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AN ABSOLUTELY CONTINUOUS FUNCTION WITH NON-σ-POROUS GRAPH

Abstract

The main aim of this paper is to construct an absolutely continuous function with non-σ-porous graph. This answers the question posed by Foran, whether there exists a function of bounded variation with non-σ-porous graph. As a consequence we obtain a positive solution of Goffman's question, whether there exists a function with non-σ-porous graph such that its Fourier series converges uniformly.

1 Introduction.

The notion of σ-porosity appears naturally in many branches of real analysis and Banach space theory and was studied in many papers from different points of view. The reader can consult Zajíček's survey paper ([Z]). Let us recall the definition.

Let (P, ρ) be a metric space, $M \subset P$, $x \in P$ and $R > 0$. Then we define

$$\begin{aligned} \theta(x, R, M) = &\sup\{r > 0;\ \text{there exists an open ball } B(z, r) \\ &\quad \text{such that } \rho(x, z) < R \text{ and } B(z, r) \cap M = \emptyset\}, \\ p(x, M) = &\limsup_{R \to 0+} \frac{\theta(x, R, M)}{R}. \end{aligned}$$

We say that $M \subset P$ is *porous*, if $p(x, M) > 0$ whenever $x \in M$. A set $M \subset P$ is said to be *σ-porous*, if it is a countable union of porous sets.

Foran constructed a continuous function defined on $[0, 1]$ whose graph considered as a subset of $\mathbb{R}^2$ is non-σ-porous ([F]). He posed a question whether

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there exists a function of bounded variation with non-σ-porous graph ([F]). The main aim of this paper is to answer this question in the positive proving the following result.

Theorem 1.1. *There exists an absolutely continuous function* $f : [-1, 1] \to [-1, 1]$ *such that the graph of* f *is a non-*σ*-porous subset of* $\mathbb{R}^2$ *(with respect to the Euclidean metric).*

As a consequence of this result we obtain a function f with non-σ-porous graph, such that its Fourier series converges uniformly to f (Theorem 3.1). This answers a question posed by Goffman ([G]).

2 Auxiliary Results.

Throughout we will work with the Euclidean metric on $\mathbb{R}^2$, which we simply denote by ρ. The closed (open, respectively) ball with center at $z \in \mathbb{R}^2$ and radius $r > 0$ is denoted by $\overline{B}(z, r)$ ($B(z, r)$, respectively). The set of positive (nonnegative, respectively) integers is denoted by $\mathbb{N}$ ($\mathbb{N}_0$, respectively).

We start with definitions of auxiliary notions and we prove some lemmas about them.

Definition 2.1. (i) We say that a set $H \subset \mathbb{R}^2$ is an *f-set* if for every $x \in \mathbb{R}$ there exists at most one $y \in \mathbb{R}$ with $[x, y] \in H$.

(ii) Let $H \subset \mathbb{R}^2$. Then we let

$$V(H) = \sup\Big\{ \sum_{i=1}^{n-1} |y_{i+1} - y_i|;\ n \in \mathbb{N},\ [x_i, y_i] \in H \text{ for some } x_i \in \mathbb{R}, \\ i = 1, \dots, n, \text{ and } x_1 < x_2 < \dots < x_n \Big\}.$$

Remark 2.2. *Let* $I \subset \mathbb{R}$ *be a closed interval and let* $f : I \to \mathbb{R}$ *be a function. If we denote the usual variation of function defined on* I *by* Var*, then* $V(\operatorname{graph} f) = \operatorname{Var}(f)$.

The following notions and results can be found in [ZP]. In that paper they are studied in complete metric spaces without isolated points. Here we restrict ourselves to the case $\mathbb{R}^2$ with the Euclidean metric.

Definition 2.3 (cf. [ZP, Definition 2.3.]).

(i) Let $\mathcal{V}$ be a system of closed balls in $\mathbb{R}^2$. Then the symbol $\operatorname{ap}(\mathcal{V})$ stands for the set of all points $x \in \mathbb{R}^2$ such that for every $\varepsilon > 0$ there exist infinitely many $B \in \mathcal{V}$ with $B \cap B(x, \varepsilon) \neq \emptyset$.

(ii) Let $B \subset \mathbb{R}^2$ be a ball. Then $c(B)$ denotes the center of B.

(iii) Let $\mathcal{V}$ be a system of closed balls in $\mathbb{R}^2$. Then $c(\mathcal{V})$ denotes the set of all centers of balls from $\mathcal{V}$.

(iv) Let $\mathcal{V}$ be a nonempty system of closed balls in $\mathbb{R}^2$ satisfying:

(a) $\mathcal{V}$ is point finite; i.e., each $x \in \mathbb{R}^2$ is contained at most in finitely many balls from $\mathcal{V}$,

(b) $\operatorname{ap}(\mathcal{V}) \subset c(\mathcal{V})$.

Then we say that $\mathcal{V}$ is a *B-system*.

Lemma 2.4 ([ZP, Lemma 2.4.(i)]). *Let $\mathcal{V}$ be a B-system and for every $B \in \mathcal{V}$ let $\mathcal{V}(B)$ be a B-system such that $\bigcup \mathcal{V}(B) \subset B$ and $c(B) \in c(\mathcal{V}(B))$. Then $\mathcal{U} = \bigcup\{\mathcal{V}(B);\ B \in \mathcal{V}\}$ is a B-system.*

Definition 2.5 ([ZP, Definition 2.5.(i)–(iv)]).

(i) Let $M \subset \mathbb{R}^2,\ x \in \mathbb{R}^2$ and $B_1,\ B_2$ be two closed balls in $\mathbb{R}^2$ with $x \in B_2 \subset B_1$. Then we put

$$\Gamma(x, B_1, B_2, M) = \sup\{r/\rho(x,z);\ z \in B_1 \setminus B_2,\ B(z,r) \subset B_1 \setminus M\}.$$

(ii) Let $M \subset \mathbb{R}^2$, $B \subset \mathbb{R}^2$ be a closed ball and $x \in B$. Then we set

$$\Gamma^\star(x, B, M) = \sup\{r/\rho(x,z);\ B(z,r) \subset B \setminus M,\ z \neq x\}.$$

(iii) Let $S \subset \mathbb{R}^2$. The set of all accumulating points of S is denoted by S'.

(iv) Let $M \subset \mathbb{R}^2$, $x \in \mathbb{R}^2$ and $\mu > 0$. We say that x is a point of *non-μ-porosity* of M, if $p(x, M) < \mu$.

Definition 2.6 ([ZP, Definition 2.6.]). Let $B \subset \mathbb{R}^2$ be a closed ball, S be a closed nonempty subset of B and $n \in \mathbb{N}$, $\delta, \kappa, \alpha \in (0,1)$. We say that S has the *$\mathcal{C}(0, \delta, \kappa, \alpha)$-property in B* if $S = \{c(B)\}$.

We say that S has the *$\mathcal{C}(n, \delta, \kappa, \alpha)$-property in B* if

$(\mathrm{C1})_n$ $\forall x \in S:\ \operatorname{dist}(x, B^c) > \delta^n \operatorname{diam} B$,

$(\mathrm{C2})_n$ $\Gamma^\star(y, B, S) \le \kappa$ whenever $y \in S'$,

$(\mathrm{C3})_n$ each point $x \in S'$ is a point of non-$\alpha\kappa$-porosity of the set S,

$(\mathrm{C4})_n$ S' has the $\mathcal{C}(n-1, \delta, \kappa, \alpha)$-property in B.

Observation 2.7 ([ZP, Observation 2.8.]). *If $B \subset \mathbb{R}^2$ is a closed ball and $S \subset \mathbb{R}^2$ is a set with the $\mathcal{C}(n,\delta,\kappa,\alpha)$-property in B for some $n \in \mathbb{N}$, $\delta,\kappa,\alpha \in (0,1)$, then S is countable.*

Definition 2.8 ([ZP, Definition 2.7.]). Let $B \subset \mathbb{R}^2$ be a closed ball, $\mathcal{V}$ be a B-system, $n \in \mathbb{N}$, $\delta,\beta,\varepsilon \in (0,1)$. We say that $\mathcal{V}$ has the $\mathcal{P}(0,\delta,\beta,\varepsilon)$-*property in B* if $\mathcal{V} = \{B_0\}$, $c(B_0) = c(B)$ and $B_0 \subset B$. We say that $\mathcal{V}$ has the $\mathcal{P}(n,\delta,\beta,\varepsilon)$*-property in B* if

(P1)$_n$ $\forall V \in \mathcal{V}:\ \operatorname{dist}(V,B^c) > \operatorname{diam} V$,

(P2)$_n$ $\forall V \in \mathcal{V}:\ \operatorname{dist}(V,B^c) > \delta^n \operatorname{diam} B$,

(P3)$_n$ $\forall V \in \mathcal{V}:\ \operatorname{diam} V \le \frac{1}{2} \operatorname{diam} B$,

(P4)$_n$ there exists a B-system $\mathcal{R} \subset \mathcal{V}$ with the $\mathcal{P}(n-1,\delta,\beta,\varepsilon)$-property in B such that, for an arbitrary set J intersecting each ball from $\mathcal{V}$, we have

$$\forall R \in \mathcal{R}\ \forall x \in R:\ \operatorname{dist}(x,R^c) > \beta \operatorname{diam} R \Rightarrow \Gamma(x,B,R,J) < \varepsilon.$$

Lemma 2.9 ([ZP, Lemma 2.13.]). *Let $B \subset \mathbb{R}^2$ be a closed ball, $m \in \mathbb{N}$, $\delta,\kappa,\alpha,\varepsilon \in (0,1)$,*

$10\kappa < \varepsilon$, $S_m \subset B$ be a set with the $\mathcal{C}(m,\delta,\kappa,\alpha)$-property in B. Then there exists a function $s : S_m \to (0,+\infty)$ such that, for every function

$r : S_m \to (0,+\infty)$ with $r \le s$, we have that $\mathcal{V}_m = \{\overline{B}(x,r(x));\ x \in S_m\}$ forms a B-system with the $\mathcal{P}(m,\delta,\alpha,\varepsilon)$-property in B.

Lemma 2.10 ([ZP, Lemma 2.22.]). *Let $\varepsilon \in (0,1/8)$, $\alpha_n,\delta_n \in (0,1)$ for every $n \in \mathbb{N}$, $B \subset \mathbb{R}^2$ be a closed ball and let $\{\mathcal{U}_n\}_{n=0}^{\infty}$ be a sequence of B-systems such that*

(i) $\mathcal{U}_0 = \{B\}$,

(ii) $\mathcal{U}_{n+1} = \bigcup\{\mathcal{U}_{n+1}(C); C \in \mathcal{U}_n\}$, *where* $\mathcal{U}_{n+1}(C)$ *has the* $\mathcal{P}(n+1,\delta_{n+1},\alpha_{n+1},\varepsilon)$*-property in* C, $n \in \mathbb{N}_0$,

(iii) *for every* $n \in \mathbb{N}$ *we have* $\alpha_n < (\delta_{n+1})^{n+1}$.

Then the set $\bigcap_{n=0}^{\infty} \bigcup \mathcal{U}_n$ is a closed non-σ-porous set.

Our aim is to construct a sequence of B-systems $\{\mathcal{U}_n\}_{n=0}^{\infty}$ satisfying the assumptions of Lemma 2.10 in such a way that the set $\bigcap_{n=0}^{\infty} \bigcup \mathcal{U}_n$ is a subset of graph of an absolutely continuous function.

Henceforth $q \in (0,1)$ will be a fixed number.

Definition 2.11. Let $S \subset \mathbb{R}^2$ and $d : S \to (0, +\infty)$ be a function. We say that

- S satisfies (A) if
$$\forall x \in S \; \exists r > 0 : \; S \cap B(x, r) \subset q(S - x) + x,$$
- S satisfies (B) if $qS \subset S$,
- the pair (S, d) satisfies (C) if
$$\forall y \in S \setminus \{[0, 0]\} : \; qy \in S \Rightarrow d(qy) = qd(y)$$
and if $S' \neq \emptyset$, then for every $y \in S \setminus S'$ we have $d(y) > \operatorname{dist}(y, S')$,
- the pair (S, d) satisfies (D) if
$$\forall x \in S \; \exists r > 0 \; \forall y_1, y_2 \in S \cap (B(x, r) \setminus \{x\}) : \\ y_2 = x + q(y_1 - x) \Rightarrow d(y_2) \leq qd(y_1)$$
and for every $\varepsilon > 0$ the set $\{x \in S; \; d(x) > \varepsilon\}$ is finite.

Let $F \subset \mathbb{R}^2$ be closed and α be an ordinal number. Using transfinite recursion we define the *iterated Cantor-Bendixson derivative* $F^{(\alpha)}$ by
$$F^{(0)} = F, \; F^{(\alpha+1)} = (F^{(\alpha)})', \; F^{(\alpha)} = \bigcap_{\xi < \alpha} F^{(\xi)}, \text{ if } \alpha \text{ is limit.}$$
The least ordinal α_0 with $F^{(\alpha_0)} = F^{(\alpha_0+1)}$ is called the *Cantor-Bendixson rank* of F and is denoted by $\operatorname{rk}(F)$ (see [K, pp. 33–34] for details). It is easy to observe that if F is a countable compact set, then $\operatorname{rk}(F)$ is a countable isolated ordinal.

The projection $[x, y] \mapsto x$ ($[x, y] \mapsto y$, respectively) of $\mathbb{R}^2$ onto $\mathbb{R}$ is denoted by π_1 (π_2, respectively).

Lemma 2.12. *Let $S \subset \mathbb{R}^2$ be a nonempty countable compact set with* (A) *and $c : S \to (0, +\infty)$ be a function.*

(i) Then there exists $d : S \to (0, +\infty)$ such that (S, d) satisfies (D) *and $d \leq c$.*

(ii) Then there exists $d : S \to (0, +\infty)$ such that (S, d) satisfies (D) *and if $S' \neq \emptyset$, then $d(y) > \operatorname{dist}(y, S')$ for every $y \in S \setminus S'$.*

PROOF. (i) We will proceed by transfinite induction on the rank of S. The case $\operatorname{rk}(S) = 1$ is obvious. Suppose that $\operatorname{rk}(S) = \lambda$, $1 < \lambda < \omega_1$, and that the assertion of the lemma holds for all sets with rank less than λ. There exists an ordinal number β such that $\lambda = \beta + 1$. The set $S^{(\beta)}$ is a finite nonempty set and so we may and do assume that $S^{(\beta)}$ contains exactly k distinct points $x_1, \ldots, x_k$. Since S is a countable set and satisfies (A), there exists $r > 0$ such that

- the balls $B(x_i, r)$, $i = 1, \ldots, k$, are pairwise disjoint,
- $\partial B(x_i, q^j r) \cap S = \emptyset$ whenever $i \in \{1, \ldots, k\}$, $j \in \mathbb{N}_0$,
- $S \cap B(x_i, r) \subset q(S - x_i) + x_i$, $i = 1, \ldots, k$.

The set $P = S \setminus \bigcup_{i=1}^{k} \overline{B}(x_i, r)$ is a countable compact set with $\operatorname{rk}(P) < \lambda$ satisfying (A). If $P \neq \emptyset$, then using the induction hypothesis we define values of the desired function d for points from P in such a way that $(P, d|_P)$ satisfies (D) and $d|_P \leq c|_P$.

Fix $i \in \{1, \ldots, k\}$. The set $A_i = S \cap (B(x_i, r) \setminus B(x_i, qr))$ is a nonempty countable compact set with $\operatorname{rk}(A_i) < \lambda$ and with (A). Using the induction hypothesis we define values of d for points from A_i in such a way that $(A_i, d|_{A_i})$ satisfies (D) and $d|_{A_i} \leq c|_{A_i}$. Now suppose that we have defined values of d for all points from the set $S \cap (B(x_i, q^{l-1} r) \setminus B(x_i, q^l r))$, $l \in \mathbb{N}$. Using the induction hypothesis we define values of d on $T := S \cap (B(x_i, q^l r) \setminus B(x_i, q^{l+1} r))$ in such a way that $(T, d|_T)$ satisfies (D), $d|_T \leq c|_T$ and $d(y) \leq q d(x_i + q^{-1}(y - x_i))$, whenever $y \in T$. Finally we define $d(x_i) = c(x_i)$. This finishes the construction of d. It is easy to see that d has the desired properties.

(ii) We will proceed by transfinite induction on the rank of S as above. The case $\operatorname{rk}(S) = 1$ is obvious. Now assume that $1 < \operatorname{rk}(S) = \lambda < \omega_1$. We define P and A_i, $i = 1, \ldots, k$, in the same way as in the part (i). If P is finite, then for $y \in P$ we define $d(y)$ so large that $d(y) > \operatorname{dist}(y, S')$. If P is infinite, then $P' \neq \emptyset$ and we define d on P using the induction hypothesis in such a way that $(P, d|_P)$ satisfies (D) and $d(y) > \operatorname{dist}(y, P')$ for every $y \in P \setminus P'$.

If A_i is finite, then for $y \in A_i$ we define $d(y)$ larger than $\rho(y, x_i)$. If A_i is infinite, then $A_i' \neq \emptyset$ and using the induction hypothesis we define d on A_i in such a way that $(A_i, d|_{A_i})$ satisfies (D) and $d(y) > \operatorname{dist}(y, A_i')$ for every $y \in A_i \setminus A_i'$. Thus we defined d on A_i.

If $y \in (x_i + q^j (A_i - x_i)) \cap S$, $j \in \mathbb{N}$, then we define $d(y) = q^j d(x_i + q^{-j}(y - x_i))$. The value $d(x_i)$ is defined arbitrarily as a positive number. This finishes the construction of d and it is easy to verify the desired properties of d. $\square$

Lemma 2.13. *Let $S \subset \mathbb{R}^2$ be a nonempty countable compact set with* (A) *and* (B). *Then there exists a function $d : S \to (0, +\infty)$ such that (S, d) satisfies* (D) *and* (C).

Proof. Since S is compact and countable, there is $R > 0$ such that we have $\partial B([0,0], q^j R) \cap S = \emptyset$ for every $j \in \mathbb{Z}$, and $S \subset B(x, R)$. Let $H_j = S \cap (B([0,0], q^j R) \setminus B([0,0], q^{j+1} R))$, $j \in \mathbb{N}_0$.

Since S is a nonempty closed set with (B), we have $[0,0] \in S$. The properties (A) and (B) of S give, that there exists $k_0 \in \mathbb{N}_0$ such that

$$H_j = q^{j-k_0} H_{k_0} \text{ for every } j \geq k_0. \tag{1}$$

Each set H_j has the property (A). Let $j \in \{0, \ldots, k_0\}$.

If $H_j' \neq \emptyset$, then using Lemma 2.12 we find a function $d_j : H_j \to (0, +\infty)$ such that (H_j, d_j) satisfies (D) and $d_j(y) > \operatorname{dist}(y, H_j')$ for every $y \in H_j \setminus H_j'$. If $H_j' = \emptyset$, then H_j is finite and we find a function d_j such that $d_j(y) > \operatorname{dist}(y, [0,0])$ for every $y \in H_j$. In this case the pair (H_j, d_j) obviously satisfies (D).

We set $T := q^{-k_0} H_{k_0}$. We have $S \subset \bigcup_{j=0}^{\infty} q^j T \cup \{[0,0]\}$ because of (B). Define $h : T \to (0, +\infty)$ by

$$h(y) = \sum_{j=0}^{k_0} q^{-j} d_j(q^j y),$$

putting $d_j(q^j y) = 0$ whenever $q^j y \notin S$. Let $x \in S$, $x \neq [0,0]$. There exists a unique $k \in \mathbb{N}_0$ with $q^{-k} x \in T$ and we define $d(x) = q^k h(q^{-k} x)$. The value $d([0,0])$ is defined as a positive number.

We have $d \geq d_j$ on H_j, $j = 0, \ldots, k_0$. It and (1) easily imply $d(y) > \operatorname{dist}(y, S')$ whenever $y \in S \setminus S'$. Now the property (C) of (S, d) follows.

The pair (T, h) satisfies (D) and it is easy to infer that (S, d) satisfies (D) as well. $\square$

Lemma 2.14. *Let $S \subset \mathbb{R}^2$ be a nonempty countable compact f-set with* (A). *Then $V(S) < \infty$.*

Proof. We will proceed by transfinite induction on the rank of S. If $\operatorname{rk}(S) = 1$, then S is finite and $V(S) < \infty$.

For $x \in \mathbb{R}^2$ and $r > 0$ we let $I(x, r) = [\pi_1(x) - r, \pi_1(x) + r] \times \mathbb{R}$. Take $S \subset \mathbb{R}^2$ with $1 < \operatorname{rk}(S) = \lambda < \omega_1$ and suppose that the assertion holds for sets with rank less than λ. There exists β with $\lambda = \beta + 1$. The set $S^{(\beta)}$ is finite. Since S is a compact f-set it is sufficient to prove that for every $x \in S$ there exists $r > 0$ with $V(S \cap I(x, r)) < \infty$.

If $x \in S \setminus S^{(\beta)}$, then we find $r > 0$ with $I(x,r) \cap S^{(\beta)} = \emptyset$. This gives that $\operatorname{rk}(I(x,r) \cap S) < \lambda$. Since $I(x,r) \cap S$ satisfies (A), we obtain $V(I(x,r) \cap S) < \infty$ by the induction hypothesis.

If $x \in S^{(\beta)}$, then we find $r > s > 0$ such that

- $qr > s$,
- $I(x,r) \cap (S^{(\beta)} \setminus \{x\}) = \emptyset$,
- $S \cap B(x,r) \subset q(S - x) + x$,
- $(I(x,s) \setminus B(x,qr)) \cap S = \emptyset$.

Put $H_1 = ([\pi_1(x) - r, \pi_1(x) - s] \times \mathbb{R}) \cap S$ and $H_2 = ([\pi_1(x) + s, \pi_1(x) + r] \times \mathbb{R}) \cap S$. The sets H_1 and H_2 are closed and satisfy (A). We have that $\operatorname{rk}(H_i) < \lambda$, $i = 1, 2$, and we obtain $V(H_i) < \infty$ by the induction hypothesis. Observe that

$$S \cap I(x,s) \subset \left(\bigcup_{k=0}^{\infty} q^k((H_1 \cup H_2) - x) + x \right) \cup \{x\}.$$

Now it is not difficult to verify the following estimate

$$\begin{aligned} V(S \cap I(x,s)) \leq \sum_{k=0}^{\infty} q^k V(H_1) + \sum_{k=0}^{\infty} q^k \operatorname{diam}(H_1 \cup (q(H_1 - x) + x)) \\ + \sum_{k=0}^{\infty} q^k V(H_2) + \sum_{k=0}^{\infty} q^k \operatorname{diam}(H_2 \cup (q(H_2 - x) + x)) < \infty. \quad \square \end{aligned}$$

Definition 2.15. Let $S, T \subset \mathbb{R}^2$ be nonempty countable compact sets, $d : S \to (0, +\infty)$ be a function and $\mu > 0$. We define a function $j_{S,T,d} : S \to \mathbb{Z} \cup \{-\infty\}$ by

$$j_{S,T,d}(x) = \begin{cases} \text{the smallest } n \in \mathbb{Z} \text{ with } x + q^n T \subset B(x, d(x)), & \text{if such } n \text{ exists,} \\ -\infty, & \text{otherwise.} \end{cases}$$

For every $x \in S$ we let

$$W_0(x, S, T, d, \mu) = \begin{cases} x + \mu q^{j_{S,T,d}(x)} T, & \text{if } j_{S,T,d}(x) \in \mathbb{Z}, \\ \{x\}, & \text{if } j_{S,T,d}(x) = -\infty. \end{cases}$$

Finally we put

$$W(S, T, d, \mu) = \bigcup \{W_0(x, S, T, d, \mu); x \in S\}.$$

Lemma 2.16. *Let $S, T \subset \mathbb{R}^2$ be nonempty countable compact f-sets such that T satisfies* (A) *and* (B)*, and S satisfies* (A)*. Let $d : S \to (0, +\infty)$ be a function such that (S, d) satisfies* (D)*. Then we have*

(a) $W(S, T, d, \mu)$ is an f-set for every $\mu > 0$ except countably many ones.

For every $\mu > 0$ we have

(b) $W(S, T, d, \mu)$ is a countable compact set,

(c) if $T' = \{[0, 0]\}$, then $W(S, T, d, \mu)' = S$,

(d) $W(S, T, d, \mu)$ satisfies (A)*,*

(e) if (S, d) satisfies (C) *and S satisfies* (B)*, then $W(S, T, d, \mu)$ satisfies* (B)*,*

(f) if $\xi > 0$ and $p([0, 0], T) < \xi$, then $p(x, W(S, T, d, \mu)) < \xi$ for every $x \in S$.

PROOF. (a) Each set $W_0(y, S, T, d, \mu)$, $y \in S$, is an f-set since T is an f-set. Fix $y_1, y_2 \in S$ with $y_1 \neq y_2$. We have

$$\pi_1(W_0(y_1, S, T, d, \mu)) \cap \pi_1(W_0(y_2, S, T, d, \mu)) = \emptyset$$

for all $\mu > 0$ except countably many ones since T is countable. This and countability of S give the conclusion.

Fix $\mu > 0$ and denote $W = W(S, T, d, \mu)$. To prove (b) and (c) we state the following Claim.

Claim. Let $\{y_n\}_{n=1}^{\infty}$ be a convergent sequence of elements of W. Then $\lim_{n\to\infty} y_n \in S$ or there exists $x^* \in S$ such that $W_0(x^*, S, T, d, \mu)$ contains infinitely many y_n's.

PROOF OF CLAIM. Suppose that the second possibility does not occur. Then there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ and a sequence $\{x_k\}_{k=1}^{\infty}$ of pairwise distinct elements of S with $y_{n_k} \in W_0(x_k, S, T, d, \mu)$. Since (S, d) satisfies (D), we have $\lim_{k\to\infty} \rho(y_{n_k}, x_k) = 0$. Thus $\lim_{n\to\infty} y_n = \lim_{k\to\infty} y_{n_k} \in S$ and Claim is proved.

(b) The set W is clearly bounded and countable. The set $W_0(x, S, T, d, \mu)$ is compact for every $x \in S$. This and Claim imply that W is closed.

(c) For every $x \in S$ we have $W_0(x, S, T, d, \mu)' = \{x\}$. This fact gives $S \subset W'$ and together with Claim implies also $W' \subset S$.

(d) Take $x \in W$. If $x \notin S$, then the ball $B(x, \frac{1}{2} \operatorname{dist}(x, S))$ is intersected only by finitely many sets of the form $W_0(y, S, T, d, \mu)$, $y \in S$. All such sets have the property (A). Thus W has the desired property at x. If $x \in S$, then we find $r > 0$ such that

- $\forall y \in S \cap B(x,r):\ x + q^{-1}(y-x) \in S$,
- $\forall y_1, y_2 \in S \cap (B(x, q^{-1}r) \setminus \{x\}):\ y_2 = x + q(y_1 - x) \Rightarrow d(y_2) \le qd(y_1)$.

Let Z be the set of all $z \in S \setminus B(x,r)$ with $W_0(z,S,T,d,\mu) \cap B(x,r/2) \neq \emptyset$. The set Z is finite and we obtain that the set $H := \bigcup\{W_0(y,S,T,d,\mu);\ y \in Z\}$ is closed and satisfies (A). Thus we can find $r_1 \in (0,r/2)$ such that $x+q^{-1}(y-x) \in H$ whenever $y \in H \cap B(x,r_1)$. (Such r_1 exists also in the case that $x \notin H$.) Take $y \in W \cap B(x,r_1)$. If $y \in H$, then we have $x + q^{-1}(y-x) \in H \subset W$. If $y \notin H$, then there exists $w \in S \cap B(x,r)$ with $y \in W_0(w,S,T,d,\mu)$. Then we have $x+q^{-1}(w-x) \in S$ and $x+q^{-1}(y-x) \in W_0(x+q^{-1}(w-x),S,T,d,\mu)$ since $d(w) \le qd(x+q^{-1}(w-x))$.

(e) The assertion is obvious.

(f) We have $p(x,W) \le p(x,W_0(x,S,T,d,\mu)) < \xi$ for every $x \in W$. $\square$

Lemma 2.17. *Let $\xi > 0$. Then there exists a nonempty countable compact f-set $T \subset B([0,0],1)$ such that*

(a) T *satisfies* (A),

(b) T *satisfies* (B),

(c) $T' = \{[0,0]\}$,

(d) $\Gamma^\star([0,0],\overline{B}([0,0],1),T) < \xi$.

PROOF. Choose $\eta > 0$ such that $\eta/q < \xi$. It is not difficult to find a finite f-set $T_0 \subset B([0,0],1) \setminus (B([0,0],q) \cup (\{0\} \times \mathbb{R}))$ such that

- for every $x \in \overline{B}([0,0],1) \setminus B([0,0],q)$ there exists $t \in T_0$ with $\rho(t,x) < \eta$,
- $\pi_1(q^m t_1) \neq \pi_1(q^n t_2)$ for every $t_1, t_2 \in T_0,\ t_1 \neq t_2$, and every $n, m \in \mathbb{N}_0$.

Now we put $T = \{[0,0]\} \cup \bigcup_{k=0}^\infty q^k T_0$. Then T is a countable compact f-set and the properties (a) – (c) are clearly satisfied. To prove (d) take $B(z,s) \subset \overline{B}([0,0],1) \setminus T$. There exists $k \in \mathbb{N}_0$ such that $z \in \overline{B}([0,0],q^k) \setminus \overline{B}([0,0],q^{k+1})$. Then there exists $t \in q^k T_0 \subset T$ with $\rho(t,z) < q^k\eta$. This gives

$$\frac{s}{\rho([0,0],z)} < \frac{\rho(t,z)}{q^{k+1}} < \frac{q^k\eta}{q^{k+1}} < \xi. \qquad \square$$

Lemma 2.18. *Let $n \in \mathbb{N}_0,\ \delta, \kappa, \alpha \in (0,1)$, $3\delta < \kappa$. Then there exists an f-set S_n such that*

(a) S_n *satisfies* (A),

(b) S_n *satisfies* (B)*,*

(c) S_n *has the* $\mathcal{C}(n,\delta,\kappa,\alpha)$*-property in* $\overline{B}([0,0],1)$*.*

PROOF. We will proceed by induction on n. The case $n = 0$ is obvious. Now suppose that we are able to construct S_n for fixed $n \in \mathbb{N}_0$ and arbitrary $\delta,\kappa,\alpha \in (0,1)$ with $3\delta < \kappa$. We will deal with the case $n+1$. Let $\delta,\kappa,\alpha \in (0,1)$ with $3\delta < \kappa$ be fixed. Choose $\kappa' \in (0,1)$ with $3\delta < \kappa' < \kappa$. According to the induction hypothesis there exists an f-set S_n satisfying (a) – (c) of Lemma 2.18, where κ is replaced by κ'. Let T be as in Lemma 2.17, where $\xi \in (0, \min\{\alpha\kappa, \delta^{n+1}\})$. We employ the following notation

$$K = \overline{B}([0,0],1),\ H = \{z \in \mathbb{R}^2;\ \rho(z,K^c) > 2\delta^{n+1}\}.$$

Using Lemma 2.13 we find a function $d : S_n \to (0,+\infty)$ such that (S_n,d) satisfies (D) and (C). Let $j = j_{S_n,T,d}$. By Lemma 2.16 (a) there exists

$$\mu > \max\{q^{-j([0,0])}, 2\kappa'/(q(\kappa-\kappa'))\}$$

such that the set $W := W(S_n,T,d,\mu)$ is an f-set. We put $S_{n+1} := W \cap H$. Our set S_{n+1} is closed since using Lemma 2.16 (c) we have $S'_{n+1} = S_n \subset H$. The set S_{n+1} satisfies (A) and (B) since $S'_{n+1} \subset H$ and the set W satisfies (A) and (B) according to Lemma 2.16 (d) and (e).

Now we verify the $\mathcal{C}(n+1,\delta,\kappa,\alpha)$-property in K of the set S_{n+1}.

$(C1)_{n+1}$: This condition is satisfied since $S_{n+1} \subset H$.

$(C2)_{n+1}$: We know that $\Gamma^\star(y,K,S_{n+1}) \le \Gamma^\star(y,K,S_n) < \kappa' < \kappa$ for every $y \in S'_n$. Take $y \in S_n \setminus S'_n$. If $n = 0$, then $y = [0,0]$ and $\Gamma^\star(y,K,S_1) = \Gamma^\star([0,0],K,T) < \xi < \kappa$. If $n > 0$, then $S'_n \ne \emptyset$ and $d(y) > \operatorname{dist}(y,S'_n)$. Let $B(z,s) \subset K \setminus S_{n+1}$. Then we distinguish the following possibilities.

1) The case $B(z,s) \subset H$. If $B(z,s) \subset B(y,\mu q^{j(y)})$, then $s/\rho(y,z) < \xi < \kappa$. If $B(z,s) \not\subset B(y,\mu q^{j(y)})$, then $\rho(z,y) > \frac{1}{2}\mu q^{j(y)} \ge \frac{1}{2}\mu q d(y)$. The last inequality follows from the definition of $j(y)$ and from the fact that $T \subset B([0,0],1)$. We find $x \in S'_n$ with $\rho(x,y) < d(y)$. We estimate

$$\begin{aligned}\frac{s}{\rho(y,z)} &= \frac{s}{\rho(z,x)} \cdot \frac{\rho(z,x)}{\rho(z,y)} < \kappa' \frac{\rho(z,x)}{\rho(z,y)} \le \kappa' \frac{\rho(z,y)+\rho(y,x)}{\rho(z,y)} \\ &= \kappa'\left(1 + \frac{\rho(y,x)}{\rho(z,y)}\right) \le \kappa'\left(1 + \frac{d(y)}{\frac{1}{2}\mu q d(y)}\right) = \kappa'\left(1+\frac{2}{\mu q}\right) < \kappa.\end{aligned}$$

2) The case $B(z,s) \not\subset H$. We have $K \subset B([0,0],\mu q^{j([0,0])})$ and therefore

$$\Gamma^\star([0,0],K,W_0([0,0],S_n,T,d,\mu)) < \xi.$$

This gives

$$s \le \operatorname{dist}(z, K^c) \le \operatorname{dist}(z, H^c) + 2\delta^{n+1} < \xi \cdot \rho(z, [0,0]) + 2\delta^{n+1} < 3\delta^{n+1}.$$

We have $\rho(y,z)+\operatorname{dist}(z,K^c) \ge \rho(y,K^c) > 2\delta^n$ and thus $\rho(z,y) > 2\delta^n - 3\delta^{n+1}$. We estimate

$$\frac{s}{\rho(z,y)} \le \frac{3\delta^{n+1}}{2\delta^n - 3\delta^{n+1}} = \frac{3\delta}{2-3\delta} < 3\delta < \kappa.$$

(C3)$_{n+1}$: If $x \in S'_{n+1} = S_n$, then $p(x, S_{n+1}) \le p(x, W_0(x, S_n, T, d, \mu)) < \xi < \alpha\kappa$.

(C4)$_{n+1}$: Using $S'_{n+1} = S_n$ and $\kappa' < \kappa$ we obtain the desired property by the induction hypothesis. □

Lemma 2.19. *Let P be a nonempty countable compact f-set with $V(P) < \infty$. Let $\mathcal{I}$ be the set of intervals contiguous to $\pi_1(P)$. Then*

$$V(P) = \sum_{I \in \mathcal{I}} V(P \cap (\overline{I} \times \mathbb{R})).$$

PROOF. We will proceed by transfinite induction on the rank of P.

1) If $\operatorname{rk}(P) = 1$, then P is finite and the assertion obviously holds.

2) Assume that $1 < \operatorname{rk}(P) = \lambda < \omega_1$ and that the statement holds for sets with rank less than λ. Take β such that $\lambda = \beta + 1$. The set $P^{(\beta)}$ is finite. Assume that $P^{(\beta)}$ contains exactly k distinct points $x_1, \dots, x_k$. Fix $\varepsilon > 0$. We find closed intervals $J_1, \dots, J_k$ such that

(i) $J_i \cap J_j = \emptyset$ whenever $i, j \in \{1, \dots, k\}$, $i \ne j$,

(ii) the set $J_i \cap \pi_1(P)$ is open in $\pi_1(P)$ and contains $\pi_1(x_i)$, $i = 1, \dots, k$,

(iii) the endpoints of J_i are contained in $\pi_1(P)$, $i = 1, \dots, k$,

(iv) $V(P \cap (J_i \times \mathbb{R})) < \varepsilon/k$, $i = 1, \dots, k$.

Let $\mathcal{H}$ be the set of all intervals contiguous to $\bigcup_{i=1}^k J_i$. The set $\mathcal{H}$ is finite. Using this and (iii) we can write

$$V(P) = \sum_{i=1}^k V(P \cap (J_i \times \mathbb{R})) + \sum_{L \in \mathcal{H}} V(P \cap (\overline{L} \times \mathbb{R})). \tag{2}$$

According to (ii) we have $\operatorname{rk}(P \cap (\overline{L} \times \mathbb{R})) < \lambda$ for every $L \in \mathcal{H}$ and using the induction hypothesis we obtain

$$V(P \cap (\overline{L} \times \mathbb{R})) = \sum_{I \in \mathcal{I}, I \subset \overline{L}} V(P \cap (\overline{I} \times \mathbb{R}))$$

for every $L \in \mathcal{H}$. This, (2) and (iv) imply

$$V(P) < \frac{\varepsilon}{k} \cdot k + \sum_{I \in \mathcal{I}} V(P \cap (\overline{I} \times \mathbb{R})).$$

Since ε was chosen arbitrarily we conclude

$$V(P) \leq \sum_{I \in \mathcal{I}} V(P \cap (\overline{I} \times \mathbb{R})).$$

Since the opposite inequality obviously holds, we are done. □

Lemma 2.20. *Let $S, T \subset \mathbb{R}^2$ be nonempty countable compact f-sets such that T satisfies* (A) *and* (B), *and S satisfies* (A). *Let $d : S \to (0, +\infty)$ be a function such that (S, d) satisfies* (D). *Let $\mu > 0$ be such that $W(S, T, d, \mu q^n)$ is an f-set for every $n \in \mathbb{N}_0$. Then we have $\lim_{n\to\infty} V(W(S, T, d, \mu q^n)) = V(S)$.*

Proof. Let $W(\tau) = W(S, T, d, \tau)$ for $\tau > 0$. Let $\mathcal{I}$ be the system of all intervals contiguous to $\pi_1(S)$. By Lemmas 2.14 and 2.16 (d) we have $V(S) < \infty$ and $V(W(\mu q^n)) < \infty$, $n \in \mathbb{N}_0$. Using Lemma 2.19 we obtain

$$V(S) = \sum_{I \in \mathcal{I}} V(S \cap (\overline{I} \times \mathbb{R})).$$

Let $n \in \mathbb{N}_0$ and $\mathcal{I}_n$ be the system of all intervals contiguous to $W(\mu q^n)$. Since $S \subset W(\mu q^n)$, we have that for every $J \in \mathcal{I}_n$ there is an interval $I \in \mathcal{I}$ with $J \subset I$. Using Lemma 2.19 we get

$$\begin{aligned} V(W(\mu q^n)) &= \sum_{J \in \mathcal{I}_n} V(W(\mu q^n) \cap (\overline{J} \times \mathbb{R})) \\ &= \sum_{I \in \mathcal{I}} \sum_{J \in \mathcal{I}_n, J \subset I} V(W(\mu q^n) \cap (\overline{J} \times \mathbb{R})) \\ &= \sum_{I \in \mathcal{I}} V(W(\mu q^n) \cap (\overline{I} \times \mathbb{R})). \end{aligned}$$

Since T satisfies (B), we have $W(q\tau) \subset W(\tau)$ for every $\tau > 0$. Then we have

$$\lim_{n\to\infty} V(W(\mu q^n) \cap (\overline{I} \times \mathbb{R})) = V(S \cap (\overline{I} \times \mathbb{R})) \text{ for every } I \in \mathcal{I}.$$

We also have

$$V(W(\mu q^n) \cap (\overline{I} \times \mathbb{R})) \leq V(W(\mu) \cap (\overline{I} \times \mathbb{R}))$$

for every $n \in \mathbb{N}_0$ and $I \in \mathcal{I}$. This and the fact $V(W(\mu)) < \infty$ imply

$$\begin{aligned} V(S) &= \sum_{I\in\mathcal{I}} V(S \cap (\overline{I} \times \mathbb{R})) = \sum_{I\in\mathcal{I}} \lim_{n\to\infty} V(W(\mu q^n) \cap (\overline{I} \times \mathbb{R})) \\ &= \lim_{n\to\infty} \sum_{I\in\mathcal{I}} V(W(\mu q^n) \cap (\overline{I} \times \mathbb{R})) = \lim_{n\to\infty} V(W(\mu q^n)) \\ &= \lim_{n\to\infty} V(W(S,T,d,\mu q^n)). \end{aligned}$$ □

Lemma 2.21. *Let $\psi > 0$, $S \subset \mathbb{R}^2$ be a nonempty countable compact f-set with* (A). *Let $c : S \to (0, +\infty)$ be a function and $n, \delta, \kappa, \alpha$ be as in Lemma 2.18. Let S_n be an f-set satisfying* (A) *and* (B). *Then there exist $\mu_0 \in (0, 1)$ and a function $d : S \to (0, +\infty)$ such that*

- *$d \le c$,*
- *$W(S, S_n, d, \mu_0)$ is an f-set,*
- *$V(W(S, S_n, d, \mu_0)) < V(S) + \psi$.*

PROOF. According to Lemma 2.12 (i) there exists a function $d : S \to (0, +\infty)$ such that (S, d) satisfies (D) and $d \le c$. Now we find $\mu \in (0, 1)$ such that all sets $W(S, S_n, d, \mu q^n)$, $n \in \mathbb{N}_0$, are f-sets. Such μ exists according to Lemma 2.16 (a). According to Lemma 2.20 there exists $m \in \mathbb{N}_0$ with $V(W(S, S_n, d, \mu q^m)) < V(S) + \psi$. Put $\mu_0 = \mu q^m$ and we are done. □

3 Proofs of the Main Results.

PROOF OF THEOREM 1.1. The symbol λ stands for Lebesgue measure on $\mathbb{R}$. Let $A \subset \mathbb{R}^2$ and $x \in \mathbb{R}$. Then we set $A^x = \{y \in \mathbb{R};\ [x, y] \in A\}$. We fix $\varepsilon, \kappa, \delta, \alpha_n \in (0, 1)$ such that $\varepsilon < 1/8$, $10\kappa < \varepsilon$, $3\delta < \kappa$ and for every $n \in \mathbb{N}$ we have $\delta^{n+1} > \alpha_n$. We will construct inductively a sequence of B-systems $\{\mathcal{U}_n\}_{n=0}^{\infty}$ such that $\mathcal{U}_0 = \{\overline{B}([0, 0], 1)\}$ and for every $n \in \mathbb{N}_0$ we have

(i) $\mathcal{U}_{n+1} = \bigcup\{\mathcal{U}_{n+1}(C);\ C \in \mathcal{U}_n\}$, where $\mathcal{U}_{n+1}(C)$ has the $\mathcal{P}(n+1, \delta, \alpha_{n+1}, \varepsilon)$-property in C,

(ii) $c(\mathcal{U}_n)$ is a compact f-set with $V(c(\mathcal{U}_n)) < \Theta$, where Θ is a fixed constant, which does not depend on n,

(iii) $\operatorname{diam}((\bigcup\mathcal{U}_n)^x) \le 2/(n + 1)$ for every $x \in \pi_1(\bigcup\mathcal{U}_n)$,

(iv) there exists a system $\mathcal{D}_{n+1} = \{D_{n+1}(C);\ C \in \mathcal{U}_n\}$ of closed subsets of $\mathbb{R}^2$ such that $D_{n+1}(C)$ has the $\mathcal{C}(n+1, \delta, \kappa, \alpha_{n+1})$-property in $C \in \mathcal{U}_n$, $\bigcup \mathcal{D}_{n+1}$ is a compact f-set with the property (A) and $V(\bigcup \mathcal{D}_{n+1}) < \Theta$,

(v) $\mathcal{U}_n$ is countable,

(vi) $\lambda(\pi_2(\bigcup \mathcal{U}_n)) \leq 2/(n+1)$.

We put $\mathcal{U}_0 = \{\overline{B}([0,0],1)\}$. According to Lemma 2.18 there exists an f-set D_1 with the $\mathcal{C}(1, \delta, \kappa, \alpha_1)$-property in $\overline{B}([0,0],1)$ and with the property (A). According to Lemma 2.14 we have $V(D_1) < \infty$. Put $\Theta = V(D_1) + 1$, $D_1(\overline{B}([0,0],1)) = D_1$, $\mathcal{D}_1 = \{D_1\}$. Suppose that we have constructed B-systems $\mathcal{U}_0, \dots, \mathcal{U}_n$ with the desired properties.

Let $\mathcal{D}_{n+1}$ be a system witnessing (iv) of $\mathcal{U}_n$. Let $D_{n+1} = \bigcup \mathcal{D}_{n+1}$. There exists an open set H containing D_{n+1} such that $\lambda(\pi_2(H)) \leq 2/(n+2)$ and $\operatorname{diam}(H^x) \leq 2/(n+2)$ for every $x \in \pi_1(H)$. The first condition can be satisfied since D_{n+1} is countable (condition (v) and Observation 2.7). The second one can be satisfied since D_{n+1} is a compact f-set. Fix $\psi > 0$ with $V(D_{n+1}) + \psi < \Theta$. The B-system $\mathcal{U}_n$ is point finite. Using this and Lemma 2.9 we find a function $c : D_{n+1} \to (0, +\infty)$ such that

(a) for every $C \in \mathcal{U}_n$ and every function $d : D_{n+1} \to (0, +\infty)$, $d \leq c$, we have that $\{\overline{B}(x, d(x));\ x \in D_{n+1}(C)\}$ is a B-system with the property $\mathcal{P}(n+1, \delta, \alpha_{n+1}, \varepsilon)$ in C,

(b) $\overline{B}(x, c(x)) \subset H$ for every $x \in D_{n+1}$.

According to Lemma 2.18 there exists an f-set S_{n+2} such that

- S_{n+2} satisfies (A),
- S_{n+2} satisfies (B),
- S_{n+2} has the property $\mathcal{C}(n+2, \delta, \kappa, \alpha_{n+2})$ in $\overline{B}([0,0],1)$.

According to Lemma 2.21 we find $\mu_0 \in (0,1)$ and a function $d : D_{n+1} \to (0, +\infty)$ such that

- $d \leq c$,
- $W(D_{n+1}, S_{n+2}, d, \mu_0)$ is an f-set,
- $V(W(D_{n+1}, S_{n+2}, d, \mu_0)) < V(D_{n+1}) + \psi$.

Let $j = j_{D_{n+1},S_{n+2},d}$. We put

$$\begin{aligned}\mathcal{U}_{n+1}(C) &= \{\overline{B}(x, \mu_0 q^{j(x)});\ x \in D_{n+1}(C)\},\ C \in \mathcal{U}_n,\\ \mathcal{U}_{n+1} &= \bigcup\{\mathcal{U}_{n+1}(C);\ C \in \mathcal{U}_n\},\\ D_{n+2}(C) &= W_0(c(C), D_{n+1}, S_{n+2}, d, \mu_0),\ C \in \mathcal{U}_{n+1},\\ \mathcal{D}_{n+2} &= \{D_{n+2}(C);\ C \in \mathcal{U}_{n+1}\}.\end{aligned}$$

Using (a) and the fact that $\mu_0 < 1$ we obtain that $\mathcal{U}_{n+1}(C)$ has the property $\mathcal{P}(n+1, \delta, \alpha_{n+1}, \varepsilon)$ in C. We have also $\overline{B}(x, \mu_0 q^{j(x)}) \subset B(x, d(x)) \subset H$ for every $x \in D_{n+1}$ and therefore $\mathcal{U}_{n+1}$ satisfies (iii) and (vi), where n is replaced by $n+1$. We have $c(\mathcal{U}_{n+1}) = D_{n+1}$ and therefore $V(c(\mathcal{U}_{n+1})) < \Theta$. The set $D_{n+2}(C)$ has the $\mathcal{C}(n+2, \delta, \kappa, \alpha_{n+2})$-property in $C \in \mathcal{U}_{n+1}$ since S_{n+2} has the $\mathcal{C}(n+2, \delta, \kappa, \alpha_{n+2})$-property in $\overline{B}([0,0], 1)$. The set $D_{n+2} = \bigcup \mathcal{D}_{n+2}$ is a countable compact f-set with property (A) since $D_{n+2} = W(D_{n+1}, S_{n+2}, d, \mu_0)$. The system $\mathcal{U}_{n+1}$ is a B-system by Lemma 2.4. The B-system $\mathcal{U}_{n+1}$ is countable by Observation 2.7 and the induction hypothesis. This finishes the construction of $\mathcal{U}_n$'s.

Lemma 2.10 shows that the set $F = \bigcap_{n=0}^{\infty} \bigcup \mathcal{U}_n$ is non-σ-porous. Condition (iii) gives that F is an f-set. We have $c(\mathcal{U}_j) \subset c(\mathcal{U}_{j+1})$, $j \in \mathbb{N}_0$, the set $\bigcup_{n=0}^{\infty} c(\mathcal{U}_n)$ is dense in F and $V(c(\mathcal{U}_n)) < \Theta$ for every $n \in \mathbb{N}$. This implies that $V(F) < \infty$. We have also $\lambda(\pi_2(F)) = 0$ since $\lambda(\pi_2(F)) \le \lambda(\pi_2(\bigcup \mathcal{U}_n)) \le 2/(n+1)$ for every $n \in \mathbb{N}$.

Let $\mathcal{I}$ be the set of all bounded intervals contiguous to $\pi_1(F)$. We define a continuous function $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} y, & \text{if } x \in \pi_1(F),\ \text{where } [x,y] \in F,\\ y + \frac{z-y}{b-a}(x-a), & \text{if } x \in (a,b),\ \text{where } (a,b) \in \mathcal{I}, [a,y] \in F,\ [b,z] \in F,\\ y, & \text{if } x \in [-1, \min(\pi_1(F))), \text{ where } [\min(\pi_1(F)), y] \in F,\\ y, & \text{if } x \in (\max(\pi_1(F)), 1],\ \text{where } [\max(\pi_1(F)), y] \in F. \end{cases}$$

Our function f is continuous, $\operatorname{Var}(f) = V(F) < \infty$ and $F \subset \operatorname{graph}(f)$. The Banach-Zarecki theorem (see [S]) says that continuous function of bounded variation with Luzin's (N) property (i.e., each measure zero set is mapped onto a measure zero set) is absolutely continuous. Our function has Luzin's (N) property since $\lambda(f(\pi_1(F))) = \lambda(\pi_2(F)) = 0$ and f is linear on each $I \in \mathcal{I}$. Thus f is an absolutely continuous function with non-σ-porous graph. □

The interval $[0, 2\pi]$, where the endpoints are identified, is denoted by $\mathbb{T}$.

Theorem 3.1. *There exists a continuous function $f : \mathbb{T} \to \mathbb{R}$ with non-σ-porous graph such that its Fourier series converges uniformly on $\mathbb{T}$.*

PROOF. According to Theorem 1.1 there exists a continuous function $f : \mathbb{T} \to \mathbb{R}$ of bounded variation with non-σ-porous graph. Fourier series of f converges uniformly to f on $\mathbb{T}$ by Jordan – Dirichlet theorem. $\square$

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