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SYMMETRIC DERIVATIVES ON SUBSETS OF THE REAL LINE AND MONOTONICITY

Abstract

We define and investigate the symmetric derivative for functions defined on a subset of the real line. We give an example of a continuous function with a positive symmetric derivative everywhere which is not monotonic. When the domain is measurable or has the Baire property, then a positive symmetric derivative does imply monotonicity on a big set.

Definition 1. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is symmetrically differentiable at a point $x \in \mathbb{R}$ if the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists. (We allow infinite values.) The value of this limit is called the symmetric derivative of f.

Many properties of ordinary derivative are preserved by its symmetric counterpart. We state a few monotonicity theorems.

Theorem 2. [5, Cor. 5.2] *Let the function $f : \mathbb{R} \to \mathbb{R}$ have a positive symmetric derivative everywhere. Then f is increasing on the set of its points of continuity. More precisely there is an increasing function g such that f agrees with g on its points of continuity.*

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Theorem 3. [5, Thm. 5.5] *Let the function $f\colon \mathbb{R} \to \mathbb{R}$ have a positive symmetric derivative everywhere on a measurable set E. Then f is locally increasing at almost every point of E (i.e., almost every point x of E has a neighborhood U such that whenever $y < x < z$ and $y, z \in U$ then $f(y) < f(x) < f(z)$).*

Theorem 4. [5, Thm. 5.4] *Let the function $f\colon \mathbb{R} \to \mathbb{R}$ have a positive symmetric derivative on a set E that has the Baire property. Then there is an open set G so that $E \setminus G$ is meager and f is increasing on each component of G.*

Note that in Definition 1 the value of f at the point x is not involved, in fact the function need not even even be defined at x to be symmetrically differentiable there. This gives rise to the definition of the symmetric derivative of a function defined on a subset of the real line.

Definition 5. Let $A \subset \mathbb{R}$. We say that a function $f\colon A \to \mathbb{R}$ is symmetrically differentiable at a point $x \in \mathbb{R}$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists, taking h from the set $\{h \in \mathbb{R} : x \pm h \in A\}$. (We assume that this set has an accumulation point at 0.) We allow infinite values.

Not surprisingly, in the general case, theorems for the functions defined on the entire real line do not hold for functions defined only on a subset of $\mathbb{R}$. We illustrate how different the two cases can be by the following theorem.

Fact 6. [4, Thm. 2.1.3] *There is a set $X \subset \mathbb{R}$ such that for any function $f\colon \mathbb{R} \to [-\infty, \infty]$ there is a function $g\colon X \to \mathbb{R}$ which is symmetrically differentiable and its symmetric derivative is equal to f. (Note that although g is not defined everywhere it has symmetric derivative everywhere).*

We give now a stronger counterexample as we assume the function to be continuous and its domain to be measurable.

Theorem 7. *There is a set $A \subset (0,1)$, measurable (with measure arbitrarily close to 1), having the Baire property and a function $f : A \to \{0,1\}$ continuous at every point in A with symmetric derivative zero at every point in $(0,1)$.*

Proof. Let C be the Cantor ternary set; i.e., C is the set of points $x = 0.x_1x_2\ldots$ which have only 0's and 2's in ternary expansion. ($x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$. In case of non-uniqueness of representation, it is enough that at least one of them contains only 0's and 2's (e.g. $\frac{1}{3} = 0.100\cdots = 0.0022\cdots \in C$)).

The set $A_0 = (0,1) \setminus C$ is open and consists of components being intervals $\{x = 0.x_1x_2\dots x_n1y_1y_2\dots; x_i \in \{0,2\}, y_i \in \{0,1,2\}\}$. We group them into two types:

- type I: first digit 1 is at an even place,
- type II: first digit 1 is at an odd place.

Define the function $f_0 : A_0 \to \{0,1\}$, where f_0 is 0 on intervals of type I and f_0 is 1 on type II intervals. Then f_0 is continuous. We will define our function f as a restriction of f_0.

CLAIM. *For every $a \in C$ there is a sequence of components of A_0 of the same type converging to a and symmetric reflections of these components about a overlap with each other.*

To argue for the claim fix $a \in C$ and take any component I of A_0 which upon symmetric reflection around a (i.e., $2a - I$) is not contained entirely in some other component of A_0. The interval $2a - I$ must contain infinitely many components of A_0 of both types. Clearly we may select, in this way, a sequence of intervals of one type only, satisfying the hypotheses of the claim.

Fact 8. [3, Thm.9 and Cor.10] *There is a set $K \subset \mathbb{R}$ (a countable sum of perfect (Cantor-like) sets) of measure zero and first category satisfying $\forall_{x\in\mathbb{R}}\forall_{\varepsilon>0}\forall_{\delta>0}\ K \cap (2x - K) \cap (x + \varepsilon, x + \varepsilon + \delta)$ has the power of the continuum.*

Lemma 9. *There is a set $X \subset K$ (thus of measure zero and first category) such that for each $a \in \mathbb{R}$ we have:*

(a) there exists a sequence $\{x^n\} \subset X$ converging to a and symmetric about a (i.e., $\{x^n\} = 2a - \{x^n\}$)

(b) any sequence $\{y^n\} \subset X$ satisfying (a) is almost contained in $\{x^n\}$; that is, $\{y^n\} \setminus \{x^n\}$ is finite

(c) all x^n can be chosen from one type of components of A_0.

PROOF. List $\{x_\alpha : \alpha < c\}$ all real numbers. We will construct the set X by transfinite induction. We start by choosing a sequence $\{x^n\}$ converging to x_0 satisfying *(a)* and *(c)*. To do this let $\{I_n\}$ be a sequence of intervals converging to x_0 such that $I_{2n} = 2x_0 - I_{2n-1}$ (Thus $\{I_n\}$ is symmetric about x_0.) and all I_n are contained in components of A_0 of the same type (say, type I). The Claim guarantees the existence of such a sequence.

Let K be the set from Fact 8. In every I_{2n-1} pick a point $x^{2n-1} \in K$ such that $x^{2n} = 2x_0 - x^{2n-1}$ also belongs to K. (It also belongs to I_{2n}.) By Fact 8 it is possible.

Let $X_0 = \{x^n\}$, $X_0' = \{x^n\} \cup \{x_0\}(= X_0 \cup \{x_0\})$ and $B_0 = \{\frac{x+y}{2} : x, y \in X_0'\}$, Also define $g_0 : \mathbb{R} \to [0, \infty]$ by $g_0(x) = \inf\{|x-a| : a \in X_0' \ \& \ \exists_{b \in X_0} \ \frac{a+b}{2} = x\}$. Note that g_0 is zero only for x_0.

Assume we have defined sets X_β, X_β' B_β for $\beta < \alpha$ and $g_\beta : \mathbb{R} \to [0, \infty]$ which is zero only for $\{x_\gamma : \gamma \leq \beta\}$. Let $V = \mathrm{lin}_{\mathbb{Q}}\{\bigcup_{\beta<\alpha} X_\beta' \ \cup \ \{x_\alpha\}\}$ - the linear space over $\mathbb{Q}$ (rationals) generated by numbers in $\bigcup_{\beta<\alpha} X_\beta' \ \cup \{x_\alpha\}$. Cardinality of V is less than continuum ($\alpha < c$). We will find a sequence $\{x^n\}$ satisfying *(a)* and *(c)* for x_α.

Let $\{I_n\}$ be a sequence of intervals converging to x_α with $I_{2n} = 2x_\alpha - I_{2n-1}$ (So $\{I_n\}$ is symmetric about x_α.) and all I_n's are contained in components of A_0 of one type. By the Claim such a sequence exists. Let a point $x^1 \in I_1 \cap K$ and $x^1 \notin V$ be such that $x^2 = 2x_\alpha - x^1$ belongs to $I_2 \cap K$ (by definition of V, $x^2 \notin V$). By Fact 8 and the fact that V has cardinality smaller than continuum such a point exists.

We proceed inductively, having points $x^1, \ldots x^{2n}$ we let $V_n = \mathrm{lin}_{\mathbb{Q}}\{V \cup \{x^k : k \leq 2n\}\}$ and select $x^{2n+1} \in (I_{2n+1} \cap K) \setminus V_n$ so that $x^{2n+2} = 2x_\alpha - x^{2n+1}$ is in $I_{2n+2} \cap K$. (Then it is also not in V_n.) Again since V_n has smaller cardinality than the continuum and by Fact 8 the point x^{2n+1} (and x^{2n+2}) can be found. This way we construct a sequence $\{x^n\}$ satisfying *(a)* and *(c)*.

Again we define $X_\alpha = \bigcup_{\beta<\alpha} X_\beta \cup \{x^n\}$, $X_\alpha' = X_\alpha \cup \{x_\alpha\}$, $B_\alpha = \{\frac{x+y}{2} : x, y \in X_\alpha\}$ and $g_\alpha : \mathbb{R} \to [0, \infty]$, $g_\alpha(x) = \min\{|a-x| : a \in X_\alpha' \ \& \ \exists_{b \in X_\alpha} \ \frac{a+b}{2} = x\}$. g_α is zero only for x_β where $\beta \leq \alpha$ (because we selected x^n's outside V thus outside B_β), which guarantees that (b) is satisfied. The set $X = \bigcup_{\alpha<c} X_\alpha$ is the set we are looking for. □

We return to the proof of Theorem 7. By the Fact 8 we may require $K \subset A_0$ and by Lemma 9 if we restrict the function f_0 to the set X, we obtain a continuous function with symmetric derivative zero everywhere. To enlarge the domain (of $f_0|_X$) it is enough to add to X intervals concentric with every component of A_0 and of arbitrarily big measure (in these components, say $\frac{9}{10}$, to obtain a domain which is of measure $\frac{9}{10}$ on the interval $[0, 1]$.) Theorem 7 is proved. □

One can expect that we should confine ourselves to symmetric sets. (A set $A \in \mathbb{R}$ is symmetric if for all $x \in A$ and all numbers h we have $x + h \in A \iff x - h \in A$.) However, even this restriction does not help. (We drop, however the continuity assumption.)

Example 10. *Let H be a Hamel base containing 1 and let S be a linear space over $\mathbb{Q}$ spanned by $H \setminus \{1\}$. Define $A = \{x + k \colon x \in S,\ k \in \mathbb{Z}\}$ and $f \colon A \to \{0, 1\}$ by*

$$f(x+k) = \begin{cases} 0 & \text{when } k \text{ is even,} \\ 1 & \text{when } k \text{ is odd.} \end{cases}$$

Then f has symmetric derivative zero on A, A is symmetric, and $f^{-1}(0)$ and $f^{-1}(1)$ are both dense in A.

Proof. The set A is a group, thus symmetric, and f is symmetrically differentiable (we preserve the parity of k in reflections). □

Even for the ordinary derivative, there is a function $f \colon \mathbb{Q} \to \mathbb{R}$ ($\mathbb{Q}$ stands for rational numbers. with $f'(x) = 1$ at every $x \in \mathbb{Q}$ (thus continuous) which is not monotonic in any interval.

In contrast to the above (especially Theorem 7), when taking the domain which is "big and nice enough," we do have positive results similar to Theorems 3 and 4. The following theorems are generalizations of these two.

Theorem 11. *Let the function $f \colon E \to \mathbb{R}$, where E has the Baire property, have a positive symmetric derivative everywhere on the set E. Then there is an open set G so that $E \setminus G$ is meager and f is increasing on each component of G.*

Proof. The proof is a repetition of the proof of [5, Thm. 5.5] (Theorem 3), we only use a strengthened lemma (Lemma 12). Let $V \subset \mathbb{R}^2$ be a relation defined by

$$V = \{(x-t, x+t) \colon x \pm t \in E \Rightarrow \frac{f(x+t) - f(x-t)}{t} > 0\}. \qquad \square$$

Lemma 12. [2, Thm. 16] *Let $E \subset \mathbb{R}$ have the Baire property and suppose $W \subset \mathbb{R}^2$ is a relation having the property that for every $x \in E$ there is a positive number $\delta(x)$ so that for every $t \in \mathbb{R}$*

$$0 < t < \delta(x) \Rightarrow (x-t, x+t) \in W.$$

Then there is an open set G such that $E \setminus G$ is of the first category and for every interval $[a, b]$ contained in G and there is a sequence of points $x_0 = a < x_1 < \cdots < x_5 = b$ belonging to E such that $(x_i, x_{i+1}) \in W$ for $i = 0, 1, 2, 3, 4$.

Clearly our V satisfies the assumptions of the lemma. Now if an interval $[a, b]$ is contained in G, then by the lemma we get points x_i and by definition of V we get $\dfrac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} > 0$ so $f(x_{i+1}) - f(x_i) > 0$, and $f(a) - f(b) > 0$. □

Theorem 13. *Let the function $f\colon E \to \mathbb{R}$, where E is measurable, have a positive symmetric derivative everywhere on the set E. Then f is locally increasing (see Theorem 3) at almost every point of E.*

PROOF. The proof is almost identical to that of Theorem 11. Let $V = \{(x-t, x+t)\colon x \pm t \in E \Rightarrow \frac{f(x+t)-f(x-t)}{t} > 0\}$.

Lemma 14. [2, Thm. 17] *Let $E \subset \mathbb{R}$ be measurable and suppose $W \subset \mathbb{R}^2$ is a relation having the property that for every $x \in E$ there is a positive number $\delta(x)$ so that for every $t \in \mathbb{R}$, $0 < t < \delta(x) \Rightarrow (x-t, x+t) \in W$. Then, for almost all $x \in E$ (in the sense of measure) there is a neighborhoods U_x of x such that whenever $x + t \in U_x \cap E$, then there is a monotonic sequence of points $x_0 = x, x_1, \ldots, x_5 = x + t$ all belonging to E and $(x_i, x_{i+1}) \in W$ for $i = 0, 1, 2, 3, 4$.*

By the lemma almost every $x \in E$ has a neighborhood U_x such that for $x + t \in U_x \cap E$ we have $f(x+t) - f(x) > 0$ (or $f(x+t) - f(x) < 0$ if $t < 0$) and f is locally increasing at x. $\square$

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