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STRONG CONVERGENCE IN KENSTOCK-KURZWEIL-PETTIS INTEGRATION UNDER AN EXTREME POINT CONDITION

Abstract

In the present paper, some Olech and Visintin-type results are obtained in Henstock-Kurzweil-Pettis integration. More precisely, under extreme or denting point condition, one can pass from weak convergence (i.e. convergence with respect to the topology induced by the tensor product of the space of real functions of bounded variation and the topological dual of the initial Banach space) or from the convergence of integrals to strong convergence (i.e. in the topology of Alexiewicz norm or, even more, of Pettis norm). Our results extend the results already known in the Bochner and Pettis integrability setting.

1 Introduction.

It is known that, for a given weakly convergent sequence of Bochner integrable functions, after imposing an extreme point condition that eliminates persistent oscillations, one can obtain strong convergence. Analogous results hold for Pettis integrable functions (see [2], [4] and [5]).

The aim of the present work is a further extension to Henstock-Kurzweil-Pettis integrability setting. We prove that, given a sequence of Henstock-Kurzweil-Pettis integrable functions convergent with respect to the topology induced by the tensor product of the space of real functions of bounded variation and the topological dual of Banach space, we can deduce its convergence

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with respect to the Alexiewicz norm topology. To this end we impose an extreme point condition, an interval-tightness condition, as well as an uniform integrability assumption appropriate to Henstock-Kurzweil integral. If, in particular, the sequence consists of selections of a Henstock-Kurzweil-Pettis integrable multifunction, we obtain the convergence in the Pettis norm topology.

The Alexiewicz norm convergence is obtained also for a sequence of selections of a multifunction such that their integrals converge to an extreme point of the set-valued integral.

Using Komlós-type results, as well as a convergence result for Henstock integral, allows us to pass from the w-HKP convergence of a sequence of Henstock-Kurzweil-Pettis (resp. Henstock) integrable functions to the convergence with respect to the τ_{HKP} topology (resp. the Alexiewicz norm one), under denting point assumptions.

2 Notations and Preliminary Facts.

Let $[0,1]$ be the real unit interval provided with the σ-algebra Σ of Lebesgue measurable sets and with the Lebesgue measure μ. We begin by introducing the Henstock-Kurzweil integral, a concept that extends the classical Lebesgue integral on the real line. A gauge δ on $[0,1]$ is a positive function; a partition of $[0,1]$ (that is, a finite family $(I_i,t_i)_{i=1}^n$ of nonoverlapping intervals covering $[0,1]$ with the tags $t_i \in I_i$) is said to be δ-fine if for each $i \in \{1,\ldots,n\}$, $I_i \subset \,]t_i-\delta(t_i), t_i+\delta(t_i)[$.

A function $f:[0,1]\to\mathbb{R}$ is called Henstock-Kurzweil (or simply, HK-) integrable if there exists a real number, denoted by $(\text{HK})\int_0^1 f(t)\,dt$, such that, for every $\varepsilon>0$, one can find a gauge δ_ε such that, for any δ_ε-fine partition $\mathcal{P}=(I_i,t_i)_{i=1}^n$ of $[0,1]$, $\left|\sum_{i=1}^n f(t_i)\mu(I_i) - (\text{HK})\int_0^1 f(t)\,dt\right| < \varepsilon$.
For more on this integral, we refer the reader to [11].

Through this paper, X is a real separable Banach space, X^* denotes its topological dual, B^* the closed unit ball of X^* and $\mathcal{P}_0(X)$ (resp. $\mathcal{P}_{fc}(X)$, $\mathcal{P}_{wkc}(X)$, $\mathcal{P}_{kc}(X)$, $\mathcal{P}_{lwc}(X)$) stands for the family of its nonempty (resp. closed convex, weakly compact convex, strongly compact convex, closed convex locally weakly compact containing no lines) subsets. If $A\in\mathcal{P}_{wkc}(X)$, the support functional of A is $\sigma(\cdot,A)$ and is defined by

$$\sigma(x^*,A)=\sup\{\langle x^*,x\rangle, x\in A\},$$

for all $x^*\in X^*$. By D, we will indicate the Hausdorff distance and by $|\cdot| = D(\cdot,\{0\})$.

Let K be a convex subset of X and $e\in K$.

(i) e is an extreme point if there are no $x, y \in K \setminus \{e\}$ such that $e = \dfrac{x+y}{2}$.

(ii) e is a strong extreme point if any $x_n, y_n \in K$ such that $\dfrac{x_n + y_n}{2} \to e$ satisfies $x_n \to e$. (It can be proved (see [5], p. 3) that e is a strong extreme point iff, for every $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ such that $x, y \in K$, $\left\| \dfrac{x+y}{2} - e \right\| < \delta_\varepsilon$ imply $\|x - e\| < \varepsilon$.)

(iii) We say that e is a denting point of K (provided with some topology τ) if, for every neighborhood V of e (with respect to τ), $e \notin \overline{co}\,(K \setminus V)$ (here $\overline{co}$ denotes the closed convex hull). In particular, if the topology is not specified, then it is understood to be the norm topology of X.

Denote by $\partial_e(K)$ (resp. $\partial_{se}(K)$, $\partial_{\tau-d}(K)$) the set of all extreme (resp. strong extreme, denting with respect to τ) points of K. It can be easily seen that $\partial_d(K) \subset \partial_{se}(K) \subset \partial_e(K)$ whenever K is closed.

Let us recall that a function $f : [0,1] \to X$ is Pettis integrable if:

1) f is scalarly integrable; i.e., for all $x^* \in X^*$, $\langle x^*, f(\cdot)\rangle \in L^1([0,1])$;

2) for each $A \in \Sigma$, there exists $x_A \in X$ such that $\langle x^*, x_A\rangle = \int_A \langle x^*, f(s)\rangle\, ds$, for all $x^* \in X^*$.

We let $x_A = (\mathrm{P}) \int_A f(s)\, ds$ and call it the Pettis integral of f.

We denote by $Pe(\mu, X)$ the space of Pettis integrable functions. On $Pe(\mu, X)$, we consider the Pettis norm, $\|f\|_{Pe} = \sup\limits_{x^* \in B^*} \displaystyle\int_0^1 |\langle x^*, f(s)\rangle|\, ds$, that is equivalent to $\sup\limits_{A \in \Sigma} \left\| (\mathrm{P}) \displaystyle\int_A f(s)\, ds \right\|$.

A subset $\mathcal{K} \subset Pe(\mu, X)$ is said to be Pettis uniformly integrable (PUI) if the family $\{\langle x^*, f\rangle, x^* \in B^*, f \in \mathcal{K}\}$ is uniformly integrable, or, equivalently, for every $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ such that, for any $A \in \Sigma$ with $\mu(A) < \delta_\varepsilon$ and any $f \in \mathcal{K}$, $\|f\chi_A\|_{Pe} < \varepsilon$.

The Pettis integral can be generalized, by considering, on the real line, the Henstock-Kurzweil integral instead of Lebesgue one, as follows.

Definition 1. A function $f : [0,1] \to X$ is said to be Henstock-Kurzweil-Pettis (or simply, HKP-) integrable if:

1) f is scalarly HK-integrable; i.e., for any $x^* \in X^*$, $\langle x^*, f(\cdot)\rangle$ is HK-integrable,

2) for each $[a,b] \subset [0,1]$, there is $x_{[a,b]} \in X$ such that, for every $x^* \in X^*$, $\left\langle x^*, x_{[a,b]} \right\rangle = (\mathrm{HK}) \int_a^b \langle x^*, f(s) \rangle \, ds$.

Let $x_{[a,b]} = (\mathrm{HKP}) \int_a^b f(s)\, ds$.

On the space of all HKP-integrable X-valued functions we can consider:

i) the Alexiewicz norm $\|\cdot\|_A$, where $\|f\|_A = \sup\limits_{[a,b]\subset[0,1]} \left\| (\mathrm{HKP}) \int_a^b f(s)\, ds \right\|$,

ii) the τ_{HKP} topology, defined by the following convergence of nets—$f_\alpha \to f$ iff $\|\langle x^*, f_\alpha - f\rangle\|_A = \sup\limits_{[a,b]\subset[0,1]} \left\| (\mathrm{HK}) \int_a^b \langle x^*, f_\alpha(s) - f(s)\rangle \, ds \right\| \to 0$, for each $x^* \in X^*$,

iii) the topology induced by the tensor product of the space of real functions of bounded variation and X^*. (We call it the weak-Henstock-Kurzweil-Pettis topology and denote it by w-HKP.) That is, the net $(f_\alpha)_\alpha$ converges to f if $\left((\mathrm{HK}) \int_0^1 g(s) \langle x^*, f_\alpha(s) \rangle \, ds \right)_\alpha$ converges to $(\mathrm{HK}) \int_0^1 g(s) \langle x^*, f(s) \rangle \, ds$, for every $g : [0,1] \to \mathbb{R}$ of bounded variation and every $x^* \in X^*$.

Our consideration arises naturally from the Pettis integrability setting, where by weak-Pettis topology we mean the one induced by $L^\infty([0,1]) \otimes X^*$.

A family $\mathcal{K}$ of real HK-integrable functions on $[0,1]$ is uniformly HK-integrable if, for any $\varepsilon > 0$, there is a gauge δ_ε such that, for every δ_ε-fine partition and any $f \in \mathcal{K}$, $\left| \sum_{i=1}^n f(t_i)\mu(I_i) - (\mathrm{HK}) \int_0^1 f(t)\, dt \right| < \varepsilon$.

The family $\mathcal{H}$ of HKP-integrable functions is said to be uniformly HKP-integrable if the set $\{\langle x^*, f\rangle : x^* \in B^*, f \in \mathcal{H}\}$ is uniformly HK-integrable.

The following straightforward generalization of Henstock-Kurzweil integral to Banach spaces (see [7]) will be used.

Definition 2. A function $f : [0,1] \to X$ is Henstock integrable if we can find $(\mathrm{H}) \int_0^1 f(s)\, ds \in X$ such that, for every $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ with $\left\| \sum_{1=1}^n f(t_i)\mu(I_i) - (\mathrm{H}) \int_0^1 f(s)\, ds \right\| < \varepsilon$ for every δ_ε-fine partition of $[0,1]$.

Note that this concept is stronger than HKP-integrability.

Let us now recall the definitions of set-valued integrals which will be of use later.

i) $\Gamma : [0,1] \to \mathcal{P}_0(X)$ is said to be Aumann-Henstock-Kurzweil-Pettis (or AHKP-) integrable if it has at least one HKP-integrable selection. In this case,

$$(\text{AHKP}) \int_0^1 \Gamma(s)\, ds = \left\{ (\text{HKP}) \int_0^1 f(s)\, ds, f \text{ HKP-integrable selection of } \Gamma \right\},$$

ii) Γ is called scalarly (resp. scalarly HK-) integrable if, for every $x^* \in X^*$, $\sigma(x^*, \Gamma(\cdot))$ is Lebesgue (resp. HK-) integrable,

iii) A $\mathcal{P}_{wkc}(X)$-valued function Γ is "Pettis integrable in $\mathcal{P}_{wkc}(X)$" (or, simply, Pettis integrable) if it is scalarly integrable and for every $A \in \Sigma$, there exists $I_A \in \mathcal{P}_{wkc}(X)$ such that, for each $x^* \in X^*$, $\sigma(x^*, I_A) = \int_A \sigma(x^*, \Gamma(t))\, dt$. We denote I_A by (P) $\int_A \Gamma(t)\, dt$,

iv) A $\mathcal{P}_{wkc}(X)$-valued function Γ is "HKP-integrable in $\mathcal{P}_{wkc}(X)$" (or, simply, HKP-integrable) if it is scalarly HK-integrable and for every $[a,b] \subset [0,1]$, there exists $I_a^b \in \mathcal{P}_{wkc}(X)$ such that $\sigma\left(x^*, I_a^b\right) = (\text{HK}) \int_a^b \sigma(x^*, \Gamma(t))\, dt$, for each $x^* \in X^*$. We denote I_a^b by (HKP) $\int_a^b \Gamma(t)\, dt$.

Theorem 1 in [9] states the following characterizations of $\mathcal{P}_{wkc}(X)$-valued HKP-integrable multifunctions.

Theorem 3. *Let $\Gamma : [0,1] \to \mathcal{P}_{wkc}(X)$ be scalarly HK-integrable. Then the following conditions are equivalent:*

i) *Γ is HKP-integrable.*

ii) *Γ has at least one HKP-integrable selection and for any HKP-integrable selection f there exists a Pettis integrable multifunction $G : [0,1] \to \mathcal{P}_{wkc}(X)$ such that, for every $t \in [0,1]$, $\Gamma(t) = f(t) + G(t)$.*

iii) *Each measurable selection of Γ is HKP-integrable.*

iv) *For all $[a,b] \subset [0,1]$, (AHKP) $\int_a^b \Gamma(t)\, dt$ belongs to $\mathcal{P}_{wkc}(X)$ and, for every $x^* \in X^*$, $\sigma\left(x^*, (\text{AHKP}) \int_a^b \Gamma(t)\, dt\right) = (\text{HK}) \int_a^b \sigma(x^*, \Gamma(t))\, dt$.*

Remark 4. i) An immediate consequence of condition iv) is that, under the previously mentioned hypothesis, the Aumann-Henstock-Kurzweil-Pettis integral coincides with the Henstock-Kurzweil-Pettis integral.
ii) (Remark 1 in [9]) The previous result still holds if one replaces everywhere "weakly compact" by "compact".

The notations S_Γ^{Pe}, S_Γ^{H}, resp. S_Γ^{HKP} stand for the family of Pettis, Henstock, resp. HKP-integrable selections of the set-valued function Γ.

3 Strong Convergence Results under Extreme and Denting Point Condition.

Our first two results require the convergence of the sequence of integrals.

Theorem 5. *Let $(f_n)_n$ be a sequence of HKP-integrable selections of a measurable multifunction $\Gamma : [0,1] \to \mathcal{P}_{fc}(X)$ satisfying:*

1) $\lim_{n\to\infty} (\mathrm{HKP}) \int_0^1 f_n(s)\, ds = e$.

2) $e \in \partial_{se} \left((\mathrm{AHKP}) \int_0^1 \Gamma(s)\, ds \right)$.

Then there is an unique HKP-integrable selection f of Γ with $(\mathrm{HKP}) \int_0^1 f(s)\, ds = e$ and $(f_n)_n$ converges to f with respect to the Alexiewicz norm topology.

Proof. Let us begin by proving that there exists at most one (therefore, exactly one, since e is an element of the AHKP-integral) selection f such that $(\mathrm{HKP}) \int_0^1 f(s)\, ds = e$. (This is true even for extreme points of the integral, not necessarily strong extreme.) Suppose that we can find two selections f_1 and f_2 with this property. Consider an arbitrary subinterval $[a,b] \subset [0,1]$ and the elements of the set-valued integral of Γ defined by

$$e_{1,2} = (\mathrm{HKP}) \int_a^b f_1(s)\, ds + (\mathrm{HKP}) \int_{[0,1]\setminus[a,b]} f_2(s)\, ds \text{ and}$$
$$e_{2,1} = (\mathrm{HKP}) \int_a^b f_2(s)\, ds + (\mathrm{HKP}) \int_{[0,1]\setminus[a,b]} f_1(s)\, ds.$$

They satisfy the equality $e = \dfrac{e_{1,2} + e_{2,1}}{2}$. Therefore $e = e_{1,2} = e_{2,1}$. Hence $(\mathrm{HKP}) \int_a^b f_1(s)\, ds = (\mathrm{HKP}) \int_a^b f_2(s)\, ds$ and, as the subinterval was arbitrarily chosen, it follows that $f_1 = f_2$ a.e.

Now, since e is a strong extreme point of $(\mathrm{AHKP}) \int_0^1 \Gamma(s)\, ds$, for every $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that $\max(\|x - e\|, \|y - e\|) < \varepsilon$ whenever x and y are elements of the integral with $\left\| \dfrac{x+y}{2} - e \right\| < \delta_\varepsilon$. Hypothesis 1) gives $n_\varepsilon \in \mathbb{N}$ such that $\left\| (\mathrm{HKP}) \int_0^1 f_n(s)\, ds - e \right\| < \delta_\varepsilon, \forall n \geq n_\varepsilon$. We claim that $\sup_{t\in[0,1]} \left\| (\mathrm{HKP}) \int_0^t (f_n(s) - f(s))\, ds \right\| \leq \varepsilon, \forall n \geq n_\varepsilon$. Indeed, if we take an arbitrary $t \in [0,1]$, then for every $n \geq n_\varepsilon$ we have

$$\left\| \frac{(\mathrm{HKP})\int_0^1 (f_n\chi_{[0,t]} + f\chi_{[t,1]})(s)\, ds + (\mathrm{HKP})\int_0^1 (f\chi_{[0,t]} + f_n\chi_{[t,1]})(s)\, ds}{2} - e \right\| < \delta_\varepsilon,$$

and, by the choice of δ_ε, $\left\|(\text{HKP})\int_0^1 \left(f_n\chi_{[0,t]} + f\chi_{[t,1]}\right)(s)\,ds - e\right\| < \varepsilon$; that is to say, $\left\|(\text{HKP})\int_0^t (f_n(s) - f(s))\,ds\right\| < \varepsilon$. It follows that, for any $n \geq n_\varepsilon$, $\|f_n - f\|_A \leq 2 \sup_{t\in[0,1]} \left\|(\text{HKP})\int_0^t (f_n(s) - f(s))\,ds\right\| \leq 2\varepsilon$, and thus the convergence in the Alexiewicz norm topology is proved. □

We can replace the strong extreme condition by the extreme one, but we have to assume that Γ is HKP-integrable.

Proposition 6. *Let $\Gamma : [0,1] \to \mathcal{P}_{kc}(X)$ be HKP-integrable and $(f_n)_n \subset S_\Gamma^{HKP}$ such that*

1) $(\text{HKP})\int_0^1 f_n(s)\,ds \to e$ *with respect to the weak topology of X.*

2) $e \in \partial_e \left((\text{HKP})\int_0^1 \Gamma(s)\,ds\right)$.

Then there exists an unique HKP-integrable selection f of Γ satisfying that $(\text{HKP})\int_0^1 f(s)\,ds = e$ *and* $\|f_n - f\|_{Pe} \to 0$.

Proof. For the uniqueness of f see the proof of Theorem 5. (By Theorem 3 and Remark 4, the AHKP-integral coincides in this case with the HKP-integral.) By Remark 4, there exist $\gamma \in S_\Gamma^{HKP}$ and $G : [0,1] \to \mathcal{P}_{kc}(X)$ Pettis integrable such that $\Gamma(t) = \gamma(t) + G(t)$, for every $t \in [0,1]$. Then $(f_n - \gamma)_n$ is a sequence of measurable selections of G. Thus, by Theorem 5.4 in [10], it is a sequence in S_G^{Pe}. Obviously, $\left((\text{P})\int_0^1 (f_n - \gamma)(s)\,ds\right)_n$ converges with respect to the weak topology of X to the element $e - (\text{HKP})\int_0^1 \gamma(s)\,ds$; that is, an extreme point of $(\text{P})\int_0^1 G(s)\,ds$. Applying Theorem 2.3 in [2] gives that $\|f_n - f\|_{Pe} = \|(f_n - \gamma) - (f - \gamma)\|_{Pe} \to 0$. □

In what follows, the convergence with respect to the w-HKP topology will be imposed in order to obtain a stronger convergence. Proposition 8 below assumes that the limit function is an extreme point of the selections set.

The following lemma gives a property of extreme points of the selections family similar to those already known in Bochner and Pettis integrability.

Lemma 7. *Let $\Gamma:[0,1]\to\mathcal{P}_{wkc}(X)$ be HKP-integrable. Then $\partial_e(S_\Gamma^{HKP}) = S_{\partial_e(\Gamma)}^{HKP}$.*

Proof. Consider $\gamma \in S_\Gamma^{HKP}$. Then the multifunction $G : [0,1] \to \mathcal{P}_{wkc}(X)$ defined by $G(t) = \Gamma(t) - \gamma(t), \forall t \in [0,1]$ is Pettis integrable. Moreover, $S_\Gamma^{HKP} = \gamma + S_G^{Pe}$. By Theorem 1.3 in [2], we have $\partial_e(S_G^{Pe}) = S_{\partial_e(G)}^{Pe}$. It follows that

$$\partial_e(S_\Gamma^{HKP}) = \partial_e(\gamma + S_G^{Pe}) = \gamma + \partial_e(S_G^{Pe}) = \gamma + S_{\partial_e(G)}^{Pe} = S_{\partial_e(\Gamma)}^{HKP}. \qquad \square$$

Proposition 8. *Let $\Gamma : [0,1] \to \mathcal{P}_{kc}(X)$ be HKP-integrable and $(f_n)_n \subset S_\Gamma^{HKP}$ be* w-*HKP convergent to $f \in \partial_e(S_\Gamma^{HKP})$. Then $\|f_n - f\|_{Pe} \to 0$.*

Proof. We can find $\gamma \in S_\Gamma^{HKP}$ and $G : [0,1] \to \mathcal{P}_{kc}(X)$ Pettis integrable such that $\Gamma(t) = \gamma(t) + G(t)$, for any $t \in [0,1]$. Then $(f_n - \gamma)_n$ is a sequence of Pettis integrable selections of G. By the sequential $\sigma\left(Pe(\mu, X), L^\infty([0,1]) \times X^*\right)$-compactness of the family of selections of a $\mathcal{P}_{wkc}(X)$-valued Pettis integrable multifunction (Theorem 1.1 in [2]), any subsequence has a further subsequence $(f_{k_n})_n$ such that $f_{k_n} - \gamma \to g$ in the weak-Pettis topology, where g is a Pettis integrable selection of G. Hence, for every $t \in [0,1]$ and any $x^* \in X^*$, $\left\langle x^*, (\mathrm{P}) \int_0^t f_{k_n}(s) - \gamma(s)\, ds \right\rangle$ converges to $\left\langle x^*, (\mathrm{P}) \int_0^t g(s)\, ds \right\rangle$ and also to $\left\langle x^*, (\mathrm{P}) \int_0^t f(s) - \gamma(s)\, ds \right\rangle$, whence

$$\left\langle x^*, (\mathrm{P}) \int_0^t g(s)\, ds \right\rangle = \left\langle x^*, (\mathrm{P}) \int_0^t f(s) - \gamma(s)\, ds \right\rangle, \forall t \in [0,1], \forall x^* \in X^*.$$

So the equality $(\mathrm{P}) \int_0^t g(s)\, ds = (\mathrm{P}) \int_0^t f(s) - \gamma(s)\, ds$ holds for any $t \in [0,1]$ and then $g(t) = f(t) - \gamma(t)$ a.e. Thus, any subsequence contains a further subsequence such that $(f_{k_n} - \gamma)_n$ converges to $f - \gamma$ with respect to the weak-Pettis topology. Applying Lemma 2.2 in [2] and Lemma 7, we obtain that $\|f_{k_n} - f\|_{Pe} \to 0$. Finally, since this is true for any subsequence, it follows that $\|f_n - f\|_{Pe} \to 0$. $\square$

More generally, an extreme point condition and a tightness assumption appropriate to Henstock-Kurzweil integral permits Alexiewicz norm convergence to follow from the w-HKP convergence.

Definition 9. A family $\mathcal{K}$ of X-valued functions is called interval-tight (resp. interval-s-tight) if for every $\varepsilon > 0$ there exists a $\mathcal{P}_{wkc}(X)$ (resp. $\mathcal{P}_{kc}(X)$)-valued HKP-integrable multifunction Γ_ε such that, for any $f \in \mathcal{K}$, there is an interval $I_{f,\varepsilon}$ with $\mu(I_{f,\varepsilon}) < \varepsilon$ and $\{t : f(t) \notin \Gamma_\varepsilon(t)\} \subset I_{f,\varepsilon}$.

By $Ls\{x_n : n \in \mathbb{N}\}$ we denote the pointwise Kuratowski lim sup of the sequence $(x_n)_n$; i.e., $Ls\{x_n : n \in \mathbb{N}\} = \cap_{p=1}^{\infty} \overline{\{x_n : n \geq p\}}$.

The following result asserts that a sequence of HKP-integrable functions that converges with respect to the w-HKP topology is Alexiewicz-norm convergent if the limit function has extreme points as values.

Theorem 10. *Let $(f_n)_n$ be a sequence of HKP-integrable functions such that:*

1) $(f_n)_n$ is interval-s-tight.

2) $(f_n)_n$ w-HKP converges to a HKP-integrable function f_∞ and, for almost every $t \in [0,1]$, $f_\infty(t) \in \partial_e(\overline{co}(Ls\{f_n(t) : n \in \mathbb{N}\}))$.

3) For every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ satisfying that $\|(\mathrm{HKP})\int_{t_1}^{t_2} f_n(s)\,ds\| < \varepsilon$, for all $n \in \mathbb{N} \cup \{\infty\}$ and t_1, t_2 with $|t_1 - t_2| < \delta_\varepsilon$.

If $\overline{co}(Ls\{f_n(t) : n \in \mathbb{N}\}) \in \mathcal{P}_{lwc}(X)$ a.e., then $\|f_n - f_\infty\|_A \to 0$.

PROOF. By the interval-s-tightness hypothesis we deduce that, for every $\varepsilon > 0$, there exists a $\mathcal{P}_{kc}(X)$-valued HKP-integrable multifunction Γ_ε with the property that, for every $n \in \mathbb{N}$ there is an interval $I_\varepsilon^n \subset [0,1]$ with $\mu(I_\varepsilon^n) < \dfrac{\delta_\varepsilon}{3}$ such that $f_n(t) \in \Gamma_\varepsilon(t)$ for any $t \in [0,1] \setminus I_\varepsilon^n$. We claim that, for any subsequence $(f_{k_n})_n$ of $(f_n)_n$, there is a further subsequence $(f_{\widetilde{k}_n})_n$ and an interval $I_\varepsilon \subset [0,1]$ such that $\mu(I_\varepsilon) < \delta_\varepsilon$ and $f_{\widetilde{k}_n}(t) \in \Gamma_\varepsilon(t)$ for every $t \in [0,1] \setminus I_\varepsilon$ and every $n \in \mathbb{N}$. Indeed, denote by N_ε the integer part of the real $\frac{3}{\delta_\varepsilon}$ and consider the partition $(J_i)_{i=0}^{N_\varepsilon}$ of the unit interval given by

$$J_i = \begin{cases} \left[\frac{\delta_\varepsilon}{3} i, \frac{\delta_\varepsilon}{3}(i+1)\right[& \text{if } 0 \leq i \leq N_\varepsilon - 1 \\ \left[\frac{\delta_\varepsilon}{3} N_\varepsilon, 1\right] & \text{if } i = N_\varepsilon. \end{cases}$$

For the sequence $\left(I_\varepsilon^{k_n}\right)_n$ of intervals of $[0,1]$ there is an $i_0 \in \{0, \ldots, N_\varepsilon\}$ and a further subsequence $\left(I_\varepsilon^{\widetilde{k}_n}\right)_n$ with $I_\varepsilon^{\widetilde{k}_n} \cap J_{i_0} \neq \emptyset, \forall n \in \mathbb{N}$. Then the interval $I_\varepsilon = J_{i_0-1} \cup J_{i_0} \cup J_{i_0+1}$ has $\mu(I_\varepsilon) < \delta_\varepsilon$ and $\left(f_{\widetilde{k}_n}(t)\right)_n \subset \Gamma_\varepsilon(t)$, for any $t \in [0,1] \setminus I_\varepsilon$. Remark 4 gives a HKP-integrable function γ_ε and a $\mathcal{P}_{kc}(X)$-valued Pettis integrable multifunction G_ε such that $\Gamma_\varepsilon(t) = \gamma_\varepsilon(t) + G_\varepsilon(t), \forall t \in [0,1]$. On $[0,1] \setminus I_\varepsilon$, the sequence $\left(f_{\widetilde{k}_n} - \gamma_\varepsilon\right)_n$ of Pettis integrable functions is PUI, since $\left(f_{\widetilde{k}_n} - \gamma_\varepsilon\right)_n(t) \subset G_\varepsilon(t)$, for every $t \in [0,1] \setminus I_\varepsilon$. Moreover, as the family of all Pettis integrable selections of a $\mathcal{P}_{wkc}(X)$-valued Pettis integrable multifunction is sequentially $\sigma(Pe(\mu, X), L^\infty([0,1]) \otimes X^*)$-compact, we can find a subsequence (not relabeled) such that $(f_{\widetilde{k}_n} - \gamma_\varepsilon)_n$ converges

with respect to this topology to a Pettis integrable selection g. It follows that $g = f_\infty - \gamma_\varepsilon$ a.e. on $[0,1] \setminus I_\varepsilon$. We apply Theorem 3.1 in [4] to obtain that $\left\| (f_{\widetilde{k}_n} - f_\infty)\chi_{[0,1]\setminus I_\varepsilon} \right\|_{Pe} \to 0$.
Since

$$\begin{aligned}\left\| f_{\widetilde{k}_n} - f_\infty \right\|_A &\le \left\| \left(f_{\widetilde{k}_n} - f_\infty \right) \chi_{I_\varepsilon} \right\|_A + \left\| \left(f_{\widetilde{k}_n} - f_\infty \right) \chi_{[0,1]\setminus I_\varepsilon} \right\|_{Pe} \\ &\le \left\| f_{\widetilde{k}_n} \chi_{I_\varepsilon} \right\|_A + \left\| f_\infty \chi_{I_\varepsilon} \right\|_A + \left\| \left(f_{\widetilde{k}_n} - f_\infty \right) \chi_{[0,1]\setminus I_\varepsilon} \right\|_{Pe} \\ &\le 2\varepsilon + \left\| \left(f_{\widetilde{k}_n} - f_\infty \right) \chi_{[0,1]\setminus I_\varepsilon} \right\|_{Pe},\end{aligned}$$

it follows that $\left\| f_{\widetilde{k}_n} - f_\infty \right\|_A \to 0$.

Finally, as any subsequence of $(f_n)_n$ possess a further subsequence satisfying the condition $\left\| f_{\widetilde{k}_n} - f_\infty \right\|_A \to 0$, we deduce that the whole sequence has the same feature. Thus $(f_n)_n$ converges to f_∞ with respect to the Alexiewicz norm topology. □

The following result is a Henstock-Kurzweil variant of Theorem 3.1 in [4].

Proposition 11. *Let the sequence $(f_n)_n$ of HKP-integrable functions satisfy:*

1) $(f_n)_n$ is interval-s-tight.

2) $(f_n)_n$ w-HKP converges to a HKP-integrable function f_∞ and, for almost every $t \in [0,1]$, $f_\infty(t) \in \partial_e(\overline{co}(Ls\{f_n(t) : n \in \mathbb{N}\}))$.

3) $(f_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is uniformly HKP-integrable and pointwise bounded.

If $\overline{co}(Ls\{f_n(t) : n \in \mathbb{N}\}) \in \mathcal{P}_{lwc}(X)$ a.e., then $\|f_n - f_\infty\|_A \to 0$.

Proof. It suffices to prove that the uniform HKP-integrability assumption implies the hypothesis 3) of Theorem 10. In other words, it suffices to show that, if $(x_k^*)_k$ is a w^*-dense sequence of B^*, the double indexed sequence $\widetilde{f}_{n,k} = (\mathrm{HK}) \int_0^\cdot \langle x_k^*, f_n(s)\rangle\, ds$ is equicontinuous on $[0,1]$. Consider $\widetilde{f} : [0,1] \to l_\infty$ defined by $\widetilde{f}(t) = \left(\widetilde{f}_{n,k}(t)\right)_{n,k}, \forall t \in [0,1]$. Let us first show that $\widetilde{f}$ is l_∞-valued. Take $c \in [0,1]$. By the uniform HKP-integrability hypothesis, there is a partition of $[0,c]$ such that, for all $k \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$, $\left|\sum_{i=1}^N \langle x_k^*, f_n(t_i)\rangle (c_{i+1} - c_i) - (\mathrm{HK}) \int_0^c \langle x_k^*, f_n(s)\rangle\, ds\right| < 1$. By the pointwise boundedness assumption there is $M < \infty$ such that $\|f_n(t_i)\| \le M$, for every $i \in \{1, \ldots, N\}$ and $n \in \mathbb{N} \cup \{\infty\}$. Thus $\left|(\mathrm{HK}) \int_0^c \langle x_k^*, f_n(s)\rangle\, ds\right| \le 1 + Mc$, for every n and k, so the statement is proved.

To show the equicontinuity of the above defined sequence is equivalent to proving that the function $\widetilde{f}$ is continuous with respect to the $\|\cdot\|_\infty$-topology on l_∞ (thus uniformly continuous, since the definition domain is compact) fix $c \in [0,1]$ and $\varepsilon > 0$. By hypothesis, we can find $M_c < +\infty$ such that, for all $n \in \mathbb{N} \cup \{\infty\}$, $\|f_n(c)\| \leq M_c$ and a gauge δ_ε satisfying that

$$\left|\sum_{i=1}^{N} \langle x_k^*, f_n(t_i)\rangle (c_{i+1} - c_i) - \left(\int_0^{c_{i+1}} \langle x_k^*, f_n(s)\rangle \, ds - \int_0^{c_i} \langle x_k^*, f_n(s)\rangle \, ds\right)\right| < \varepsilon$$

for every n, k and every δ_ε-fine partition. Then, by Saks-Henstock's Lemma (Lemma 9.11 in [11]), any t with $|t - c| \leq \eta_{\varepsilon,c}$, where $\eta_{\varepsilon,c} = \min\left(\delta_\varepsilon(c), \dfrac{\varepsilon}{M_c}\right)$, satisfies

$$\left|(\mathrm{HK})\int_0^t \langle x_k^*, f_n(s)\rangle \, ds - (\mathrm{HK})\int_0^c \langle x_k^*, f_n(s)\rangle \, ds\right| \leq |\langle x_k^*, f_n(c)\rangle (t-c)|$$

$$+\left|(\mathrm{HK})\int_c^t \langle x_k^*, f_n(s)\rangle \, ds - \langle x_k^*, f_n(c)\rangle (t-c)\right| \leq 2\varepsilon, \forall k \in \mathbb{N}, \forall n \in \mathbb{N} \cup \{\infty\},$$

since (t, c) with the tag c is an element of a δ_ε-fine partition of $[0,1]$. Consequently, for every t with $|t - c| \leq \eta_{\varepsilon,c}$, $\left\|\widetilde{f}(t) - \widetilde{f}(c)\right\|_\infty \leq 2\varepsilon$, and so $\widetilde{f}$ is continuous. □

Under denting point assumptions, using the characterizations of $\mathcal{P}_{wkc}(X)$-valued HKP-integrable multifunctions (Theorem 3), we get the following.

Theorem 12. *Let $\Gamma : [0,1] \to \mathcal{P}_{wkc}(X)$ be HKP-integrable and $(f_n)_n$ a sequence of HKP-integrable selections* w-*HKP convergent to f. If a.e. $f(t)$ is a denting point of $\Gamma(t)$, then $\|f_n - f\|_{Pe} \to 0$.*

Proof. Let γ be a HKP-integrable function and let G be a $\mathcal{P}_{wkc}(X)$-valued Pettis integrable multifunction so that, for all $t \in [0,1]$, $\Gamma(t) = \gamma(t) + G(t)$. As in the proof of Proposition 8, for any subsequence we can find a further subsequence $(f_{k_n})_n$ such that $f_{k_n} - \gamma \to f - \gamma$ with respect to the $\sigma(Pe(\mu, X), L^\infty([0,1]) \otimes X^*)$-topology. Since, obviously, $f(t) - \gamma(t)$ is a.e. a denting point of $G(t)$, the hypotheses of Theorem 3.4 in [5] are satisfied, so $\|f_{k_n} - f\|_{Pe} = \|(f_{k_n} - \gamma) - (f - \gamma)\|_{Pe} \to 0$. Hence, $\|f_n - f\|_{Pe} \to 0$. □

Using Komlós-type results, as well as a convergence theorem given in [8] for Henstock integral, one can obtain further results under denting point assumptions on the family of selections this time. Let us first of all recall the concept of Komlós convergence.

Definition 13. A sequence of functions $(g_n)_n$ is said to be Komlós-convergent to g if for every subsequence $(g_{k_n})_n$ there is a μ-null set $N \subset [0,1]$ (depending on the subsequence), such that for every $t \in [0,1]\setminus N$, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} g_{k_i}(t) = g(t)$.

Theorem 14. *Let the sequence $(f_n)_n$ of Henstock-Kurzweil-Pettis integrable selections of a $\mathcal{P}_{wkc}(X)$-valued function Γ satisfy:*

1 *$(f_n)_n$ is pointwise bounded and, for every $x^*\in X^*$, the set $\{\langle x^*, f_n\rangle, n\in\mathbb{N}\}$ is uniformly HK-integrable.*

2 *$f_n \to f$ with respect to the* w*-HKP topology.*

3 *$f \in \partial_{\tau_{\mathrm{HKP}}-d}\left(S_\Gamma^{HKP}\right)$.*

Then $(f_n)_n$ converges in the τ_{HKP}-topology to f.

Proof. Suppose that $(f_n)_n$ does not converge to f in the τ_{HKP} topology. Then we are able to find a neighborhood V (with respect to this topology) of f and a subsequence (not relabeled) such that $f_n \notin V, \forall n\in\mathbb{N}$. Consider the sequence $(\widetilde{E}_m)_m$ of measurable subsets of $[0,1]$ defined by $\widetilde{E}_m = = \{t\in[0,1] : \|f_n(t))\| \le m, \forall n\in\mathbb{N}\}$ which, by the hypothesis of pointwise boundedness, covers the unit interval. Let $(E_m)_m$ be the associated pairwise disjoint sequence, which means that $E_1 = \widetilde{E}_1$ and, for every $m\ge 2$, $E_m = \widetilde{E}_m \setminus \bigcup_{i=1}^{m-1}\widetilde{E}_i$. On each E_m, the sequence $(f_n)_n$ is $L^1(E_m, X)$-bounded and the closed convex hull $\overline{co}\{f_n(t); n\in\mathbb{N}\}$ is weakly compact, so, by Corollary 2.2 in [3], one can find a subsequence Komlós-convergent (with respect to the weak topology of X) on E_m to a Bochner integrable function g_m. Applying successively this method on E_1, E_2, …we obtain that the diagonal subsequence, denoted by $(f_{k_n})_n$, Komlós-converges on $[0,1]$ to the function g that coincides with g_m on E_m, for each m. Otherwise stated, on any further subsequence, not relabeled, $\left(\frac{1}{n}\sum_{i=1}^{n} f_{k_i}\right)_n$ weakly a.e. converges to g. Since the Banach space is separable, it follows that g is measurable. We are able to apply Theorem 4 in [8] (in the particular case of real valued functions) and obtain that g is scalarly HK-integrable and, for each $x^*\in B^*$, $\left\|\left\langle x^*, \frac{1}{n}\sum_{i=1}^{n} f_{k_i} - g\right\rangle\right\|_A \to 0$. By hypothesis 2), the equality $\int_0^t \langle x^*, g(s)\rangle\, ds = \int_0^t \langle x^*, f(s)\rangle\, ds$ holds for any $x^*\in X^*$ and any $t\in[0,1]$, whence, as X is separable, it follows that $f{=}g$ a.e. and $f\in\overline{co}\left(S_\Gamma^{HKP}\setminus V\right)$. This contradicts our denting assumption. □

As it was noticed in [11], p. 209, the concept of uniform HK-integrability ignoring μ-null sets isn't allowed. Whereas we have the following property.

Lemma 15. *Any pointwise bounded sequence of functions $f_k : [0,1] \to \mathbb{R}$ which are null except on a set of null measure is uniformly HK-integrable.*

PROOF. Let N be the μ-null set from the hypothesis. For every $n \in \mathbb{N}$, put $N'_n = \{t \in N : 0 < |f_k(t)| \leq n, \forall k \in \mathbb{N}\}$ and let $(N_n)_n$ be the associated pairwise disjoint sequence. By the pointwise boundedness assumption, the sequence $(N_n)_n$ covers the set N. Let $\varepsilon > 0$ be arbitrary. One can find, for each n, an open set O_n such that $N_n \subset O_n$ and $\mu(O_n) < \frac{\varepsilon}{n2^n}$. Define the gauge $\delta_\varepsilon : [0,1] \to \mathbb{R}$ by

$$\delta_\varepsilon(t) = \begin{cases} 1 & \text{if } t \in [0,1] \setminus N \\ d(t, (O_n)^c) & \text{if } t \in N_n. \end{cases}$$

Then for every δ_ε-fine partition $\mathcal{P}$ of $[0,1]$, denote by $\mathcal{P}_n$ the subset of $\mathcal{P}$ that has tags in N_n. If I is an interval of $\mathcal{P}_n$, then $I \subset O_n$. If we denote by $f(\mathcal{P})$ the HK-integral sum associated to f and to the partition $\mathcal{P}$, then, for every k, $|f_k(\mathcal{P})| \leq \sum_{n=1}^{\infty} |f_k(\mathcal{P}_n)| \leq \sum_{n=1}^{\infty} n\mu(O_n) < \varepsilon$. Thus the considered sequence is uniformly HK-integrable. □

This allows us to deduce another strong convergence result under denting point condition.

Proposition 16. *Let the sequence $(f_n)_n$ of Henstock-Kurzweil-Pettis integrable selections of a $\mathcal{P}_{wkc}(X)$-valued scalarly HK-integrable multifunction Γ satisfy:*

1') *$(f_n)_n$ is pointwise bounded.*

2) *$f_n \to f$ with respect to the* w-*HKP topology.*

3) *$f \in \partial_{\tau_{\mathrm{HKP}}-d}\left(S_\Gamma^{HKP}\right)$.*

Then $(f_n)_n$ converges in the τ_{HKP}-topology to f.

PROOF. The proof is similar to that of Theorem 14. As it was done there, we obtain a subsequence $(f_{k_n})_n$ that Komlós-converges. It suffices to show that, by the scalar HK-integrability assumption on Γ, this subsequence satisfies that, for every $x^* \in X^*$, the set $\left\{\left\langle x^*, \frac{1}{n}\sum_{i=1}^n f_{k_i}\right\rangle, n \in \mathbb{N}\right\}$ is uniformly HK-integrable. Fix $x^* \in X^*$. Then one can find a μ-null set $N \subset [0,1]$ on whose complement $\left(\left\langle x^*, \frac{1}{n}\sum_{i=1}^n f_{k_i}(t)\right\rangle\right)_n$ converges to $\langle x^*, f(t)\rangle$ and

$$-\sigma\left(-x^*, \Gamma(t)\right) \leq \left\langle x^*, \frac{1}{n}\sum_{i=1}^n f_{k_i}(t)\right\rangle \leq \sigma\left(x^*, \Gamma(t)\right), \ \forall n \in \mathbb{N}.$$

Then the sequence defined, for each $n \in \mathbb{N}$, by $\widetilde{f_n} = \left(\frac{1}{n}\sum_{i=1}^{n} f_{k_i}\right)\chi_{[0,1]\setminus N}$, satisfies that $\left(\left\langle x^*, \widetilde{f_n}\right\rangle\right)_n$ converges on $[0,1]$ to $\left\langle x^*, \widetilde{f}\right\rangle$, where $\widetilde{f} = f\chi_{[0,1]\setminus N}$. Moreover, for every $t \in [0,1]$ and any $n \in \mathbb{N}$,

$$-\sigma\left(-x^*, \Gamma(t)\right)\chi_{[0,1]\setminus N}(t) \leq \left\langle x^*, \widetilde{f_n}(t)\right\rangle \leq \sigma\left(x^*, \Gamma(t)\right)\chi_{[0,1]\setminus N}(t)$$

and the two extreme functions are HK-integrable. By Corollary 13.17 in [11], the sequence $\left(\left\langle x^*, \widetilde{f_n}\right\rangle\right)_n$ is uniformly HK-integrable and, by Lemma 15, the set $\left\{\left\langle x^*, \frac{1}{n}\sum_{i=1}^{n} f_{k_i}\right\rangle, n \in \mathbb{N}\right\}$ is uniformly HK-integrable. The rest of the proof goes then as in Theorem 14. □

In the sequel, the Alexiewicz norm convergence is obtained, under denting point assumption, in Henstock integrability setting.

Theorem 17. *Let the sequence $(f_n)_n$ of Henstock integrable selections of a set-valued function Γ satisfy:*

1) *$(f_n)_n$ is uniformly HKP-integrable, pointwise bounded and, for a.e. $t \in [0,1]$, $\overline{co}\left(\{f_n(t), n \in \mathbb{N}\}\right)$ is ball-compact (that is, its intersections with closed balls are compact).*

2) *$f_n \to f$ with respect to the* w-*HKP topology.*

3) *$f \in \partial_{\|\cdot\|_A - d}\left(S_\Gamma^H\right)$.*

Then $(f_n)_n$ converges in Alexiewicz-norm to f.

PROOF. As in the proof of Theorem 14, there exists a measurable partition $(E_m)_{m\in\mathbb{N}}$ of $[0,1]$ such that, on each E_m, the sequence $(f_n)_n$ is $L^1(E_m, X)$-bounded. Suppose that $(f_n)_n$ does not converge with respect to the Alexiewicz-norm topology to f. Then one can find $\varepsilon > 0$ and a subsequence, not relabeled, such that, for every $n \in \mathbb{N}$, $\|f_n - f\|_A \geq \varepsilon$. On each E_m we are able to apply to $(f_n)_n$ the remark which postponed Komlós-type Corollary 2.2 in [3] (on the Banach space considering the norm topology) in order to obtain a subsequence that Komlós-converges. By doing this successively on E_1, E_2, ... we obtain the diagonal subsequence $\left(f_{k_n}\right)_n$ that is Komlós-convergent to a function g on $[0,1]$. Therefore $\left(\frac{1}{n}\sum_{i=1}^{n} f_{k_i}\right)_n$ converges a.e. to g; whence, by Theorem 4 in [8], g is Henstock-integrable and $\left\|\frac{1}{n}\sum_{i=1}^{n} f_{k_i} - f\right\|_A \to 0$. This implies that $f = g$ a.e. and $f \in \overline{co}(S_\Gamma^H \setminus B_{\|\cdot\|_A}(f, \varepsilon))$, which contradicts the denting point assumption on f. □

Corollary 18. *Let* $(f_n)_n$ *be a sequence of Henstock integrable selections of a* $\mathcal{P}_{kc}(X)$*-valued function* Γ *satisfying hypothesis* 2), 3) *in* Theorem 17 *and* 1') $(f_n)_n$ *is uniformly HKP-integrable and pointwise bounded. Then* $(f_n)_n$ *converges in Alexiewicz-norm to* f*.*

Proof. It suffices to note that the ball-compactness holds as a consequence of the compactness assumption on the values of Γ and then the statement follows from Theorem 17. $\square$

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