}{5}\right) + \sec\left(\frac{2\pi}{5}\right)}{\sec\left(\frac{2\pi}{5}\right) + \tan\left(\frac{\pi}{5}\right)}\right], \\ & \frac{1}{2}\sec\left(\frac{2\pi}{5}\right)\left[\frac{\cot\left(\frac{\pi}{5}\right) + \csc\left(\frac{\pi}{5}\right) + \sec\left(\frac{2\pi}{5}\right)}{\sec\left(\frac{2\pi}{5}\right) + \tan\left(\frac{\pi}{5}\right)} - 1\right]\Bigg\} \\ & \simeq \{.796666\ldots,\ .96003\ldots\}, \end{aligned}$$

$$v_4^* = \left\{-\frac{1}{2},\ 0\right\},$$

$$v_5^* = \left\{\frac{1}{2},\ 0\right\}.$$

A sketch of lines and vertices of the pentagon is shown in Fig. 1. The lines forming the pentagon may be written as $l_i : a_i x + b_i y + c_i = 0$, where

$$\begin{aligned} \{a_i\} &= \left\{0, \sec\left(\frac{2\pi}{5}\right), -\sec\left(\frac{2\pi}{5}\right), \tan\left(\frac{\pi}{5}\right), -\tan\left(\frac{\pi}{5}\right)\right\} \\ &= \{0, 3.23606\ldots, -3.23606\ldots, .726542\ldots, -.726542\ldots\}, \\ \{b_i\} &= \{-1, -1, -1, 1, 1\}, \\ \{c_i\} &= \Bigg\{0, -\frac{1}{2}\sec\left(\frac{2\pi}{5}\right), -\frac{1}{2}\sec\left(\frac{2\pi}{5}\right), -\frac{1}{2}\left[\csc\left(\frac{\pi}{5}\right) + \cot\left(\frac{\pi}{5}\right)\right], \\ &\quad -\frac{1}{2}\left[\csc\left(\frac{\pi}{5}\right) + \cot\left(\frac{\pi}{5}\right)\right]\Bigg\} \\ &= \{0, -1.61803\ldots, -1.61803\ldots, -1.53884\ldots, -1.53884\ldots\}, \end{aligned}$$

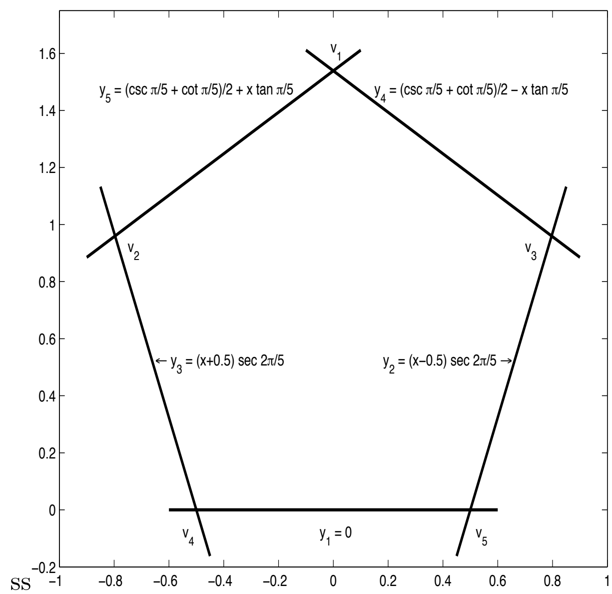

Figure 1: Bounding lines and vertices of the pentagon of section 3.

where $\{a_i\}$ denotes $a_1, a_2, \ldots, a_5$ etc. The exact values in algebraic form of the secant, cosecant, tangent, and cotangent functions are given in Spanier and Olham [8]. The following table gives their values.

function	value
$\sec(\frac{2\pi}{5})$	$\sqrt{\frac{8}{3-\sqrt{5}}}$
$\csc(\frac{\pi}{5})$	$\sqrt{\frac{8}{40-\sqrt{5}}}$
$\tan(\frac{\pi}{5})$	$\sqrt{\frac{40-\sqrt{5}}{3+\sqrt{5}}}$
$\cot(\frac{\pi}{5})$	$\sqrt{\frac{3+\sqrt{5}}{40-\sqrt{5}}}$

Now the functions

$$f_i(x,y) = 1 - \frac{2(a_i x + b_i y + c_i)}{a_i x_i^* + b_i y_i^* + c_i}, \quad i = 1,2,3,4,5$$

evaluate to

$$\begin{aligned}
f_1(x,y) =& 1 - \frac{4y}{\cot(\frac{\pi}{5}) + \csc(\frac{\pi}{5})}, \\
f_2(x,y) =& 1 - \frac{2}{\alpha}\left[-y - (\frac{1}{2} - x)\sec\left(\frac{2\pi}{5}\right)\right], \\
f_3(x,y) =& 1 - \frac{2}{\alpha}\left[-y - (\frac{1}{2} + x)\sec\left(\frac{2\pi}{5}\right)\right], \\
f_4(x,y) =& 1 - \frac{1}{\beta}\left[2y - \cot\left(\frac{\pi}{5}\right) - \csc\left(\frac{\pi}{5}\right) + 2x\tan\left(\frac{\pi}{5}\right)\right], \\
f_5(x,y) =& 1 - \frac{1}{\beta}\left[2y - \cot\left(\frac{\pi}{5}\right) - \csc\left(\frac{\pi}{5}\right) - 2x\tan\left(\frac{\pi}{5}\right)\right],
\end{aligned}$$

where

$$\alpha = -\sec\left(\frac{2\pi}{5}\right)\left[\frac{\cot\left(\frac{\pi}{5}\right) + \csc\left(\frac{\pi}{5}\right) + \sec\left(\frac{2\pi}{5}\right)}{\sec\left(\frac{2\pi}{5}\right) + \tan\left(\frac{\pi}{5}\right)}\right]$$

and

$$\beta = -\frac{1}{2}\left(\cot\left(\frac{\pi}{5}\right) + \csc\left(\frac{\pi}{5}\right) + \tan\left(\frac{\pi}{5}\right)\right).$$

Maps of the summation $F_r(x,y) = \sum_{i=1}^{5}[f_i(x,y)]^{2r} \leq 1$ are shown in Fig. 2 for various r.

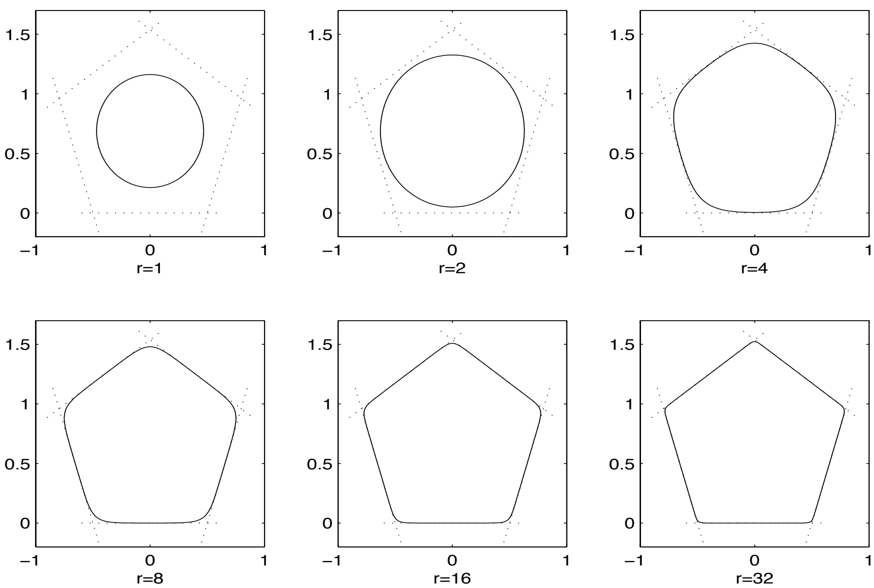

Figure 2: Plots of the pseudo-characteristic function $F_r(x, y)$ for different values of r for the pentagon of section 3. The boundary of F_r "balloons out" to fill the polygon P; we note that it is possible to construct a modified function $F_r^{'}(x, y) = \eta F_r(x, y)$ which "shrink-wraps" onto P by choosing $\eta = 1/\max_j (F_1(v_j))$ where the v_j are the polyhedron vertices.

4 N Dimensions

The method for $N > 2$ dimensions is similar to that of two dimensions. Determine the bounding (hyper) planes, followed by a set of polynomials $f_j(\underline{x})$ which has magnitude less than one between pairs of parallel bounding (hyper) planes, where $\underline{x} \in \mathbb{R}^N$. The pseudo-characteristic polynomials are then given by

$$F_r(\underline{x}) = \sum_{i=1}^{k} [f_i(\underline{x})]^{2r},$$

similar to equation (2).

In N dimensions, each face of a polyhedron contains a set of vertices $\underline{v}_i$ which lie in a plane of dimension $N-1$. The equation of a plane in $\mathbb{R}^N$ is

$$\underline{w} \cdot \underline{x} + a = 0, \tag{3}$$

where $\underline{x} \in \mathbb{R}^N$ and $\underline{w} \in \mathbb{R}^N$ is a constant vector. The dot $\cdot$ represents inner product.

The vector $\underline{w}$ is perpendicular to the plane, which may be seen by translating the plane to the origin. Let $\underline{x}_0$ be some point in the plane, i.e. a point which satisfies (3). Points in the translated plane are given by $\underline{x}' = \underline{x} - \underline{x}_0$, and hence satisfy the equation $\underline{w} \cdot \underline{x}' = 0$. Since the translated plane is parallel to the original plane, $\underline{w}$ is perpendicular to the plane.

This also gives an easy method for determining $\underline{w}$ if the vertices of the face are known. In three dimensions, the cross-product may be used to determine a vector perpendicular to each face, but a more general method is needed for $N > 4$. Without loss of generality, let $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ be the vertices of a face; for the face to lie in an $(N-1)$-dimensional plane, we require that $k > N-1$. Since each point satisfies (3) we may translate by one of the vertices, say $\underline{v}_k$, giving a system of equations

$$\begin{aligned} \underline{w} \cdot (\underline{v}_1 - \underline{v}_k) &= 0, \\ \underline{w} \cdot (\underline{v}_2 - \underline{v}_k) &= 0, \\ &\cdots \\ \underline{w} \cdot (\underline{v}_{k-1} - \underline{v}_k) &= 0, \end{aligned}$$

which may be written in matrix form as

$$\begin{pmatrix} \underline{v}_1 - \underline{v}_k \\ \underline{v}_2 - \underline{v}_k \\ \cdots \\ \underline{v}_{k-1} - \underline{v}_k \end{pmatrix} \underline{w}^T = 0. \tag{4}$$

Since the translated plane is a vector subspace of dimension $N-1$, there is a set of $N-1$ basis vectors which span the subspace, and the Gauss-Jordan method may be used to reduce (4) to a row echelon form matrix.This gives a $k-1 \times N$ matrix of rank $N-1$, which in turn determines the null vector $\underline{w}$ up to an arbitrary constant. Finally, the choice of $\underline{w}$ determines the value of the constant a in (3).

As in the two-dimensional case, the next step is to determine the polynomials $f_j(x) = 1+\gamma_j(a_j+\underline{w}_j \cdot \underline{x})$ for each face j of the object, where γ_j is again given by

$$\gamma_j = \frac{-2}{a_j + \underline{w}_j \cdot \underline{v}_j^*},$$

where $\underline{v}_j^*$ is a vertex farthest away from the face, so that $f_j(\underline{x}) = 1$ for points in the jth face and $f_j(\underline{x}) = -1$ for points in the opposing bounding plane (centered on $\underline{v}_j^*$). The vertex $\underline{v}_j^*$ is simply the vertex which maximizes $|a_j + \underline{w} \cdot \underline{v}_j|$, since $f_j(\underline{x}) = -1$ at this maximum (and $f_j(\underline{x}) = 1$ at the minimum). Thus, once $\underline{w}_j$, a_j, and $\underline{v}_j^*$ are known for each face, the pseudo-characteristic polynomial is known for the polyhedron.

4.1 Example: Tetrahedron in 3 Dimensions

As a simple example, consider a tetrahedron in $\mathbb{R}^3$ whose vertices are given by $\underline{v}_1 = (1,-1,1)$, $\underline{v}_2 = (-1,1,1)$, $\underline{v}_3 = (1,1,-1)$, and $\underline{v}_4 = (-1,-1,-1)$. Let face 1 be given by $(\underline{v}_1,\underline{v}_2,\underline{v}_3)$, face 2 by $(\underline{v}_1,\underline{v}_3,\underline{v}_4)$, face 3 by $(\underline{v}_1,\underline{v}_2,\underline{v}_4)$, and face 4 by $(\underline{v}_2,\underline{v}_3,\underline{v}_4)$. This in turn determines the various values as

face	a_j	$\underline{w}_j$	$\underline{v}_j^*$	γ_j
1	-1	$(1,1,1)$	$\underline{v}_4$	$1/2$
2	-1	$(1,-1,-1)$	$\underline{v}_2$	$1/2$
3	1	$(1,1,-1)$	$\underline{v}_3$	$-1/2$
4	1	$(1,-1,1)$	$\underline{v}_1$	$-1/2$

The pseudo-characteristic polynomials for the tetrahedron in 3 dimensions are therefore

$$F_r(x_1,x_2,x_3) = \sum_{i=1}^{4} [f_i(x_1,x_2,x_3)]^{2r}, \; r = 1,2,\ldots \tag{5}$$

where

$$\begin{aligned} f_1(x_1,x_2,x_3) &= 1-(1-x_1-x_2-x_3)/2,\\ f_2(x_1,x_2,x_3) &= 1-(1-x_1+x_2+x_3)/2,\\ f_3(x_1,x_2,x_3) &= 1-(1+x_1+x_2-x_3)/2,\\ f_4(x_1,x_2,x_3) &= 1-(1+x_1-x_2+x_3)/2. \end{aligned}$$

Plots of the boundary of F_r are given in Fig. 3 for $r=2$ and $r=16$.

Examples of values for F_r are given by

$$\begin{aligned} F_1(x_1,x_2,x_3) =& x_1^2+x_2^2+x_3^2,\\ F_2(x_1,x_2,x_3) =& \frac{1}{4}(x_1^4+x_2^4+x^4)+\frac{3}{2}(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)\\ &+6x_1x_2x_3+\frac{3}{2}(x_1^2+x_2^2+x_3^2)+\frac{1}{4}. \end{aligned}$$

4.2 Example: Tetrahedron in 4 Dimensions

Next, consider a tetrahedron in $\mathbb{R}^4$ whose vertices are given by

$$\begin{aligned} \underline{v}_1 =&(1,1,1,0),\\ \underline{v}_2 =&(1,-1,-1,0),\\ \underline{v}_3 =&(-1,1,-1,0),\\ \underline{v}_4 =&(-1,-1,1,0), \end{aligned}$$

and whose faces (3 dimensional tetrahedra) are given in terms of the vertices of the face by:

$$\begin{aligned} \text{face } 1 &: \underline{v}_1,\underline{v}_2,\underline{v}_3,\underline{v}_4,\\ \text{face } 2 &: \underline{v}_1,\underline{v}_2,\underline{v}_3,\underline{v}_5,\\ \text{face } 3 &: \underline{v}_1,\underline{v}_2,\underline{v}_4,\underline{v}_5,\\ \text{face } 4 &: \underline{v}_1,\underline{v}_3,\underline{v}_4,\underline{v}_5,\\ \text{face } 5 &: \underline{v}_2,\underline{v}_3,\underline{v}_4,\underline{v}_5. \end{aligned}$$

To compute $\underline{w}_1$, we use the matrix formed from $\underline{v}_1-\underline{v}_2$, $\underline{v}_2-\underline{v}_3$, and $\underline{v}_3-\underline{v}_4$:

$$\begin{pmatrix} 0 & 2 & 2 & 0\\ 2 & -2 & 0 & 0\\ 0 & 2 & -2 & 0 \end{pmatrix}$$

which upon row-reduction becomes

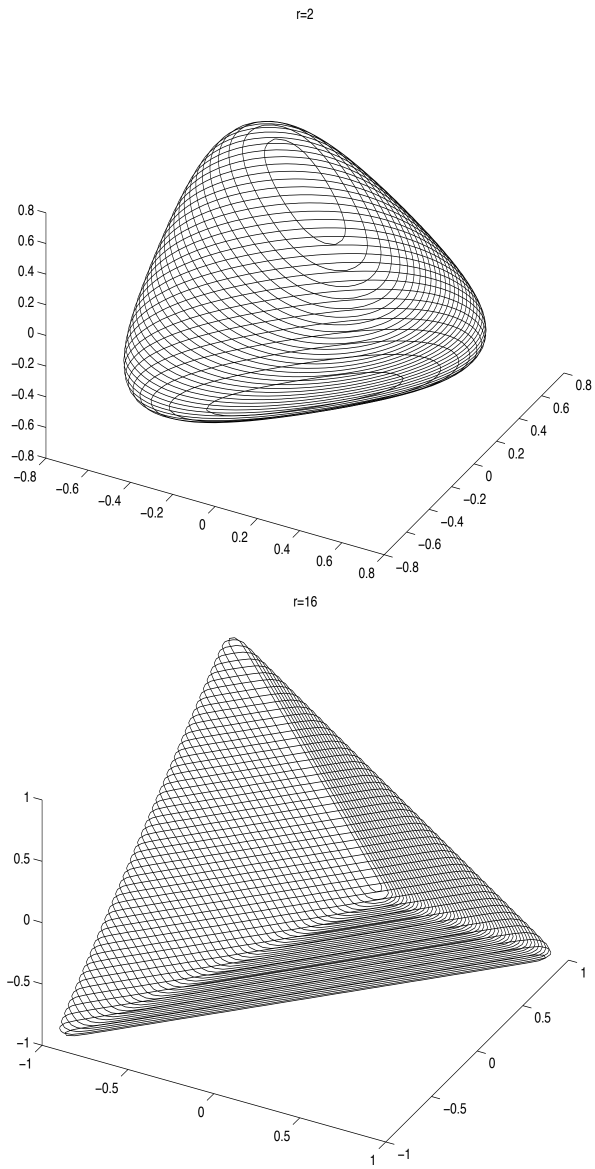

Figure 3: Plots of the pseudo-characteristic function F_r for the tetrahedron in three dimensions. The first figure is for $r = 2$; the second is for $r = 16$.

$$\begin{pmatrix} 0 & 2 & -2 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

giving

$$\underline{w}_1 = \begin{pmatrix} 0, \\ 0, \\ 0, \\ 1 \end{pmatrix}$$

and hence $a_1 = -1$. Note that this is the expected result, since if we consider the coordinate system to be $\underline{x} = (x_1, x_2, x_3, x_4)$, face 1 is an $\mathbb{R}^3$ tetrahedron at $x_1 = 1$. Note also that a different choice of $\underline{w}_1$ would force a different choice of a_1. Vertices $\underline{v}_5$, $\underline{v}_6$, $\underline{v}_7$, and $\underline{v}_8$ are equidistant from face 1, and choosing $\underline{v}_1^*$ to be any of these gives $\gamma_1 = -2/\sqrt{5}$ and hence

$$f_1(\underline{x}) = 1 + (-1 + w_1 \cdot \underline{x}) = 1 - \frac{2}{\sqrt{5}} x_4.$$

Similar analysis of the other four faces gives the following summary.

face	a_j	$\underline{w}_j$	$\underline{v}_j^*$	γ_j
1	0	$(0, 0, 0, 1)$	$\underline{v}_5$	$-2/\sqrt{5}$
2	-1	$(1, 1, -1, 1/\sqrt{5}\ )$	$\underline{v}_4$	1/2
3	-1	$(1, -1, 1, 1/\sqrt{5}\ )$	$\underline{v}_3$	1/2
4	-1	$(-1, 1, 1, 1/\sqrt{5}\ )$	$\underline{v}_2$	1/2
5	-1	$(-1, -1, -1, 1/\sqrt{5}\ )$	$\underline{v}_1$	1/2

Thus the pseudo-characteristic functions for the four-dimensional tetrahedron are $F_r(x_1, x_2, x_3, x_4) = \sum_{i=1}^{5}, \ [f_i(x_1, x_2, x_3, x_4)]^{2r}$, where

$$\begin{aligned}
f_1(x_1, x_2, x_3, x_4) =& 1 - 2\sqrt{5} x_4/5, \\
f_2(x_1, x_2, x_3, x_4) =& 1 + \frac{1}{2}(-1 + x_1 + x_2 - x_3 + \sqrt{5} x_4/5), \\
f_3(x_1, x_2, x_3, x_4) =& 1 + \frac{1}{2}(-1 + x_1 - x_2 + x_3 + \sqrt{5} x_4/5), \\
f_4(x_1, x_2, x_3, x_4) =& 1 + \frac{1}{2}(-1 - x_1 + x_2 + x_3 + \sqrt{5} x_4/5), \\
f_5(x_1, x_2, x_3, x_4) =& 1 + \frac{1}{2}(-1 - x_1 - x_2 - x_3 + \sqrt{5} x_4/5).
\end{aligned}$$

For $r = 1$, we have

$$F_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2\sqrt{5}x_4 - \frac{2}{5}x_4\sqrt{5} + 1.$$

5 Conjectures and Remarks

We conjecture that the region determined by the pseudo-characteristic polynomial of a convex polyhedron is convex.

We conjecture that for $r = 1$, the set where the pseudo-characteristic function is less than 1 is a sphere and the center of the sphere is the center of mass of the polyhedron.

Another way of constructing a pseudo characteristic function for polygons is to use scaled and translated box functions arising in the theory of wavelets. See page 13 of [9]. If one used a finite number of such boxes, the boundaries would be less smooth than with the methods used in this paper and the calculations would be longer.

Reference [6] discusses the problem of determining if a point in the plane or in higher dimensions is inside a given polygon or polyhedron. These figures need not be convex.

To apply the results of this paper to nonconvex polytopes, one could divide the nonconvex polytopes into a finite number of convex polytopes and apply the method of this paper to each of the convex polytopes.

We acknowledge a letter from Michael Hawrylycz to David Torney that gives an algorithm for determining whether a given plane point is inside a given convex polygon.

What do the polynomials that are developed in this paper tell us about the convex polytopes that produce the polynomials?

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