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ON THE MAXIMUM OF TWO UNILATERALLY CONTINUOUS REGULATED FUNCTIONS

Abstract

We prove that if f is the maximum of two unilaterally continuous regulated functions, then the set $D_{un}(f) = \{x : f$ is not unilaterally continuous at $x\}$ is unilaterally isolated and for $x \in D_{un}(f)$ the inequality $f(x) < \max(f(x+), f(x-))$ holds. Moreover, for a regulated function f such that $D_{un}(f)$ is isolated and for $x \in D_{un}(f)$ the inequality $f(x) < \max(f(x+), f(x-))$ holds, there are two unilaterally continuous regulated functions g, h with $f = \max(g, h)$.

Let $\mathbb{R}$ be the set of all reals. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a regulated function if for each point $x \in \mathbb{R}$ there are both finite unilateral limits

$$f(x+) = \lim_{t\to x^+} f(t) \text{ and } f(x-) = \lim_{t\to x-} f(t).$$

In paper [6] and in my article [4] such functions are called jump functions.

The regulated functions play an important role in some theorems of Goffman and Waterman on Fourier series ([2, 3, 5]).

It is known that each regulated function f may be discontinuous only on a countable set, i.e., the set $D(f)$ of all discontinuity points of f is countable.

All bounded variation functions are regulated and for each countable set A there is a bounded monotone function $f : \mathbb{R} \to \mathbb{R}$ with $D(f) = A$. Thus the family

$$\{A : \text{there is a regulated } f \text{ with } D(f) = A\}$$

is the family of all countable sets.

The regulated functions form an uniformly closed algebra of functions for the pointwise operations ([1]).

The most transparent example of a regulated function is a *step function* and each regulated function restricted to a closed interval $[a, b]$ is the uniform limit of a sequence of step functions ([1]).

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Remark 1. *The maximum of two regulated functions $f, g : \mathbb{R} \to \mathbb{R}$ is also a regulated function and*

$$(\max(f,g))(x+) = \max(f(x+), g(x+)),$$

$$(\max(f,g))(x-) = \max(f(x-), g(x-)).$$

PROOF. Let $h = \max(f,g)$. Fix a point $x \in \mathbb{R}$. If $f(x+) > g(x+)$, then there is a positive real r such that $f(t) > g(t)$ for $t \in (x, x+r)$, and consequently $h(t) = f(t)$ for $\in (x, x+r)$ and $h(x+) = f(x+)$. The same we can prove that if $g(x+) > f(x+)$, then $h(x+) = g(x+)$. If $f(x+) = g(x+)$, then we observe that for each positive real η there is a positive real s such that $|f(t) - f(x+)| < \eta$ and $|g(t) - g(x+)| < \eta$ for $t \in (x, x+s)$. So for $t \in (x, x+s)$ we have

$$|h(t) - f(x+)| \leq \max(|g(t) - g(x+)|, |f(t) - f(x+)|) < \eta.$$

Thus $h(x+) = f(x+) = g(x+)$. The proof that $h(x-) = \max(f(x-), g(x-))$ is analogous. So the proof is completed. □

Obviously, the maximum of two functions continuous from the right (from the left) at a point x is also continuous on the right (on the left) hand at x. However the maximum of two unilaterally continuous regulated functions may be discontinuous on the right and on the left hand at some points. For example the function

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

is the maximum of two functions

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \text{ and } h(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0, \end{cases}$$

which are regulated and unilaterally continuous at each point $x \in \mathbb{R}$.

Since the maximum of two regulated functions is also a regulated function, the set $D(\max(f,g))$ of all discontinuity points of the maximum $\max(f,g)$ of two unilaterally continuous regulated functions f and g is countable.

For a regulated function $f : \mathbb{R} \to \mathbb{R}$ let

$$D_{un}(f) = \{x \in \mathbb{R} : f \text{ is not unilaterally continuous at } x\}.$$

Theorem 1. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the maximum of two unilaterally continuous regulated functions $g, h : \mathbb{R} \to \mathbb{R}$, then for each point $x \in D_{un}(f)$ the inequality $\max(f(x+), f(x-)) > f(x)$ holds.*

PROOF. Since $x \in D_{un}(f)$, we obtain $f(x) \neq f(x+)$ and $f(x) \neq f(x-)$. We have

$$f(x+) = \max(g(x+), h(x+)) \text{ and } f(x-) = \max(g(x-), h(x-)).$$

Assume, to the contrary, that $f(x) > \max(f(x+), f(x-))$. Then either

$$g(x) = f(x) > \max(f(x+), f(x-)) \geq \max(g(x+), g(x-))$$

or

$$h(x) = f(x) > \max(f(x+), f(x-)) \geq \max(h(x+), h(x-)).$$

This means that at least one of functions g and h is not unilaterally continuous at x, contrary to the hypothesis. This contradiction shows that $f(x) < \max(f(x+), f(x-))$ and the proof is completed. □

Theorem 2. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the maximum of two unilaterally continuous regulated functions $g, h : \mathbb{R} \to \mathbb{R}$ and a point $x \in D_{un}(f)$ is such that $f(x) < \min(f(x+), f(x-))$, then the point x is isolated in $D_{un}(f)$.*

PROOF. Assume that $f(x) = g(x) = g(x+)$. Since $f(x) < f(x+)$, we have

$$f(x+) = h(x+) > f(x) \geq h(x) \text{ and } h(x) = h(x-).$$

There is a positive real r such that $g(t) < h(t)$ for $t \in (x, x+r)$. Consequently, in this case $f(t) = h(t)$ for $t \in (x, x+r)$, and f is unilaterally continuous on the interval $(x, x+r)$. So, $D_{un}(f) \cap (x, x+r) = \emptyset$. Since $h(x-) = h(x) \leq f(x) < f(x-)$, we obtain $f(x-) = g(x-)$ and there is a positive real s such that $f(t) = g(t) > h(t)$ for $t \in (x-s, x)$, and consequently, the function f is unilaterally continuous on $(x-s, x)$. Thus $D_{un}(f) \cap (x-s, x) = \emptyset$ and consequently $(x-s, x+r) \cap D_{un}(f) = \{x\}$. So the point x is isolated in $D_{un}(f)$. Since proofs in other cases are analogous, the proof is completed. □

Theorem 3. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the maximum of two unilaterally continuous regulated functions $g, h : \mathbb{R} \to \mathbb{R}$ and a point $x \in D_{un}(f)$ is such that $\min(f(x+), f(x-)) < f(x) < \max(f(x+), f(x-))$, then the point x is unilaterally isolated in $D_{un}(f)$.*

PROOF. Assume that

$$f(x) = g(x) = g(x+) < f(x+) = \max(f(x+), f(x-)) = h(x+).$$

Then there is a positive real r such that $g(t) < h(t)$ for $t \in (x, x+r)$. Consequently, in this case

$$f(t) = h(t) \text{ for } t \in (x, x+r) \text{ and } D_{un}(f) \cap (x, x+r) = \emptyset.$$

Since proofs in other cases are analogous, our theorem is proved. □

The following example shows that in the hypothesis of the previous theorem the point $x \in D_{un}(f)$ need not be isolated in $D_{un}(f)$.

Example. For $n = 1, 2, \ldots$ let

$$I_n = \left[-\frac{1}{n}, -\frac{1}{n+1}\right) \text{ and } J_n = \left(-\frac{1}{n}, -\frac{1}{n+1}\right].$$

Define

$$g(x) = \begin{cases} x & \text{for } x \in I_{2n-1}, \quad n \geq 1 \\ \frac{1}{2} & \text{for } x \geq 0 \\ 0 & \text{otherwise on } \mathbb{R}, \end{cases}$$

and

$$h(x) = \begin{cases} x & \text{for } x \in J_{2n}, \quad n \geq 1 \\ 1 & \text{for } x > 0 \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Then g and h are unilaterally continuous regulated functions and $f = \max(g, h)$ is of the form

$$f(x) = \begin{cases} x & \text{for } x = -\frac{1}{2n+1}, \quad n \geq 1 \\ 1 & \text{for } x > 0 \\ \frac{1}{2} & \text{for } x = 0 \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

So,

$$D_{un}(f) = \left\{-\frac{1}{2n+1} : n \geq 1\right\} \cup \{0\}$$

and 0 is not isolated from the left in $D_{un}(f)$.

Observe that in the above example we have

$$\max(f(0+), f(0-)) = f(0+) = h(0+) = 1 \text{ and } g(0-) = h(0-) = 0.$$

Theorem 4. *Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is the maximum of two unilaterally continuous regulated functions $g, h : \mathbb{R} \to \mathbb{R}$ and $x \in D_{un}(f)$ is a point such that*

$$\min(f(x+), f(x-)) < f(x) < \max(f(x+), f(x-)) = g(x+)$$

$$(resp.\ \min(f(x+), f(x-)) < f(x) < \max(f(x+), f(x-)) = g(x-)).$$

If $g(x-) \neq h(x-)$ (resp. $g(x+) \neq h(x+)$), then the point x is isolated in $D_{un}(f)$.

PROOF. Consider the case

$$\min(f(x+), f(x-)) < f(x) < \max(f(x+), f(x-)) = g(x+).$$

As in the proof of last theorem, we can prove that there is a positive real r such that $f(t) = g(t) > h(t)$ for $t \in (x, x+r)$, and consequently $D_{un}(f) \cap (x, x+r) = \emptyset$. Since $g(x-) \neq h(x-)$, there is a real $s > 0$ such that

$$\text{either } f \restriction (x-s, x) = g \restriction (x-s, x) \text{ or } f \restriction (x-s, x) = h \restriction (x-s, x).$$

Consequently, the restricted function $f \restriction (x-s, x)$ is unilaterally continuous and

$$D_{un}(f) \cap (x-s, x+r) = \{x\},$$

and the point x is isolated in $D_{un}(f)$. In the other case the proof is analogous, so our theorem is proved. □

Theorem 5. *If $f : \mathbb{R} \to \mathbb{R}$ is a regulated function such that the set $D_{un}(f)$ is isolated and for each $x \in D_{un}(f)$ the inequality $f(x) < \max(f(x+), f(x-))$ holds, then there are two unilaterally continuous regulated function $g, h : \mathbb{R} \to \mathbb{R}$ such that $f = \max(g, h)$.*

PROOF. If the set $D_{un}(f) = \emptyset$, then we can define $g = h = f$. So we suppose that $D_{un}(f) \neq \emptyset$ and observe that the set $D_{un}(f) \subset D(f)$ is countable. Let $D_{un}(f) = \{a_n : n \in \mathcal{N}_1\}$, where the set $\mathcal{N}_1$ of positive integers is finite or infinite. Since the set $D_{un}(f)$ is isolated, there are pairwise disjoint closed intervals $I_n = [b_n, c_n]$, $n \in \mathcal{N}_1$, such that:

1. all endpoints b_n, c_n are continuity points of f,
2. $|c_n - b_n| < \frac{1}{n}$ and $a_n \in (b_n, c_n)$ for $n \in \mathcal{N}_1$,
3. if $f(a_n) < \min(f(a_n+), f(a_n-))$, then $f(t) > f(a_n)$ for $t \in [b_n, c_n]$,
4. if $f(a_n-) = \min(f(a_n+), f(a_n-)) < f(a_n) < \max(f(a_n+), f(a_n-)) = f(a_n+)$, then $f(t) < f(a_n)$ for $t \in [b_n, a_n)$ and $f(a_n) < f(t)$ for $t \in (a_n, c_n]$,
5. if $f(a_n+) = \min(f(a_n+), f(a_n-)) < f(a_n) < \max(f(a_n+), f(a_n-)) = f(a_n-)$, then $f(t) > f(a_n)$ for $t \in [b_n, a_n)$ and $f(a_n) > f(t)$ for $t \in (a_n, c_n]$.

If $f(a_n) < \min(f(a_n+), f(a_n-))$, then we put

$$g_n(x) = \begin{cases} f(a_n) & \text{for } x \in [a_n, c_n] \\ f(x) & \text{for } x \in [b_n, a_n) \end{cases}$$

and

$$h_n(x) = \begin{cases} f(a_n) & \text{for } x \in [b_n, a_n] \\ f(x) & \text{for } x \in (a_n, c_n]. \end{cases}$$

If $f(a_n+) = \min(f(a_n+), f(a_n-)) < f(a_n) < \max(f(a_n+), f(a_n-)) = f(a_n-)$, then we put

$$g_n(x) = \begin{cases} f(a_n) & \text{for } x \in [b_n, a_n] \\ f(x) & \text{for } x \in (a_n, c_n] \end{cases}$$

and

$$h_n(x) = \begin{cases} f(x) & \text{for } x \in [b_n, a_n) \\ f(x) & \text{for } x \in (a_n, c_n] \\ f(a_n+) & \text{for } x = a_n. \end{cases}$$

If $f(a_n-) = \min(f(a_n+), f(a_n-)) < f(a_n) < \max(f(a_n+), f(a_n-)) = f(a_n+)$ then we put

$$g_n(x) = \begin{cases} f(a_n) & \text{for } x \in [a_n, c_n] \\ f(x) & \text{for } x \in (a_n, c_n] \end{cases}$$

and

$$h_n(x) = \begin{cases} f(x) & \text{for } x \in [b_n, a_n) \\ f(x) & \text{for } x \in (a_n, c_n] \\ f(a_n-) & \text{for } x = a_n. \end{cases}$$

Now let

$$g(x) = \begin{cases} g_n(x) & \text{for } x \in [b_n, c_n],\ n \in \mathcal{N}_1 \\ f(x) & \text{otherwise on } \mathbb{R} \end{cases}$$

and

$$h(x) = \begin{cases} h_n(x) & \text{for } x \in [b_n, c_n],\ n \in \mathcal{N}_1 \\ f(x) & \text{otherwise on } \mathbb{R}. \end{cases}$$

Then g and h are unilaterally continuous regulated functions and $f = \max(g, h)$. This finishes the proof. □

As above, we can prove analogous theorems for the minimum of two unilaterally continuous regulated functions. In particular, we have the following.

Theorem 6. *Let a function* $f : \mathbb{R} \to \mathbb{R}$ *be the minimum of two unilaterally continuous regulated functions* $g, h : \mathbb{R} \to \mathbb{R}$. *Then*

(a) *for each point* $x \in D_{un}(f)$ *the inequality* $f(x) > \min(f(x+), f(x-))$ *holds,*

(b) *if* $x \in D_{un}(f)$, *then* x *is unilaterally isolated in* $D_{un}(f)$,

(c) *if* $x \in D_{un}(f)$ *and* $f(x) > \max(f(x+), f(x-))$, *then the point* x *is isolated in* $D_{un}(f)$.

Theorem 7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a regulated function such that the set* $D_{un}(f)$ *is isolated and for each point* $x \in D_{un}(f)$ *the inequality* $f(x) > \min(f(x+), f(x-))$ *holds. Then there are two unilaterally continuous regulated functions* $g, h : \mathbb{R} \to \mathbb{R}$ *such that* $f = \min(g, h)$.

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