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ON SPARSE SUBSPACES OF $C[0,1]$

Abstract

We prove the existence of three subspaces of $C[0,1]$, each homeomorphic to $C[0,1]$; the first consists only of infinitely many times differentiable functions, the second consists only of singular functions of bounded variation, and the third consists only of nowhere differentiable functions.

In this paper $C[0,1]$ denotes the space of continuous real valued functions on $[0,1]$ under the uniform metric. We seek apparently "sparse" subspaces of $C[0,1]$ that are nevertheless homeomorphic to $C[0,1]$. We offer:

Theorem I. *There is a subspace H of $C[0,1]$, composed exclusively of everywhere infinitely many times differentiable functions, such that for each integer $n \geq 0$, the subspace*

$$H_n = \left\{ f^{(n)} : f \in H \right\}$$

is homeomorphic to $C[0,1]$.

Proof. For $0 < x < 1$, let

$$W(x) = \exp\Big(-x^{-2}(1-x)^{-2} \Big)$$

and for $x \leq 0$ or $x \geq 1$, let $W(x) = 0$. It follows easily that W is infinitely many times differentiable on $[0,1]$, W' vanishes outside $(0,1)$, and on $(0,1)$ $W'(x)$ is the product of $\exp\big(-x^{-2}(1-x)^{-2}\big)$ and a rational function of x. For any integer $j \geq 1$, put $h_j(x) = W(2^j x - 1)$. Then h_j is infinitely many times differentiable, h_j and all its derivatives vanish outside the interval $\big(2^{-j},\, 2^{1-j}\big)$, and on $\big(2^{-j},\, 2^{1-j}\big)$ each derivative of h_j vanishes at at most finitely many points.

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For each $j \geq 1$, let a_j be the largest positive number for which

$$\sup\Big\{\big|h_i^{(k)}\big| : i = 1, \ldots, j,\ k = 0, 1, \ldots, j\Big\} \leq \frac{1}{ja_j}. \tag{1}$$

For each $i \geq 1$, put

$$g_i = a_i h_i \tag{2}$$

and

$$G = \sum_{i \geq 1} g_i. \tag{3}$$

Then G is an infinitely many times differentiable function on $(0, 1]$, and

$$G^{(k)} = \sum_{i \geq 1} g_i^{(k)}$$

on $(0, 1]$.

Now let j and k be indices, $j \geq k$, let $2^{-j} \leq x \leq 2^{1-j}$ and let $0 < u < x$. Say $2^{-i} \leq u \leq 2^{1-i}$ where $i \geq j$. We deduce from (1), (2) and (3) that

$$\big|G^{(k)}(u)\big| = \big|g_i^{(k)}(u)\big| \leq \frac{1}{i} \leq \frac{1}{j}.$$

For $k = 0$ we conclude that G is continuous at 0. From the Mean Value Theorem we obtain a value $u \in (0, x)$ such that

$$\left|\frac{G(x) - G(0)}{x}\right| = \big|G'(u)\big| \leq \frac{1}{j}. \tag{4}$$

We deduce from the preceding paragraph that G is differentiable at 0, that $G'(0) = 0$, and moreover that G' is continuous at 0. In a similar way we prove that $G''(0) = 0$ and that G'' is continuous at 0. From an induction argument on k, it is clear that $G^{(k)}(0) = 0$ for all $k \geq 1$. Thus G is infinitely many times differentiable on the closed interval $[0, 1]$.

By the "Hilbert cube" we mean the cartesian product of countably infinitely many copies of the interval $[0, 1]$ (consult [D, 8.4, p.193]).

Let $c(c_1, c_2, c_3, \ldots)$ be a sequence of numbers in the interval $[0, 1]$ regarded as a point in the Hilbert cube. Put

$$G_c = \sum_{i \geq 1} c_i g_i \tag{5}$$

on $[0,1]$. By an argument analogous to the previous one, we find that G_c is infinitely many times differentiable on $[0,1]$. In fact for $k>0$,

$$G_c^{(k)} = \sum_{i\geq 1} c_i g_i^{(k)}. \tag{6}$$

Put

$$\begin{aligned} F =& \big\{ G_c : c \text{ is a point in the Hilbert cube} \big\}, \\ F_k =& \big\{ f^{(k)} : f \in F \big\} \end{aligned}$$

for $k>0$.

Let $c(n)$ be a sequence of points in the Hilbert cube and let d be another point in the Hilbert cube. Fix $k>0$ and assume that $G_{c(n)}$ converges to G_d in $C[0,1]$. It follows that $c_i(n)g_i \to d_i g_i$ in $C[0,1]$ for each $i \geq 1$, and hence $c_i(n) \to d_i$ in $\mathbb{R}$. It follows that in $C[0,1]$

$$c_i(n) g_i^{(k)} \to d_i g_i^{(k)} \quad (i \geq 1) \tag{7}$$

But $G^{(k)}$ is continuous at 0 and $G^{(k)}(0)=0$, so

$$\lim_{u\to 0^+} G^{(k)}(u) = 0 \tag{8}$$

Now from (3) and (6) we easily deduce that

$$\big|G_d^{(k)}\big| \leq \big|G^{(k)}\big|, \ \big|G_{c(n)}^{(k)}\big| \leq \big|G^{(k)}\big| \tag{9}$$

for all n. It follows routinely from (5), (6), (7), (8) and (9) that $G_{c(n)}^{(k)} \to G_d^{(k)}$ in $C[0,1]$. (To see this treat the tails of the expansions for $G_{c(n)}^{(k)}$ and $G_d^{(k)}$ separately.) Thus

$$G_{c(n)} \to G_d \text{ implies } G_{c(n)}^{(k)} \to G_d^{(k)} \text{ in } C[0,1].$$

By an analogous argument we see that

$$G_{c(n)}^{(k)} \to G_d^{(k)} \text{ implies } G_{c(n)} \to G_d \ \text{ in } C[0,1].$$

Again by essentially (part of) this argument we have that $G_{c(n)} \to G_d$ in $C[0,1]$ if and only if $c(n) \to d$ in the Hilbert cube.

All the topological spaces under consideration are second countable [D, p. 173]. Thus we have a homeomorphism of the Hilbert cube onto F where $c \to G_c$, and a homeomorphism of F onto F_k where $f \to f^{(k)}$.

Now $C[0,1]$ is a separable metric space and hence $C[0,1]$ is a second countable regular Hausdorff space. By a Theorem of Uryson [D, 9.2, p. 195], $C[0,1]$ is homeomorphic to a subspace of the Hilbert cube. The conclusion follows where H is the image of $C[0,1]$ in F. □

Note that in our proof of Theorem I each function in H can be expanded in a power series in any open interval that contains none of the points 2^{-j}, $j = 0, 1, 2, 3, \ldots$.

A continuous function f on $[0,1]$ is said to be *singular* if $f' = 0$ almost everywhere on $[0,1]$. We turn now to singular functions of bounded variation on $[0,1]$.

Theorem II. *There is a subspace H_1 of $C[0,1]$, composed exclusively of singular functions of bounded variation, such that H_1 is homeomorphic to $C[0,1]$.*

PROOF. For $0 \leq x \leq 1/2$, let $W(x) = L(x)$ where L denotes Lebesgue's singular function [HS, (8.28)]. For $1/2 \leq x \leq 1$, let $W(x) = 1 - L(x)$. Then W is a singular function with total variation 1 on $[0,1]$. We define h_i as in the proof of Theorem I. This time put $a_j = 2^{-j}$. We define g_i and G as before, and then G is a singular function of bounded variation on $[0,1]$. The remainder of the proof is contained in the proof of Theorem I, so we leave it. □

We note that there is an open subset U of $(0,1)$ of measure 1 such that in our proof of Theorem II, $f(U)$ is a denumerable set for each function f in H_1.

In [M], Anthony Morse constructed a continuous nowhere differentiable function f on $[0,1]$ satisfying this curious property:

$$\liminf_{t\to x^+}\left|\frac{f(t)-f(x)}{t-x}\right| < \limsup_{t\to x^+}\left|\frac{f(t)-f(x)}{t-x}\right| = \infty \text{ for } 0 \leq x < 1,$$

and

$$\liminf_{t\to x^-}\left|\frac{f(t)-f(x)}{t-x}\right| < \limsup_{t\to x^-}\left|\frac{f(t)-f(x)}{t-x}\right| = \infty \text{ for } 0 < x \leq 1,$$

Let M denote the family of all functions in $C[0,1]$ enjoying this property.

Theorem III. *There is a subset H_2 of M such that the subspace H_2 is homeomorphic to $C[0,1]$.*

PROOF. Let W denote the function constructed in [M]. By adding a linear function to W if necessary, we assume, without loss of generality, that $W(0) = W(1) = 0$. Redefine W so that $W(x) = 0$ for $x < 0$ or $x > 1$.

We complete the argument as in the proof of Theorem II with two changes. This time put $a_j = j2^{-j}$, and replace $0 \leq c_i \leq 1$ with $1 \leq c_i \leq 2$. We leave the rest. □

References

[D] J. Dugundji, *Topology*, Allyn & Bacon, Boston, 1966.

[HS] E. Hewitt, K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York, 1965.

[M] A. P. Morse, *A Continuous Function With No Unilateral Derivatives*, Trans. Amer. Math. Soc., **44** (1938), 496–507.