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REMARKS ON SOME THEOREMS ON CONVOLUTION

Abstract

We discuss relations between the Titchmarsh convolution theorem, the Kierat-Skornik theorem on convolutions and the Brodski-Sakhnovich-Donoghue theorem on cyclic vectors of the Volterra integration operator V given by $Vf(x) = \int_0^x f(t)\,dt$, on a space $L^p[0,1],\ 1 \le p < \infty$.

1 Introduction.

The symbol $L^p[0,1],\ 1 \le p < \infty$, will denote the Banach space of complex measurable functions f on $[0,1]$ for which the norm

$$\|f\|_p = \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p}$$

is finite. The convolution

$$\int_0^x f(x-t)\,g(t)\,dt$$

will be denoted by $f * g$. It is well-known that $f * g \in L^p$ and $\|f * g\|_p \le \|f\|_p \|g\|_p$ for all $f, g \in L^p$. Therefore, for each $f \in L^p$ the convolution operator $\mathcal{K}_f$, defined by $(\mathcal{K}_f g)(x) = (f * g)(x)$, is bounded on L^p. In particular, the Volterra operator $(Vg)(x) = \int_0^x g(t)\,dt$; i.e., the convolution operator $\mathcal{K}_{\mathbf{1}}$, is bounded on each of the L^p spaces. We will say that the function $f \in L^p[0,1]$ posses the property (Λ) (briefly $f \in (\Lambda)$), if $\int_0^\lambda |f(t)|\,dt > 0$ for each $\lambda \in (0,1)$; that is, if there is no right neighborhood of 0 in which the function f vanishes almost everywhere.

One version (from which others can be derived) of the classical Titchmarsh convolution theorem (see [1] and its references) is the following.

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Theorem 1. *If f and g are functions of $L^p[0,1]$ such that $f*g$ vanishes on $[0,1]$ and if $f\in(\Lambda)$ (i.e., f vanishes on no interval $(0,\lambda)$), then g must vanish on $[0,1]$; i.e.,* $\ker\mathcal{K}_f=\{0\}$.

In [2] Kierat and Skornik gave a proof of the Foias theorem (see [3]); i.e., if $f\in L^1[0,1]$ $f\in(\Lambda)$, then $\overline{\mathcal{K}_f L^1}=L^1$, in a short and elegant way without using transfinite methods. The following result was essentially proved by Kierat and Skornik [2] (see also [4]).

Theorem 2. *If f is a function of $L^p[0,1]$ such that $f\in(\Lambda)$, then $\overline{\mathcal{K}_f L^p}=L^p$.*

The following well-known result is due to Brodski, Sakhnovich and Donoghue (see [1]).

Theorem 3. *If $f\in L^p[0,1]$ and $f\in(\Lambda)$, then f is a cyclic vector for the Volterra integration operator V on $L^p[0,1]$; that is,* $\operatorname{span}\{V^n f: n\geq 0\}=L^p[0,1]$.

The main purpose of this note is to point out relations between these three theorems. Theorem 2 is of interest partly because of its connection with the convolution theorem of Titchmarsh (Theorem 1) and the Brodski, Sakhnovich and Donoghue theorem (Theorem 3). It has been known for some time that a proof of Theorem 2 can be based on Theorem 1. Here we shall show that conversely, Theorem 1 can be derived very simply from Theorem 2. We shall also obtain new proofs of Theorem 2 and Theorem 1 (see Remark 1 below).

2 Proofs.

Our arguments are essentially based on the author's paper [4]. For the sake of completeness, we provide the details.

2.1 Theorem 2 $\Longrightarrow$ Theorem 1.

Indeed, let the functions f, $g\in L^p$ satisfy the conditions of Theorem 1. Then it follows from the equality $(f*g)(x)=0$ a.a. $x\in[0,1]$ that $(h*f)*g=0$ for all $h\in L^p$. Since by virtue of Theorem 2 $\{h*f: f\in L^p\}$ is a dense set in L^p, we have from the last equality that $\mathbf{1}*g=0$; that is, $\int_0^x g(t)\,dt=0$ for a.a. $x\in[0,1]$, which implies that $g(x)=0$ for a.a. $x\in[0,1]$, as desired.

2.2 Theorem 2 $\Longrightarrow$ Theorem 3.

Indeed, if $f \in L^p$ and $f \in (\Lambda)$, then by Theorem 2 we have

$$\begin{aligned} L^p =& \overline{\mathcal{K}_f L^p} = \overline{\mathcal{K}_f \operatorname{span}\{\frac{x^n}{n!} : n \geq 0\}} \\ =& \operatorname{span}\{\mathcal{K}_f \left(\frac{x^n}{n!}\right) : n \geq 0\} = \operatorname{span}\{\frac{x^n}{n!} * f : n \geq 0\} \\ =& \operatorname{span}\{V^{n+1} f : n \geq 0\} \end{aligned}$$

(here span$\{\dots\}$means the closed linear hull of the set $\{\dots\}$), or

$$\operatorname{span}\{V^{n+1} f : n \geq 0\} = L^p,$$

which implies that $\operatorname{span}\{V^n f : n \geq 0\} = L^p$, because $\operatorname{span}\{V^{n+1} f : n \geq 0\} \subset \operatorname{span}\{V^n f : n \geq 0\}$, which means that f is a cyclic vector for the operator V.

2.3 Theorem 3 $\Longrightarrow$ Theorem 2.

Since

$$\overline{\mathcal{K}_f L^p} = \operatorname{span}\{V^{n+1} f : n \geq 0\} \overset{def}{=} E,$$

(see 2.2. above), to prove this implication it suffices to show that

$$E = \operatorname{span}\{V^n f : n \geq 0\}.$$

Really, since $x^k * f \in E$ for all $k \geq 0$, we have $p * f \in E$ for all polynomials $p(x)$. Let $\delta_n(x) \in L^p$ be any $*-$approximate unit for the space L^p; i.e, $\delta_n * g \to g$ $(n \to \infty)$ in L^p for all $g \in L^p$. If $P_{n,m}(x)$ are polynomials such that $\lim_m P_{n,m} = \delta_n$ in L^p, then clearly $\delta_n * f \in E$. Therefore $\lim_n (\delta_n * f) \in E$, and hence $f \in E$. Consequently

$$\overline{\mathcal{K}_f L^p} = \operatorname{span}\{V^n f : n \geq 0\} = L^p \text{ (Theorem 3)},$$

which completes the proof.

Remark 1. Note that in the case $p = 2$ Brodski [5] and Sakhnovich [6] gave, in particular, a different proof of Theorem 3 without use of Titchmarsh's convolution theorem; namely, they have used Livsic's theory of characteristic functions (see [1] for more detailed information). By combining the above proofs of Theorem 2 and Theorem 1 (see Subsection 2.3., Theorem 3 $\Longrightarrow$Theorem 2 and Subsection 2.1., Theorem 2 $\Longrightarrow$Theorem 1) with these arguments of Brodski and Sakhnovich, we obtain new proofs of the Kierat-Skornik theorem and Titchmarsh's convolution theorem which is in an essential way close to one due to Kalish [7]. Kalish has shown that the Titchmarsh convolution theorem can be derived from the description of the lattice of all $V-$invariant subspaces.

We recall that there exist numerous proofs (see, for instance [8] and its references and references in [1]) of Titchmarsh's convolution theorem. Most of these proofs are based on the theory of analytic or harmonic functions. Mikusinski [8] gave a simple proof based only on methods of analysis of functions of a real variable.

Our next result can be essentially considered as an analogy of a well-known result due to Daniel (see [1] and [9, Theorem 3]) for the space $C^\infty = C^\infty[0,1]$ of infinitely differentiable functions in $[0,1]$.

Theorem 4. *Let $f \in C^\infty[0,1]$ be a function and assume the convolution operator $\mathcal{K}_f$ on the space $C^\infty[0,1]$ has a constant cyclic vector. Then a function $g \in C^\infty[0,1]$ is a cyclic vector of $\mathcal{K}_f$ if and only if $g(0) \neq 0$.*

The proof of this theorem uses the notion of the so-called Duhamel product

$$(f \circledast g)(x) \overset{def}{=} \frac{d}{dx}(f * g) = \frac{d}{dx}\int_0^x f(x-t)\, g(t)\; dt$$

of the functions $f,\ g \in C^\infty[0,1]$. With this multiplication C^∞ becomes an algebra. Let $(C^\infty, \circledast)$ denote the algebra of infinitely differentiable functions in $[0,1]$ with the Duhamel product $\circledast$ as multiplication. The following result is contained in the author's paper [10, Lemma 2. 2].

Lemma 5. *Let $f \in (C^\infty, \circledast)$. Then f is $\circledast$-invertible, if and only if $f(0) \neq 0$.*

PROOF OF THEOREM 4. Let $\mathcal{D}_g,\ \mathcal{D}_g h \overset{def}{=} g \circledast h$, be a Duhamel operator on a C^∞. Modulo Lemma 5, Theorem 4 is a simple remark. Indeed, if E_h denotes the cyclic subspace of $\mathcal{K}_f$ generated by a vector $h \in C^\infty$; i.e., $E_h = \operatorname{span}\{\mathcal{K}_f^k h : k \geq 0\}$, then $E_g * \mathbf{1} = \mathcal{K}_g E_{\mathbf{1}}$. Therefore, since the operator $\frac{d}{dx}$ is continuous in C^∞, we have

$$E_g = \frac{d}{dx}(E_g * \mathbf{1}) = \frac{d}{dx}(\mathcal{K}_g E_{\mathbf{1}}) = \mathcal{D}_g E_{\mathbf{1}} \tag{1}$$

and the assumption $E_1 = C^\infty[0,1]$ together with the invertibility of $\mathcal{D}_g$ (see Lemma 5) yields that if $g(0) \neq 0$, then g is a cyclic vector of $\mathcal{K}_f$. Conversely, let g be a cyclic vector of $\mathcal{K}_f$; i.e., $E_g = C^\infty$. If $g(0) = 0$, then it follows from (1) and the definition of C^∞ that $E_g \subset \{h \in C^\infty : h(0) = 0\}$ ($\neq C^\infty$), which is impossible. Thus $g(0) \neq 0$. □

Remark 2. Note that in such type of questions the Duhamel product can be considered as a rather useful tool because it rid us of consideration of an approximative identity when we have a " usual" one (see [11-18] for other applications of the Duhamel products).

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