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ON MODULI OF SMOOTHNESS OF FRACTIONAL ORDER

Abstract

In this paper we consider the properties of moduli of smoothness of fractional order. The main result of the paper describes the equivalence of the modulus of smoothness and a function from some class.

1 Introduction.

In 1977 P. L. Butzer, H. Dyckhoff, E. Goerlich, R. L. Stens (see [2]) and R. Tabersky (see [14]) introduced the modulus of smoothness of fractional order. This notion can be considered as a direct generalization of the classical modulus of smoothness and is more natural to use for a number of problems in harmonic analysis (see, for example, [2], [5], [7], [10]).

An important problem in approximation theory and theory of Fourier series is the description of the moduli of smoothness (see [1], [4], [8], [11]). One can consider this problem from the viewpoint of description of majorant of smoothness moduli. Recently, A. Medvedev (see [6]) has proved that for any modulus of continuity on $[0,\infty)$ there exists a concave majorant that is infinitely differentiable. In this paper, we obtain the description of the modulus of smoothness of fractional order from the viewpoint of the order of decreasing to zero of the modulus of smoothness.

Let us introduce some definitions. If $p \in [1,\infty)$, let L_p be the space of measurable, 2π-periodic functions $f(x)$ such that $\|f\|_p = \left(\int\limits_0^{2\pi} |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty$.

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Similarly, let L_∞ be the space of 2π-periodic, continuous functions $f(x)$ with $\|f\|_\infty = \max\limits_{x\in[0,2\pi]} |f(x)|$. We define the difference of fractional order $\beta\,(\beta > 0)$ of the function $f(x)$ at the point $x\,(x \in \mathbb{R})$ with increment $h\,(h \in \mathbb{R})$ by

$$\triangle_h^\beta f(x) = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\beta}{\nu} f(x + (\beta - \nu)h),$$

where $\binom{\beta}{\nu} = \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!}$ for $\nu > 1$, $\binom{\beta}{\nu} = \beta$ for $\nu = 1$, $\binom{\beta}{\nu} = 1$ for $\nu = 0$.

The modulus of smoothness of order $\beta\,(\beta > 0)$ of the function $f \in L_p$, $1 \le p \le \infty$, is given by $\omega_\beta(f,t)_p = \sup\limits_{|h|\le t} \left\| \triangle_h^\beta f(\cdot) \right\|_p$ (see [2],[14]).

Let Φ_γ $(\gamma \in \mathbb{R})$ be the set of nonnegative, bounded functions $\varphi(\delta)$ on $(0,\infty)$ such that:

a) $\varphi(\delta) \to 0$ as $\delta \to 0$,

b) $\varphi(\delta)$ is nondecreasing,

c) $\varphi(\delta)\delta^{-\gamma}$ is nonincreasing.

If for $f \in L_p$ there exists $g \in L_p$ such that $\lim\limits_{h\to 0+} \left\| h^{-\beta} \triangle_h^\beta f(\cdot) - g(\cdot) \right\|_p = 0$, then g is called the Liouville-Grunwald-Letnikov derivative of order $\beta > 0$ of the function f in the L_p-norm, denoted by $g = D^\beta f$ (see [2], [12]). Set $W_p^\beta := \left\{ f \in L_p : D^\beta f \text{ exists as element in } L_p \right\}$. The K-functional is given by $K(f,t,L_p,W_p^\beta) := \inf\limits_{g\in W_p^\beta} \left(\|f-g\|_p + t \left\| D^\beta g \right\|_p \right)$.

2 Results.

Let $f(x) \in L_p$, $p \in [1,\infty]$ and $\beta > 0$. It is clear that (see [12])

$$\left| \binom{\beta}{\nu} \right| = \left| \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!} \right| \le \frac{C(\beta)}{\nu^{\beta+1}},\ \nu \in \mathbb{N}$$

implies $C^*(\beta) := \sum\limits_{\nu=0}^{\infty} \left| \binom{\beta}{\nu} \right| < \infty$ and the fractional difference $\triangle_h^\beta f(x)$ is defined almost everywhere, belongs to L_p and

$$\|\triangle_h^\beta f(\cdot)\|_p \le C^*(\beta)\, \|f(\cdot)\|_p. \tag{1}$$

It is easy to write the following representation for $C^*(\beta)$ (see [14]).

$$C^*(\beta) = \begin{cases} 2\sum\limits_{\nu=0}^{k} \binom{\beta}{2\nu}, & \text{if } 2k < \beta \leq 2k+1 \, (k = 0,1,2,\dots), \\ 2\sum\limits_{\nu=0}^{k} \binom{\beta}{2\nu+1}, & \text{if } 2k+1 < \beta \leq 2k+2 \, (k = 0,1,2,\dots). \end{cases} \tag{2}$$

The fractional differences and moduli of smoothness have some useful properties and we shall establish some of them in the following lemmas.

Lemma 2.1. ([2], [14]) *Let $f \in L_p$, $p \in [1,\infty]$, $\alpha, \beta > 0$; $h \in \mathbb{R}$. Then*

(a) $\triangle_h^\alpha(\triangle_h^\beta f(x)) = \triangle_h^{\alpha+\beta} f(x)$ *for almost every* x;

(b) $\|\triangle_h^{\alpha+\beta} f(\cdot)\|_p \leq C^*(\alpha)\|\triangle_h^\beta f(\cdot)\|_p$;

(c) $\lim\limits_{h\to 0+} \|\triangle_h^\alpha f(\cdot)\|_p = 0$.

Lemma 2.2. *Let $f, f_1, f_2 \in L_p$, $p \in [1,\infty]$, $\alpha, \beta > 0$; $x, h \in \mathbb{R}$. Then*

(a) $\omega_\beta(f,\delta)_p$ *is nondecreasing nonnegative function of* δ *on* $(0,\infty)$ *with* $\lim\limits_{\delta\to 0+} \omega_\beta(f,\delta)_p = 0$;

(b) $\omega_\beta(f_1+f_2,\delta)_p \leq \omega_\beta(f_1,\delta)_p + \omega_\beta(f_2,\delta)_p$;

(c) $\omega_{\alpha+\beta}(f,\delta)_p \leq C^*(\alpha)\omega_\beta(f,\delta)_p$;

(d) *if* $\lambda \geq 1$, *then* $\omega_\beta(f,\lambda\delta)_p \leq C(\beta)\lambda^\beta \omega_\beta(f,\delta)_p$;

(e) *if* $0 < t \leq \delta$, *then* $\omega_\beta(f,\delta)_p \, \delta^{-\beta} \leq C(\beta)\omega_\beta(f,t)_p \, t^{-\beta}$.

Indeed, we immediately have $(a)-(c)$ from Lemma 2.1, (d) was proved in [2], and (d) implies (e).

$$\omega_\beta(f,\delta)_p = \omega_\beta\left(f, \frac{\delta}{t}t\right)_p \leq C(\beta)\left(\frac{\delta}{t}\right)^\beta \omega_\beta(f,t)_p.$$

Lemma 2.3. *Let $f \in L_p$, $p \in [1,\infty]$, $\beta > 0$.*

(a) *If* $\beta \in \mathbb{N}$, *then* $\left\|\triangle_\pi^\beta f(\cdot)\right\|_p \leq 2^{[\frac{\beta+1}{2}]} \left\|\triangle_{\frac{\pi}{2}}^\beta f(\cdot)\right\|_p$.

(b) *If* $\beta \notin \mathbb{N}$, *then* $\left\|\triangle_\pi^\beta f(\cdot)\right\|_p \leq 2^{[\frac{\beta+1}{2}]+1} \left\|\triangle_{\frac{\pi}{2}}^\beta f(\cdot)\right\|_p$.

Corollary 2.4. *For a function $\varphi(t) = t^{\alpha}\,(0 \le t \le \pi)$ to be a modulus of smoothness of order $\beta\,(\beta > 0)$ of a function $f(\cdot) \in L_p,\ 1 \le p \le \infty$ it is necessary to have $\alpha \le \left[\frac{\beta+1}{2}\right] + 1$.*

Theorem 2.5. *Let $p \in [1, \infty]$, $\beta > 0$.*

(A) *If $f(\cdot) \in L_p$, then there exists a function $\varphi(\cdot) \in \Phi_\beta$ such that*

$$\varphi(t) \le \omega_\beta(f, t)_p \le C(\beta)\varphi(t)\,(0 < t < \infty),$$

where $C(\beta)$ is a positive constant depending only on β.

(B) *If $\varphi(\cdot) \in \Phi_\beta$, then there exist a function $f(\cdot) \in L_p$ and a constant $t_1 > 0$ such that*

$$C_1(\beta)\omega_\beta(f, t)_p \le \varphi(t) \le C_2(\beta)\omega_\beta(f, t)_p\,(0 < t < t_1),$$

where $C_1(\beta), C_2(\beta)$ are positive constants depending only on β.

Corollary 2.6. *Let $p \in [1, \infty]$, $\beta > 0$.*

(A) *If $f(\cdot) \in L_p$, then there exists a function $\varphi(\cdot) \in \Phi_\beta$ such that*

$$C_1(\beta)\varphi(t) \le K(f, t^\beta, L_p, W_p^\beta) \le C_2(\beta)\varphi(t). \tag{3}$$

(B) *If $\varphi(\cdot) \in \Phi_\beta$, then there exists a function $f(\cdot) \in L_p$ such that (3) is true.*

Remark 2.7. 1). We can replace condition $f \in L_p$ by condition $f \in L_\infty$ in the part (B) of Theorem 2.5.
2). Note that theorem 2.5 for $\beta \in \mathbb{N}$ was proved in [11]. Also, for H^p-spaces the analogue of Corollary 2.6 for $\beta \in \mathbb{R}_+$ and the analogue of theorem 2.5 for $\beta \in \mathbb{N}$ were proved in [5].

3 Proofs.

Proof of Lemma 2.3. The first inequality was proved in [3]. Let $\beta > 1,\ \notin \mathbb{N}$. We shall use the following representation (see [14]).

$$\triangle_{2h}^{\beta} f(x - 2\beta h) = \sum_{\nu=0}^{\infty} \binom{\beta}{\nu} \triangle_h^{\beta} f(x - \beta h - \nu h) \text{ for almost every } x \tag{4}$$

By Lemma 2.1(a) and part (a) of this Lemma, it follows that

$$\begin{aligned}\left\|\triangle_\pi^\beta f(\cdot)\right\|_p &= \left\|\left(\triangle_\pi^{[\beta]}(\triangle_\pi^{\beta-[\beta]} f)\right)(\cdot)\right\|_p \\ &\le 2^{\left[\frac{[\beta]+1}{2}\right]} \left\|\left(\triangle_{\frac{\pi}{2}}^{[\beta]}(\triangle_\pi^{\beta-[\beta]} f)\right)(\cdot)\right\|_p .\end{aligned}$$

Here we use (4) for $h=\frac{\pi}{2}$. We have

$$\left\|\triangle_{\pi}^{\beta}f(\cdot)\right\|_p \leq 2^{\left[\frac{[\beta]+1}{2}\right]}\left\|\triangle_{\frac{\pi}{2}}^{[\beta]}\left\{\sum_{\nu=0}^{\infty}\binom{\beta-[\beta]}{\nu}\triangle_{\frac{\pi}{2}}^{\beta-[\beta]}f\right\}(\cdot-\frac{\beta\pi}{2}-\frac{\nu\pi}{2})\right\|_p$$
$$=2^{\left[\frac{[\beta]+1}{2}\right]}\left\|\sum_{\nu=0}^{\infty}\binom{\beta-[\beta]}{\nu}\left(\triangle_{\frac{\pi}{2}}^{[\beta]}(\triangle_{\frac{\pi}{2}}^{\beta-[\beta]}f)\right)(\cdot)\right\|_p.$$

Thus, by Lemma 2.1(a) and inequality (1), we get

$$\left\|\triangle_{\pi}^{\beta}f(\cdot)\right\|_p \leq C^*(\beta-[\beta])\,2^{\left[\frac{[\beta]+1}{2}\right]}\left\|\triangle_{\frac{\pi}{2}}^{\beta}f(\cdot)\right\|_p.$$

If we combine this result with $C^*(\beta-[\beta])=2$ (see (2)) and $2^{\left[\frac{[\beta]+1}{2}\right]}=2^{\left[\frac{\beta+1}{2}\right]}$, we obtain the required inequality. If $0<\beta<1$, then we use (1) and (4). □

We will need the following lemma.

Lemma 3.1. *Let $\beta>0$, $n\in\mathbb{N}$, $\delta>0$.*

(a) *If $f(x)=\sin x$ and $p\in[1,\infty]$, then there exist $t_1>0$ and $C_1(\beta),C_2(\beta)>0$ such that for any $\delta\in(0,t_1)$ we have*

$$C_1(\beta)\delta^{\beta}\leq\omega_{\beta}(f,\delta)_p\leq C_2(\beta)\delta^{\beta}. \tag{5}$$

(b) *If $f(x)=\sin nx$ and $p\in[1,\infty]$, then for any $\delta\in\left(0,\frac{\pi}{2}\right]$ we have*[1] $\|\triangle_{\delta}^{\beta}f(\cdot)\|_p\leq(2\pi)^{\frac{1}{p}}(n\delta)^{\beta}$.

(c) *If $f(x)=\sin nx$, then $\|\triangle_{\pi/n}^{\beta}f(\cdot)\|_1=2^{\beta+2}$.*

(d) *If $f(x)=\sin nx$, then for any $\delta\in\left(0,\frac{\pi}{n}\right]$ we have $\|\triangle_{\delta}^{\beta}f(\cdot)\|_1\geq 4\left(\frac{2}{\pi}\right)^{\beta}(\delta n)^{\beta}$.*

Proof of Lemma 3.1. Let $T_n(x)=\sum\limits_{\nu=-n}^{n}c_{\nu}e^{i\nu x}$. Then

$$\triangle_{\delta}^{\beta}T_n(x-\frac{\beta\delta}{2})=\sum_{\nu=-n}^{n}\left(2i\sin\frac{\nu\delta}{2}\right)^{\beta}c_{\nu}e^{i\nu x}.$$

Thus, for $f(x)=\sin nx$, $n\in\mathbb{N}$, we get

$$\triangle_{\delta}^{\beta}f(x-\frac{\beta\delta}{2})=\left(2\sin\frac{n\delta}{2}\right)^{\beta}\sin\left(nx+\frac{\beta\pi}{2}\right). \tag{6}$$

[1] Here $\frac{1}{\infty}=0$.

For $n=1$ we obviously have

$$C_1(\beta)\left(2\left|\sin\frac{\delta}{2}\right|\right)^\beta \le \|\triangle_\delta^\beta \sin(\cdot)\|_p \le C_2(\beta)\left(2\left|\sin\frac{\delta}{2}\right|\right)^\beta .$$

If we combine this inequality with $\sin t \le t\,(t\ge 0)$ and $\sin t \ge \frac{2t}{\pi}\,(0\le t\le \frac{\pi}{2})$, then we obtain (5). In the same way, by (6), we shall have the proofs of $(b)-(d)$. □

PROOF OF THEOREM 2.5. (A). Let $\varphi(t) := t^\beta \inf\limits_{0<\xi\le t}\{\xi^{-\beta}\omega_\beta(f,\xi)_p\}$. We immediately have $\varphi(t)\in\Phi_\beta$ from [13, §2]. It is trivial, that $\varphi(t)\le\omega_\beta(f,t)_p$. By Lemma 2.2(e), we have

$$\omega_\beta(f,t)_p = t^\beta t^{-\beta}\omega_\beta(f,t)_p \le C(\beta)t^\beta \inf_{0<\xi\le t}\left\{\xi^{-\beta}\omega_\beta(f,\xi)_p\right\} = C(\beta)\varphi(t).$$

Therefore, for any $t>0$ the inequality $\varphi(t)\le\omega_\beta(f,t)_p\le C(\beta)\varphi(t)$ holds and (A) follows.

(B). *Case 1.* Let $\lim\limits_{t\to 0}\frac{\varphi(t)}{t^\beta} = C\,(0\le C<\infty)$. Then, by virtue of monotonicity of $\frac{\varphi(t)}{t^\beta}$, we write
$(*)$ $\varphi(t)\le Ct^\beta$ for $0<t\le\pi$;
$(**)$ there exists $t_1>0$ such that $\varphi(t)\ge\frac{Ct^\beta}{2}$ for $0<t\le t_1$.
Indeed, $(*)$ is trivial like $(**)$ for $C=0$. If $C>0$ and $\lim\limits_{t\to 0}\frac{\varphi(t)}{t^\beta}=C$, then for any $\varepsilon>0$ there exists $t_1>0$ such that $C-\frac{\varphi(t)}{t^\beta}\le\varepsilon$ for $0<t\le t_1$. Then $\frac{\varphi(t)}{t^\beta}\ge C-\varepsilon$, and choosing small ε we have $(**)$.

Let $f(x)=C\sin x$. By Lemma 3.1(a), we have

$$\omega_\beta(f,\delta)_p \ge CC_1(\beta)\delta^\beta \ge C_2(\beta)\varphi(\delta) \text{ for } 0<\delta\le\pi,$$
$$\omega_\beta(f,\delta)_p \le CC_3(\beta)\delta^\beta \le C_4(\beta)\varphi(\delta) \text{ for } 0<\delta\le t_1,$$

completing the proof in this case.

Case 2. Let $\lim\limits_{t\to 0}\frac{\varphi(t)}{t^\beta}=+\infty$. Then $\lim\limits_{t\to 0}\varphi(t)=0$ and $\lim\limits_{t\to 0}\frac{t^\beta}{\varphi(t)}=0$. We fix $a\ge 2$. Then, following Oskolkov ([9]), we define the sequence $\{n_\nu\}_{\nu=1}^\infty$, where $n_\nu=2^{m_\nu}$ are the numbers m_ν such that

$$m_1=2,\ m_{\nu+1}=\min\left\{m\in\mathbb{N}\colon \max\left(\frac{\varphi(2^{-m})}{\varphi(2^{-m_\nu})},\frac{2^{m_\nu\beta}\varphi(2^{-m_\nu})}{2^{m\beta}\varphi(2^{-m})}\right)\le\frac{1}{a}\right\}\ (\nu\in\mathbb{N}).$$

From the definition of $\{n_\nu\}_{\nu=1}^\infty$ it follows that $m_{\nu+1}>m_\nu$, $n_{\nu+1}\ge 2n_\nu$ and for any $\nu\in\mathbb{N}$ we have

$$\varphi\left(\frac{1}{n_{\nu+1}}\right)\le\frac{1}{a}\varphi\left(\frac{1}{n_\nu}\right); \tag{7}$$

$$n_\nu^\beta \varphi\left(\frac{1}{n_\nu}\right) \le \frac{1}{a} n_{\nu+1}^\beta \varphi\left(\frac{1}{n_{\nu+1}}\right). \tag{8}$$

Let us fix $\varkappa = 2^d$ ($d \in \mathbb{N}$) such that $\varkappa > 2\pi$. Note that (7) implies

$$\sum_{\nu=1}^\infty \varphi\left(\frac{1}{n_\nu}\right) \le \varphi\left(\frac{1}{n_1}\right) \sum_{\nu=1}^\infty a^{1-\nu} < \infty,$$

and, therefore, we can define the function $f(x) = \sum\limits_{\nu=1}^\infty \varphi\left(\frac{1}{n_\nu}\right) \sin(\varkappa n_\nu x)$.

First, we shall estimate $\omega_\beta(f,\delta)_p$ from above. By the inequality

$$\|f\|_p \le (2\pi)^{\frac{1}{p}} \|f\|_\infty \le 2\pi \|f\|_\infty,\ p \in [1,\infty),$$

it is enough to prove $\omega_\beta(f,\delta)_\infty \le C(\beta)\varphi(\delta)$. Let $\delta \in (0, \frac{1}{n_1}]$. For all $h \in (0, \frac{1}{n_1}]$ we can find the number $N \in \mathbb{N}$ such that $\frac{1}{n_{N+1}} < h \le \frac{1}{n_N}$. Then

$$\begin{aligned}\left\|\triangle_h^\beta f(x)\right\|_\infty \le & \left\|\sum_{\nu=1}^N \varphi\left(\frac{1}{n_\nu}\right) \triangle_h^\beta \sin(\varkappa n_\nu x)\right\|_\infty + \left\|\sum_{\nu=N+1}^\infty \varphi\left(\frac{1}{n_\nu}\right) \triangle_h^\beta \sin(\varkappa n_\nu x)\right\|_\infty \\ =: & I_1 + I_2.\end{aligned}$$

Combining Lemma 3.1(b), inequality (8), and condition (c) in the definition of Φ_β, we get

$$\begin{aligned} I_1 \le & \sum_{\nu=1}^N \varphi\left(\frac{1}{n_\nu}\right) \left\|\triangle_h^\beta \sin(\varkappa n_\nu x)\right\|_\infty \le C(\beta)\, (\varkappa h)^\beta \varphi\left(\frac{1}{n_N}\right) n_N^\beta \sum_{\nu=1}^N a^{-(N-\nu)} \\ \le & C(\beta)\, (n_N h)^\beta \varphi\left(\frac{1}{n_N}\right) \le C(\beta)\varphi(h). \end{aligned}$$

Inequalities (1) and (7) yield

$$\begin{aligned} I_2 \le & \sum_{\nu=N+1}^\infty \varphi\left(\frac{1}{n_\nu}\right) \left\|\triangle_h^\beta \sin(\varkappa n_\nu x)\right\|_\infty \le C(\beta) \sum_{\nu=N+1}^\infty \varphi\left(\frac{1}{n_\nu}\right) \\ \le & C(\beta)\varphi\left(\frac{1}{n_{N+1}}\right) \sum_{\nu=N+1}^\infty a^{N+1-\nu} \le C(\beta)\varphi\left(\frac{1}{n_{N+1}}\right) \le C(\beta)\varphi(h). \end{aligned}$$

Therefore, if $h \in (\frac{1}{n_{N+1}}, \frac{1}{n_N}]$, $N \in \mathbb{N}$, then $\|\triangle_h^\beta f(x)\|_\infty \le C(\beta)\varphi(h)$, which implies $\omega_\beta(f,\delta)_\infty \le C(\beta)\varphi(\delta)$. Now we shall obtain the inequality $\varphi(\delta) \le C(\beta)\omega_\beta(f,\delta)_p$. From the inequality $\|f\|_1 \le 2\pi\|f\|_p$, $p \in [1,\infty]$ it is sufficient to prove $\varphi(\delta) \leq C(\beta)\omega_\beta(f,\delta)_1$. Also, we note that if the last inequality holds for $\delta = \frac{\pi}{2^k}$, $k = N, N+1, N+2, \dots$, where $N \in \mathbb{N}$, then it holds for $\delta \in \left(\frac{\pi}{2^k}, \frac{\pi}{2^{k+1}}\right)$. Indeed, from the monotonicity of $t^{-\beta}\varphi(t)$, we see that the estimate $\varphi(\delta) \leq C(\beta)\varphi\left(\frac{\pi}{2^k}\right)$ is true. By Lemma 2.2(a), we get

$$\varphi(\delta) \leq C(\beta)\varphi\left(\frac{\pi}{2^k}\right) \leq C(\beta)\omega_\beta\left(f, \frac{\pi}{2^k}\right)_1 \leq C(\beta)\omega_\beta(f,\delta)_1.$$

To go further, we suppose that $\delta = \frac{\pi}{2^k}$.

Let M be the integer, $M > 1$, and let $h_1 = \frac{\pi}{\varkappa n_M}$. We shall show that

$$\left\| \triangle_{h_1}^\beta f(x) \right\|_1 \geq 4\varphi\left(\frac{1}{n_M}\right)\left(2^\beta - \frac{\pi^{\beta+1}}{a}\right). \tag{9}$$

For this purpose, we shall use the representation of a function $f(x)$

$$\begin{aligned} f(x) = &\sum_{\nu=1}^{M-1}\varphi\left(\frac{1}{n_\nu}\right)\sin(\varkappa n_\nu x) + \varphi\left(\frac{1}{n_M}\right)\sin(\varkappa n_M x) + \sum_{\nu=M+1}^{\infty}\varphi\left(\frac{1}{n_\nu}\right)\sin(\varkappa n_\nu x) \\ =: &f_1 + f_2 + f_3. \end{aligned}$$

Note that $\sin(\varkappa n_\nu x + \frac{\pi n_\nu}{n_M}) = \sin(\varkappa n_\nu x)$ for $\nu > M$, and $f_3(x)$ has the period $T = h_1 = \frac{\pi}{\varkappa n_M}$. We therefore obtain

$$\triangle_{h_1}^\beta f_3(x) = f(x + \beta h_1)\sum_{\xi=0}^{\infty}(-1)^\xi\binom{\beta}{\xi} = 0.$$

By Lemma 3.1(b) and (8), we have

$$\begin{aligned} \left\| \triangle_{h_1}^\beta f_1(x) \right\|_1 &\leq \sum_{\nu=1}^{M-1}\varphi\left(\frac{1}{n_\nu}\right)\left\| \triangle_{h_1}^\beta \sin(\varkappa n_\nu x) \right\|_1 \leq \sum_{\nu=1}^{M-1} 2\pi\left(\varkappa n_\nu h_1\right)^\beta\varphi\left(\frac{1}{n_\nu}\right) \\ &= 2\pi\left(\frac{\pi}{n_M}\right)^\beta\sum_{\nu=1}^{M-1}\varphi\left(\frac{1}{n_\nu}\right)n_\nu^\beta \\ &\leq 2\pi\left(\frac{\pi}{n_M}\right)^\beta\varphi\left(\frac{1}{n_{M-1}}\right)n_{M-1}^\beta\sum_{\nu=1}^{M-1}a^{-(M-1-\nu)}. \end{aligned}$$

Using $\sum_{\nu=1}^{M-1}a^{-(M-1-\nu)} \leq 2$ and (8), we obtain $\left\| \triangle_{h_1}^\beta f_1(x) \right\|_1 \leq \frac{4\pi^{\beta+1}}{a}\varphi\left(\frac{1}{n_M}\right)$.

By Lemma 3.1(c), $\left\| \triangle_{h_1}^\beta f_2(x) \right\|_1 = \varphi\left(\frac{1}{n_M}\right)\left\| \triangle_{h_1}^\beta \sin(\varkappa n_M x) \right\|_1 = 2^{\beta+2}\varphi\left(\frac{1}{n_M}\right)$.

Therefore, for $h_1 = \frac{\pi}{\varkappa n_M}$, the inequality $|f| \geq |f_2| - |f_1| - |f_3|$ implies

$$\begin{aligned}\left\| \triangle_{h_1}^{\beta} f(x) \right\|_1 &\geq \left\| \triangle_{h_1}^{\beta} f_2(x) \right\|_1 - \left\| \triangle_{h_1}^{\beta} f_1(x) \right\|_1 - \left\| \triangle_{h_1}^{\beta} f_3(x) \right\|_1 \\ &= \left\| \triangle_{h_1}^{\beta} f_2(x) \right\|_1 - \left\| \triangle_{h_1}^{\beta} f_1(x) \right\|_1 \geq 4\varphi\left(\frac{1}{n_M}\right)\left(2^{\beta} - \frac{\pi^{\beta+1}}{a}\right);\end{aligned}$$

i.e., we obtain (9). Further, we choose the integer i such that

$$\frac{1}{n_{i+1}} = \frac{1}{2^{m_{i+1}}} < \delta \leq \frac{1}{2^{m_i}} = \frac{1}{n_i}.$$

Note that, by definition of m_i, at the least one of the following inequalities is true:

$$2^{\beta(m_{i+1}-1)}\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) < a2^{\beta m_i}\varphi\left(\frac{1}{2^{m_i}}\right), \tag{10}$$

$$\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) > \frac{1}{a}\varphi\left(\frac{1}{2^{m_i}}\right) \tag{11}$$

Case 2(a). Let (10) hold. Using the monotonicity of $\varphi(t)$ and (10), we get

$$\begin{aligned} n_{i+1}^{\beta}\varphi\left(\frac{1}{n_{i+1}}\right) &\leq 2^{\beta}2^{\beta(m_{i+1}-1)}\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) \\ &< a\,2^{\beta}n_i^{\beta}\varphi\left(\frac{1}{n_i}\right).\end{aligned} \tag{12}$$

We write

$$\begin{aligned} f(x) &= \sum_{\nu=1}^{i-1}\varphi\left(\frac{1}{n_\nu}\right)\sin(\varkappa n_\nu x) + \varphi\left(\frac{1}{n_i}\right)\sin(\varkappa n_i x) + \sum_{\nu=i+1}^{\infty}\varphi\left(\frac{1}{n_\nu}\right)\sin(\varkappa n_\nu x) \\ &=: f_1 + f_2 + f_3.\end{aligned}$$

It is clear, that the function f_3 has a period $T = \frac{2\pi}{\varkappa n_{i+1}}$. Then, for $\varkappa = 2^d > 2\pi$ we have $\delta = \frac{\pi}{2^r} > \frac{1}{n_{i+1}} > T$. Therefore, f_3 has a period δ and $\triangle_{\delta}^{\beta} f_3(x) = 0$.

For $0 < \delta \leq \frac{\pi}{\varkappa n_i}$, by Lemma 3.1(d), we have

$$\left\| \triangle_{\delta}^{\beta} f_2(x) \right\|_1 = \varphi\left(\frac{1}{n_i}\right)\left\| \triangle_{\delta}^{\beta}\sin(\varkappa n_i x) \right\|_1 \geq 4\left(\frac{2}{\pi}\right)^{\beta}\varphi\left(\frac{1}{n_i}\right)(\varkappa n_i \delta)^{\beta}.$$

Using Lemma 3.1(b) and inequality (8)

$$\left\|\triangle_{h_1}^{\beta} f_1(x)\right\|_1 \leq \sum_{\nu=1}^{i-1} \varphi\left(\frac{1}{n_\nu}\right) \left\|\triangle_{\delta}^{\beta} \sin(\varkappa n_\nu x)\right\|_1 \leq \sum_{\nu=1}^{i-1} 2\pi \left(\varkappa n_\nu \delta\right)^{\beta} \varphi\left(\frac{1}{n_\nu}\right)$$
$$\leq 4\pi \left(\varkappa n_{i-1}\delta\right)^{\beta} \varphi\left(\frac{1}{n_{i-1}}\right) \leq 4\pi \left(\varkappa n_i \delta\right)^{\beta} \frac{1}{a} \varphi\left(\frac{1}{n_i}\right).$$

For $\frac{1}{n_{i+1}} < \delta \leq \frac{\pi}{\varkappa n_i}$ we obtain

$$\left\|\triangle_{\delta}^{\beta} f(x)\right\|_1 \geq \left\|\triangle_{\delta}^{\beta} f_2(x)\right\|_1 - \left\|\triangle_{\delta}^{\beta} f_1(x)\right\|_1$$
$$\geq \varphi\left(\frac{1}{n_i}\right) \left(\varkappa n_i \delta\right)^{\beta} \left\{4\left(\frac{2}{\pi}\right)^{\beta} - \frac{4\pi}{a}\right\}.$$

Now we choose a such that $2^{\beta} - \frac{\pi^{\beta+1}}{a} = \gamma_1 > 0$. (Then $4\left(\frac{2}{\pi}\right)^{\beta} - \frac{4\pi}{a} = \gamma_2 > 0$.) From (12) and condition (c) in the definition of Φ_β, we have

$$\left(\delta n_i\right)^{\beta} \varphi\left(\frac{1}{n_i}\right) \geq \left(\frac{\delta n_{i+1}}{2}\right)^{\beta} \frac{1}{a} \varphi\left(\frac{1}{n_{i+1}}\right) \geq 2^{-\beta} \frac{1}{a} \varphi\left(\delta\right).$$

Thus, the inequality $\omega_\beta(f,\delta)_p \geq C(\beta)\varphi(\delta)$ holds for $\frac{1}{n_{i+1}} < \delta \leq \frac{\pi}{\varkappa n_i}$. If $\frac{\pi}{\varkappa n_i} < \delta \leq \frac{1}{n_i}$, then (9) implies

$$\omega_\beta(f,\delta)_p \geq \omega_\beta\left(f, \frac{\pi}{\varkappa n_i}\right)_p \geq C(\beta)\varphi\left(\frac{1}{n_i}\right) \geq C(\beta)\varphi\left(\delta\right).$$

The theorem has been proved in case $2(a)$.

Case 2(b). Let (11) hold. By virtue of the monotonicity of $\frac{\varphi(t)}{t^{\beta}}$, we write $\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) \leq 2^{\beta}\varphi\left(\frac{1}{2^{m_{i+1}}}\right)$.

Hence,

$$\varphi\left(\frac{1}{n_{i+1}}\right) = \varphi\left(\frac{1}{2^{m_{i+1}}}\right) \geq 2^{-\beta}\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right)$$
$$> \frac{2^{-\beta}}{a}\varphi\left(\frac{1}{2^{m_i}}\right) = \frac{2^{-\beta}}{a}\varphi\left(\frac{1}{n_i}\right). \qquad (13)$$

From (9) and (13) it follows that

$$\omega_\beta\left(f,\delta\right)_1 \geq \omega_\beta\left(f, \frac{1}{n_{i+1}}\right)_1 \geq \omega_\beta\left(f, \frac{\pi}{\varkappa n_{i+1}}\right)_1$$
$$\geq C(\beta)\varphi\left(\frac{1}{n_{i+1}}\right) \geq C(\beta)\varphi\left(\frac{1}{n_i}\right) \geq C(\beta)\varphi\left(\delta\right).$$

This completes the proof of case 2(b) and Theorem 2.5. □

Proof of Corollary 2.6. The proof follows from (see [2])

$$C_1(\beta)\omega_\beta(f,t)_p \le K(f,t^\beta,L_p,W_p^\beta) \le C_2(\beta)\omega_\beta(f,t)_p.$$

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