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AN ANSWER TO SOME QUESTIONS OF ENE

Abstract

Four approximately continuous generalizations of the wide Denjoy integral are considered and a complete chart of relations between them is given.

1 Introduction.

In 1939 Tolstov [21] gave an example of a function integrable in the wide Denjoy sense, but not integrable in the sense of Burkill's approximately continuous integral. An easy counterexample in the opposite direction was already well known by that time: an approximately differentiable discontinuous function which being a Burkill's primitive is not a wide Denjoy primitive. So the problem of defining an integral that would include both of these integrals arose.

The first solution was due to Ridder [15]. He generalized the wide Denjoy integral (β-integral [16]) and Burkill's integral (D_4-integral [15]). The same definitions (respectively AD-integral [7] and AP^*-integral [8]) one can find in papers by Kubota. Ridder and Kubota, then claimed to have proved that these two generalizations are equivalent. However, they used a similar fallacious argument to justify this claim. There were then a few incorrect attempts to repair Ridder and Kubota's proof (for details see [5, 19]). Eventually Lee [10] and Sarkhel [19] proved that Burkill's integral is included in Ridder's β-integral (Kubota's AD-integral).

It is still unknown whether the approximate Henstock integral (or the approximate Perron integral) is a *strict* generalization of Burkill's integral. So, the problem we mentioned above may be replaced by the following one. *Define an integral that includes both the wide Denjoy and the approximate Henstock integrals.* The β-integral (AD-integral) is a solution of this problem, as was

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shown by Lu [11]. Before Lu's proof was known, Gordon suggested another solution: the so-called $AK_{\mathcal{N}}$-integral [5], which turned out to be strictly more general than the β-integral (*AD*-integral).

The other (and earlier) solution of the first problem is due to Sarkhel and Kar. Using an *orderly connected topology* [20] or a more general *abstract limit process* [18] T, they defined the *TD*-, [18], and *TP*-integrals [20, 17], respectively of Denjoy (constructive) and Perron types, and proved that these integrals are equivalent. Taking for T the approximate limit process, T_{ap}, they obtained an integral strictly more general than the β-integral (*AD*-integral) [20]. A descriptive definition of this integral was also given.

In this paper we want to give a complete chart of relations between the integrals mentioned above: the β-integral (*AD*-integral), the $T_{\mathrm{ap}}D$-integral, and the $AK_{\mathcal{N}}$-integral. This will be an answer to questions asked by the late Vasile Ene.

2 Preliminaries.

Symbols $|E|$, $\operatorname{int} E$, $\operatorname{cl} E$, $\operatorname{fr} E$ denote the Lebesgue outer measure, interior, closure, and boundary of a set $E \subset \mathbb{R}$, respectively. The family of sets $\{E_n\}_{n=1}^{\infty}$ is called an *E-form* if $\bigcup_{n=1}^{\infty} E_n = E$.

If $F\colon E \to \mathbb{R}$ and $A \subset E$ is non-void, then $\omega_F(A)$, $F \restriction A$ denote the oscillation of F on A and the restriction of F to A, respectively. We write $\mathcal{C}_F$ and $\mathcal{D}_F$ for the set of points $x \in E$ at which F is respectively continuous and discontinuous. We say that F satisfies condition $\mathcal{N}$ if $|F(N)| = 0$ for every $N \subset E$ with $|N| = 0$.

We assume that notions of *AC*- and *VB*-function on a set $E \subset \mathbb{R}$ are known to the reader. We say that an $F\colon \langle a,b\rangle \to \mathbb{R}$ is [*ACG*]-, [*VBG*]-, *ACG*-, and *VBG*-function, if there is an $\langle a,b\rangle$-form $\{E_n\}_{n=1}^{\infty}$ such that for each n

- F is *AC* on E_n and E_n is closed,
- F is *VB* on E_n and E_n is closed,
- F is *AC* on E_n,
- F is *VB* on E_n,

respectively. We say that F is approximately continuous if it is continuous with respect to the density topology. The approximate derivative at a point x, $F'_{\mathrm{ap}}(x)$, is defined in the same way.

Consider the following four classes of measurable functions defined on a fixed interval $\langle a,b\rangle$.

- $\mathcal{L}_1$: [*ACG*]-functions,

- $\mathcal{L}_2$: [*VBG*]-functions satisfying condition $\mathcal{N}$,
- $\mathcal{L}_3$: *ACG*-functions,
- $\mathcal{L}_4$: *VBG*-functions satisfying condition $\mathcal{N}$.

Classes $\mathcal{L}_i$ are linear spaces for each i. For $i = 1, 3$ this is elementary and well-known, for $i = 2$ it was shown by Sarkhel and Kar, *Corollary 3.1.1* and *Theorem 3.6* in [20], for $i = 4$ it was shown by Ene, *Corollary 2* in [2]. Note that every function from $\mathcal{L}_i$ is approximately differentiable at almost every point of $\langle a, b\rangle$.

Let $\mathcal{F}$ be a linear space of Baire one Darboux functions defined on $\langle a, b\rangle$.

Definition 2.1. We say that an $f\colon \langle a, b\rangle \to \mathbb{R}$ is $\mathcal{F}_i$-integrable, $i = 1, 2, 3, 4$, if there exists an $F \in \mathcal{F}_i = \mathcal{L}_i \cap \mathcal{F}$ such that $F'_{\text{ap}}(x) = f(x)$ for almost all $x \in \langle a, b\rangle$. The integral of f is defined to be $F(b) - F(a)$.

For each i, the $\mathcal{F}_i$-integral is properly defined; this is a consequence of $\mathcal{L}_i$ being a linear space, and the following monotonicity lemma, *Theorem 1* in [9].

Lemma 2.2. *Suppose that an $F\colon \langle a, b\rangle \to \mathbb{R}$ satisfies condition $\mathcal{N}$ and has the Darboux property. If $F'(x) \geq 0$ at almost every point x of the set where F is differentiable, then F is nondecreasing.*

The proof of the above lemma relies upon a more general monotonicity theorem of Bruckner, see [1].

In the rest of this section and in the next section, we will consider the particular case where $\mathcal{F}$ is the linear space of approximately continuous functions, $\mathcal{F} = C_{\text{ap}}$. (Recall that an approximately continuous function has the Darboux property.) The $\mathcal{F}_1$-integral is Ridder's β-integral (Kubota's *AD*-integral), the $\mathcal{F}_4$-integral is Gordon's $AK_{\mathcal{N}}$-integral, while the $\mathcal{F}_2$ is equivalent to Sarkhel's $T_{\text{ap}}D$-integral. The $\mathcal{F}_3$-integral was originally considered by Kubota [6] (also known as the *AD*-integral), but it was abandoned in favor of the $\mathcal{F}_1$-integral.

Let us turn to relations between the integrals defined above. Clearly (for any $\mathcal{F}$), the $\mathcal{F}_i$-integral is more general than the $\mathcal{F}_j$-integral, iff $\mathcal{F}_i \supset \mathcal{F}_j$. The following inclusions are obvious: $\mathcal{F}_4 \supset \mathcal{F}_3 \supset \mathcal{F}_1$, $\mathcal{F}_4 \supset \mathcal{F}_2 \supset \mathcal{F}_1$. Sarkhel and Kar constructed an approximately continuous function $F \in \mathcal{L}_2 \setminus \mathcal{L}_3$, *Example 3.1* in [20]. Thus, for $\mathcal{F} = C_{\text{ap}}$, $\mathcal{F}_3 \not\supset \mathcal{F}_2$.

Ene, at the end of [2], asked the following questions:

1. $[VBG] \cap \mathcal{N} \cap C_{\text{ap}} \subsetneq VBG \cap \mathcal{N} \cap C_{\text{ap}}$ *on* $\langle a, b\rangle$*?* (i.e., $\mathcal{F}_2 \subsetneq \mathcal{F}_4$?);
2. *Is there a function $F\colon \langle a, b\rangle \to \mathbb{R}$ such that $F \in ACG \cap C_{\text{ap}}$ and $F \notin [VBG] \cap \mathcal{N} \cap C_{\text{ap}}$?* (i.e., $\mathcal{F}_3 \not\subset \mathcal{F}_2$?).

(Ene asked a related question in [3], *Remark 5(v).*) Answers to both questions are in the affirmative, as we will see in the next section.

3 Examples.

All the examples below are cases of the following result of Petruska and Laczkovich [13]: *Let* $f\colon \langle a,b\rangle \to \mathbb{R}$ *be a Baire one function,* $H \subset \langle a,b\rangle$ *a nullset. Then, the restriction* $f \restriction H$ *can be extended to an approximately continuous function.* However, the constructions we give do not follow from the above theorem nor from its proof, since we need to obtain a function from the class $\mathcal{L}_i$.

Example 3.1. *There exists a function* $F \in \mathcal{F}_3 \setminus \mathcal{F}_2$.

CONSTRUCTION. Let $\mathbb{C}$ be the Cantor ternary set and let $I_i^{(n)}$, $i = 1, \ldots, 2^{n-1}$, be the intervals of the nth rank contiguous to $\mathbb{C}$, $n = 1, 2, 3, \ldots$. Let Z denote the collection of endpoints of $I_i^{(n)}$'s. Let $J_1^{(n)}, \ldots, J_{2^n}^{(n)}$ be connected components of the set $\langle 0,1\rangle \setminus \bigcup_{i=1}^{2^{n-1}} I_i^{(n)}$. We choose a countable set $\{x_i^{(n)}\}_{i,n} \subset \mathbb{C} \setminus Z$, dense in $\mathbb{C}$, as follows:

- let $x_i^{(1)} \in J_i^{(1)}$, $i = 1, 2$;
- for $N \geq 1$, put $S = \{x_i^{(n)} : n = 1, \ldots, N, \ i = 1, \ldots, 2^n\}$, and pick an $x_i^{(N+1)} \in J_i^{(N+1)} \setminus S$, $i = 1, \ldots, 2^{N+1}$.

For any $x \in \{x_i^{(n)}\}_{i,n}$ we choose two monotone sequences of closed intervals contiguous to $\mathbb{C}$: $\{R_k(x)\}_{k=1}^{\infty}$ (decreasing), $\{L_k(x)\}_{k=1}^{\infty}$ (increasing), both converging to x, so that x is a right density point of $\bigcup_{k=1}^{\infty} R_k$ and a left density point of $\bigcup_{k=1}^{\infty} L_k$. Let R'_k and L'_k be closed intervals concentric to R_k and L_k respectively, such that

$$|R'_k| = \frac{k-1}{k}|R_k|, \ |L'_k| = \frac{k-1}{k}|L_k|.$$

Note that x is also a right density point of $\bigcup_{k=1}^{\infty} R'_k$ and a left density point of $\bigcup_{k=1}^{\infty} L'_k$. By a successive application of the above construction of sequences $\{R_k(x)\}_{k=1}^{\infty}$, $\{L_k(x)\}_{k=1}^{\infty}$, we can require that these sequences for two different $x \in \{x_i^{(n)}\}_{i,n}$ have no point in common. (This is possible thanks to the monotonicity of $\{R_k\}_k$ and $\{L_k\}_k$.) Define the function F by

$$F(x) = \begin{cases} \frac{1}{n} & \text{if } x = x_i^{(n)} \\ \frac{1}{n} & \text{if } x \in \bigcup_{k=1}^{\infty} R'_k(x_i^{(n)}) \cup \bigcup_{k=1}^{\infty} L'_k(x_i^{(n)}) \\ 0 & \text{if } x \in \mathbb{C} \setminus \{x_i^{(n)}\}_{i,n} \\ \text{linear} & \text{on the connected components of } R_k \setminus R'_k \text{ and } L_k \setminus L'_k. \end{cases}$$

Note that F is upper semicontinuous, discontinuous exactly at $x_i^{(n)}$'s, and approximately continuous. F is an *AC*-function on R_k's, L_k's, $\mathbb{C} \setminus \{x_i^{(n)}\}_{i,n}$. Thus F is an *ACG*-function. However, F is *VB* on no portion of $\mathbb{C}$ and so it is not a [*VBG*]-function. □

The idea of the above example was used by Sarkhel and Kar [20] to prove the converse; namely, that $\mathcal{F}_2 \not\subset \mathcal{F}_3$. Note that Example 3.1 also provides an affirmative answer to the first question of Ene.

Now, we will show how badly the inclusions $\mathcal{F}_3 \subset \mathcal{F}_1$, $\mathcal{F}_4 \subset \mathcal{F}_2$ fail.

Example 3.2. *There exists a function $F \in \mathcal{F}_2 \cap \mathcal{F}_3 \setminus \mathcal{F}_1$.*

CONSTRUCTION. Define F as in Example 3.1, but using $\frac{1}{2^{2n}}$ instead of the value $\frac{1}{n}$. Then F is *VB* on $\mathbb{C}$. □

For the proof of the lemma below see for instance [14].

Lemma 3.3. (Lusin-Menchov). *Let the set $D \subset \mathbb{R}$ be closed, and $E \supset D$ be measurable. Then, there exists a perfect set P, $E \supset P \supset D$, such that for each $x \in D$ all four extreme densities of E at x are equal to those of P at x.*

Lemma 3.4. *Let the set $X \subset \mathbb{C}$ be closed and nowhere dense in $\mathbb{C}$. Then, for each $c > 0$ there exists an approximately continuous function ϕ on $\langle 0, 1\rangle$ such that $\phi(\mathbb{C} \setminus X) = \{0\}$, $\phi(X) = \{c\}$, $0 \le \phi(x) \le c$ for every $x \in \langle 0, 1\rangle$, and $\mathcal{D}_\phi = X$.*

PROOF. Apply Lemma 3.3 for $D = X$, $E = X \cup (\langle 0, 1\rangle \setminus \mathbb{C})$ to obtain a suitable set P. For each closed interval I_i contiguous to $\mathbb{C}$, if $I_i \cap P \ne \emptyset$, let $J_i = \langle \inf P \cap I_i, \sup P \cap I_i\rangle \subset I_i$. We put

$$\phi(x) = \begin{cases} c & \text{if } x \in J_i \text{ or } x \in X, \\ 0 & \text{if } x \in \mathbb{C} \setminus X, \\ \text{linear} & \text{on the connected components of } I_i \setminus \operatorname{int} J_i. \end{cases}$$

Obviously, the function ϕ is approximately continuous on X and continuous on $\langle 0, 1\rangle \setminus \mathbb{C}$. It remains to check that it is continuous at each point $x \in \mathbb{C} \setminus X$. Since $x \notin P$, there is a neighborhood $U \ni x$ such that $U \cap P = \emptyset$. There are at most two intervals of the kind I_i such that $U \cap I_i \ne \emptyset$ and $P \cap I_i \ne \emptyset$. So for an $\varepsilon > 0$, if necessary, we can shrink U to obtain $\phi(U) \subset \langle 0, \varepsilon)$. That means, ϕ is continuous at x. □

Example 3.5. *There exists a function $F \in \mathcal{F}_4 \setminus (\mathcal{F}_2 + \mathcal{F}_3)$.*

CONSTRUCTION. Let G be the Cantor ternary function. For an $n = 1, 2, \dots$ let G_n be a continuous piecewise linear function such that $|G(x) - G_n(x)| < \frac{1}{n}$ for each $x \in \langle 0, 1\rangle$. Choose an ascending sequence of subsets $A_n \subset \mathbb{C}$, $n = 1, 2, \dots$, nowhere dense in $\mathbb{C}$, such that $|G(A_n)|$ tends to 1 as $n \to \infty$. (We may assume that each set $A_n \setminus A_{n-1}$ is closed and that $A_n \cap Z = \emptyset$, where Z is the set constructed in Example 3.1.) Define a function F on $\mathbb{C}$ by the formula

$$F(x) = \begin{cases} G(x) & \text{if } x \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} A_n, \\ G_n(x) & \text{if } x \in A_n \setminus A_{n-1},\ n = 1, 2, \dots, \end{cases}$$

$A_0 = \emptyset$. Notice that F is *VBG* (on $\mathbb{C}$) and (since $|F(\mathbb{C})| = 0$) it satisfies condition $\mathcal{N}$. Moreover, it is a Baire one function. Now, we will extend F to a function defined on the whole interval $\langle 0, 1\rangle$.

We proceed by induction. Assume we have defined approximately continuous functions $\phi_1, \dots, \phi_n \colon \langle 0, 1\rangle \to \mathbb{R}$ satisfying:

1. $|(F - G - \phi_1 - \dots - \phi_n)(x)| \le \frac{1}{2^n}$ for each $x \in \mathbb{C}$.
2. $F - G - \phi_1 - \dots - \phi_n = 0$ on $\mathbb{C} \setminus \bigcup_{m=1}^{\infty} A_m$.
3. The pre-image $(F - G - \phi_1 - \dots - \phi_n)^{-1}(I)$ is a closed set, for each compact interval I contained in $(0, \infty)$ or in $(-\infty, 0)$.
4. $\mathcal{D}_{F-G-\phi_1-\dots-\phi_n} \subset \bigcup_{m=1}^{\infty} A_m$.

All four conditions are fulfilled for $n = 0$ (then $F - G - \phi_1 - \dots - \phi_n$ is taken to be $F - G$). This is clear for conditions 1, 2, and 4. Condition 3 follows for $n = 0$ since $F - G_n$ is continuous on A_n and since $(F - G)^{-1}(I)$ is contained in some A_n. We will define ϕ_{n+1} and check conditions 1 – 4 for $n + 1$. Consider the sets

$$B^1_{n+1} = \left\{ x \in \mathbb{C} \colon (F - G - \phi_1 - \dots - \phi_n)(x) \ge \frac{1}{2^{n+1}} \right\},$$

$$B^2_{n+1} = \left\{ x \in \mathbb{C} \colon (F - G - \phi_1 - \dots - \phi_n)(x) \le -\frac{1}{2^{n+1}} \right\},$$

$$B^3_{n+1} = \left\{ x \in \mathbb{C} \colon (F - G - \phi_1 - \dots - \phi_n)(x) = \frac{1}{2^{n+1}} \right\},$$

$$B^4_{n+1} = \left\{ x \in \mathbb{C} \colon (F - G - \phi_1 - \dots - \phi_n)(x) = -\frac{1}{2^{n+1}} \right\}.$$

Thanks to condition 3 these sets are closed. Apply Lemma 3.4 to construct approximately continuous functions $\phi^i_{n+1} \colon \langle 0, 1\rangle \to \mathbb{R}$, $i = 1, \dots, 4$, such that

- $\phi^i_{n+1}(\mathbb{C} \setminus B^i_{n+1}) = \{0\}$,
- $\phi^i_{n+1}(B^i_{n+1}) = \left\{\frac{(-1)^{i+1}}{2^{n+1}}\right\}$,

5. $0 \le \phi^i_{n+1}(x) \le \frac{1}{2^{n+1}}$, $i = 1, 3$; $0 \ge \phi^i_{n+1}(x) \ge -\frac{1}{2^{n+1}}$, $i = 2, 4$; for each $x \in \langle 0, 1\rangle$;

- $\mathcal{D}_{\phi^i_{n+1}} = B^i_{n+1}$.

Put $\phi_{n+1} = \phi^1_{n+1} + \phi^2_{n+1} - \phi^3_{n+1} - \phi^4_{n+1}$. Notice that $\phi_{n+1}(B^3_{n+1} \cup B^4_{n+1}) = \{0\}$, and $\phi_{n+1}(B^1_{n+1} \setminus B^3_{n+1}) = \left\{\frac{1}{2^{n+1}}\right\}$, $\phi_{n+1}(B^2_{n+1} \setminus B^4_{n+1}) = \left\{-\frac{1}{2^{n+1}}\right\}$. Consider the function $F - G - \phi_1 - \cdots - \phi_{n+1}$ on $\mathbb{C}$. Conditions 1, 2, and 4 are obviously satisfied ($B^i_{n+1} \subset \bigcup_{m=1}^{\infty} A_m$). Let $0 < c \le d \le 2^{-(n+1)}$. Then, by the definition of ϕ_{n+1},

$$\begin{aligned}(F - G - \phi_1 - \cdots - \phi_{n+1})^{-1}(\langle c, d\rangle) \\ = (F - G - \phi_1 - \cdots - \phi_n)^{-1}\big(\langle c, d\rangle \cup \langle c + 2^{-n-1}, d + 2^{-n-1}\rangle\big).\end{aligned}$$

From the assumption of condition 3 holding for n, condition 3 follows for $n+1$. Similarly for the negative interval case.

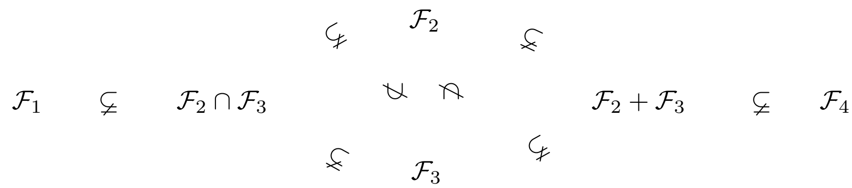
We have constructed the sequence $\phi_1, \phi_2, \ldots$. Note that by condition 5 the series $\sum_{n=1}^{\infty} \phi_n$ is uniformly convergent on $\langle 0, 1\rangle$, so its sum $S = \sum_{n=1}^{\infty} \phi_n$ is approximately continuous. Moreover, $S = F - G$ on $\mathbb{C}$. Since $Z \cap \bigcup_{n=1}^{\infty} A_n = \emptyset$, S is continuous at each point $x \in I_i$ of each closed interval I_i contiguous to $\mathbb{C}$. Let T_i on I_i be a continuous piecewise linear function such that $S = T_i$ on $\operatorname{fr} I_i$ and $|T_i - S| < \frac{1}{i}$ on I_i. Note that the function

$$\tilde{F}(x) = \begin{cases} S(x) + G(x) = F(x) & \text{if } x \in \mathbb{C}, \\ T_i(x) + G(x) & \text{if } x \in I_i, \end{cases}$$

is an approximately continuous extension of F. (In the sequel we will write F instead of $\tilde{F}$.) Moreover, it is VBG and satisfies condition $\mathcal{N}$.

Suppose that F is the sum of two approximately continuous functions, an ACG-function H_1 and a $[VBG]$-function H_2 satisfying $\mathcal{N}$. By the Baire Category Theorem there is a portion $\mathbb{C}' = I \cap \mathbb{C}$ of $\mathbb{C}$, I an open interval, such that H_2 is VB on $\mathbb{C}'$ and H_1 is AC on a dense $\mathcal{G}_\delta$ subset $O = \mathcal{C}_{F\restriction\mathbb{C}'}$ of $\mathbb{C}'$. Note that at each point $x \in \mathbb{C}'$, both limits $\lim_{t\to x,\, t\in O} F(t)$ and $\lim_{t\to x,\, t\in O} H_1(t)$ exist and are finite. That means, at this x the bilateral limit of $H_2 \restriction \mathbb{C}$ exists. Note that the set $\mathcal{D}_{H_2\restriction\mathbb{C}'}$ is at most countable; thus H_2 is AC on $\mathcal{C}_{H_2\restriction\mathbb{C}'}$. Both H_1 and H_2 are AC on $O \cap \mathcal{C}_{H_2\restriction\mathbb{C}'}$. Thus, $F = H_1 + H_2$ is AC on this set. Since $O \cap \mathcal{C}_{H_2\restriction\mathbb{C}'}$ is a dense $\mathcal{G}_\delta$ subset of $\mathbb{C}'$, it is dense also in $\mathbb{C}' \setminus \bigcup_{n=1}^{\infty} A_n$. But, we have $F = G$ on $\mathbb{C}' \setminus \bigcup_{n=1}^{\infty} A_n$, a contradiction. □

The chart below shows all relationships between the classes of approximately continuous primitives we have considered.



Question 3.6. For a given approximately continuous $[ACG]$-function F, is it possible to write F as the sum of a continuous ACG-function (i.e., a wide Denjoy primitive) and an approximate Henstock primitive?

4 Some Remarks.

We end our note with an observation related to functions from the classes $\mathcal{F}_i$. The picture of primitives from classes $\mathcal{F}_i$, $i = 3, 4$, was a little bit hazy, since these classes are defined using nonclosed $\langle a, b\rangle$-forms. We will show that these primitives are not so bad, since their discontinuity sets are nowhere dense.

Definition 4.1. We say that a real function F is quasi-continuous if the set $F \restriction \mathcal{C}_F$ is dense in F (in the sense of graphs).

One can easily define one-sided and bilateral versions of the above notion. The following lemmas were proved in [1], and [12], *Lemma 2.(a)*, respectively.

Lemma 4.2. *Every Darboux Baire one function which satisfies condition $\mathcal{N}$ is quasi-continuous.*

Lemma 4.3. *Every quasi-continuous Darboux function is bilaterally quasi-continuous.*

Theorem 4.4. *Let $F \in \mathcal{F}_4$. Then, the set $\mathcal{D}_F$ is nowhere dense.*

PROOF. Suppose not. Then by the Baire Category Theorem there is an interval I such that F is VB on a dense subset E of $\mathcal{C}_F \cap I$, and such that $\mathcal{D}_F \cap \operatorname{int} I \neq \emptyset$. Take $x \in \mathcal{D}_F \cap \operatorname{int} I$. Pick a sequence of points $x_n \in I$ such that $x_n \to x$ and $|F(x_n) - F(x)| \geq M > 0$ for all n. We may assume that $(x_n)_n$ is decreasing. By Lemmas 4.2 and 4.3, F is bilaterally quasi-continuous. Hence, there exists a decreasing sequence of points $y_n \in \mathcal{C}_F$ such that $|F(y_n) - F(x)| < \frac{M}{4}$ and $y_n \to x$. We may assume that $x_{n+1} < y_n < x_n$. Moreover, there are points $z_n \in E \cap (x_{n+1}, x_n)$ and $w_n \in E \cap (z_n, x_n)$ such that $|F(z_n) - F(y_n)| < \frac{M}{4}$, $|F(w_n) - F(x_n)| < \frac{M}{4}$. We have

$$|F(w_n) - F(z_n)| \geq |F(x) - F(x_n)| - |F(x) - F(y_n)| - |F(y_n) - F(z_n)| - |F(w_n) - F(x_n)| > M - \frac{3}{4}M = \frac{M}{4}.$$

Summing over all n's, we obtain $\sum_{n=1}^{\infty} |F(w_n) - F(z_n)| = \infty$. Since $w_n, z_n \in E$, F is not VB on E, a contradiction. □

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