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SUBDIFFERENTIABILITY OF REAL FUNCTIONS

Abstract

In this paper, we show that nowhere monotone functions are the key ingredients to construction of continuous functions, absolutely continuous functions, and Lipschitz functions with large subdifferentials on the real line. Let $\partial_c f, \partial_a f$ denote the Clarke subdifferential and approximate subdifferential respectively. We construct absolutely continuous functions on $\mathbb{R}$ such that $\partial_a f = \partial_c f \equiv \mathbb{R}$. In the Banach space of continuous functions defined on $[0,1]$, denoted by $C[0,1]$, with the uniform norm, we show that there exists a residual and prevalent set $D \subset C[0,1]$ such that $\partial_a f = \partial_c f \equiv \mathbb{R}$ on $[0,1]$ for every $f \in D$. In the space of automorphisms we prove that most functions f satisfy $\partial_a f = \partial_c f \equiv [0,+\infty)$ on $[0,1]$. The subdifferentiability of the Weierstrass function and the Cantor function are completely analyzed. Similar results for Lipschitz functions are also given.

1 Introduction.

Nonsmooth analysis deals with nondifferentiabilities. Little has been written on the subdifferentiabilities of the classical nondifferentiable examples. In this paper, we study the subdifferentiabilities of *nowhere monotone functions* as they provide the best test ground of generalized subdifferentials. Subdifferentials have been defined for lower semicontinuous functions in arbitrary Banach spaces [9, 20, 21, 27, 28]. Since we work on continuous functions on real line,

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we do not see any advantage in presenting a general definition. We recall the definitions of subdifferentials that we need and present basic comments on them. For more properties on subgradients and subderivatives of functions on $\mathbb{R}^n$, we refer the readers to [35, pages 299–348].

Let U be open in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be continuous. At $x \in U$, the *Dini-Hadamard type lower derivative and upper derivative* of f at x in the direction $v \in \mathbb{R}^n$ are defined by

$$f^-(x;v) := \liminf_{t\downarrow 0, h\to v} \frac{f(x+th)-f(x)}{t},$$

$$f^+(x;v) := \limsup_{t\downarrow 0, h\to v} \frac{f(x+th)-f(x)}{t}.$$

When $f^-(x;v) = f^+(x;v)$, we write $f'(x;v)$. We define the *Dini-Hadamard subdifferential* of f at x as

$$\partial_- f(x) := \{x^* \in \mathbb{R}^n : \langle x^*, v\rangle \leq f^-(x;v) \text{ for every } v \in \mathbb{R}^n\}.$$

The *Rockafellar directional derivative* of f at x in the direction v and the *Clarke-Rockafellar* subdifferential of f at x [35, page 337] are given respectively by

$$f^{\uparrow}(x;v) := \lim_{\epsilon\downarrow 0} \limsup_{y\to x, t\downarrow 0} \inf_{w\in v+\epsilon\mathbb{B}} \frac{f(y+tw)-f(y)}{t},$$

$$\partial_c f(x) := \{x^* \in \mathbb{R}^n : \langle x^*, v\rangle \leq f^{\uparrow}(x;v) \text{ for all } v \in \mathbb{R}^n\}.$$

When f is locally Lipschitz at x, $f^{\uparrow}(x;v)$ and $\partial_c f$ reduce to the *Clarke directional derivative* and *Clarke subdifferential* $\partial_c f$ given by

$$f^0(x;v) := \limsup_{y\to x, t\downarrow 0} \frac{f(y+tv)-f(y)}{t},$$

$$\partial_c f(x) := \{x^* \in \mathbb{R}^n : \ \langle x^*, v\rangle \leq f^0(x;v) \text{ for all } v \in \mathbb{R}^n\}.$$

The *Michel-Penot directional derivative* of f at x in the direction v and subdifferential at x are given respectively by:

$$f^{\diamond}(x;v) := \sup_{w} \limsup_{t\downarrow 0} \frac{f(x+tw+tv)-f(x+tw)}{t}, \tag{1}$$

$$\partial_{mp} f(x) := \{x^* : \langle x^*, v\rangle \leq f^{\diamond}(x;v) \text{ for all } v \in \mathbb{R}^n\}.$$

In general, the Michel-Penot subdifferential is smaller than the Clarke subdifferential. Unlike $\partial_c f$, the Michel-Penot subdifferential $\partial_{mp} f(x)$ is singleton if and only if f is Gâteaux differentiable at x.

A special type of viscosity subdifferential is the *proximal subdifferential*: $x^* \in \mathbb{R}^n$ is called a *proximal* subgradient of f at x if for some $\sigma > 0$ and $\delta > 0$ one has

$$f(y) \geq f(x) + \langle x^*, y - x \rangle - \sigma \|y - x\|^2,$$

when $\|y - x\| < \delta$. We write $x^* \in \partial_p f(x)$.

The *Mordukhovich sequential (or approximate) subdifferential* [26] of f at x, denoted by $\partial_a f(x)$, is defined by

$$\{\lim_{n\to\infty} x_n^* : x_n^* \in \partial_p f(x_n), x_n \to x\},$$

and it has an equivalent characterization given by

$$\partial_a f(x) := \{\lim_{n\to\infty} x_n^* : x_n^* \in \partial_- f(x_n), x_n \to x\}. \tag{2}$$

Both $\partial_c f$ and $\partial_a f$ enjoy nice calculus rules. Unlike $\partial_c f$, $\partial_a f$ needs not be convex-valued. Extension of the limiting subdifferential to infinite dimensional spaces (in the form of limiting Fréchet subdifferential) was done in [27]. Ioffe made another line of developments of Mordukhovich's constructions to infinite-dimensional spaces in [20, 21]. See [28, 35] for the full account of these constructions and relationships among them.

If f is convex, the subdifferential of f at x is defined as

$$\partial f(x) := \{x^* : \langle x^*, y - x \rangle \leq f(y) - f(x) \quad \text{for all } y \in U\}. \tag{3}$$

A convex function f on an open convex subset U of a Banach space X is Gâteaux differentiable at $x \in U$ if and only if f has a unique subgradient [30]. If f is a continuous convex function on an open set U, then f is locally Lipschitz [9], and all generalized subdifferentials become ∂f [30, 9, 21]. For any continuous function $f : U \to \mathbb{R}$, we have

$$\partial_p f(x) \subset \partial_- f(x) \subset \partial_a f(x) \subset \partial_c f(x), \text{ and}$$

$$\partial_c f(x) \neq \emptyset \Rightarrow \partial_a f(x) \neq \emptyset.$$

When f is locally Lipschitz at x, both $\partial_c f$ and $\partial_a f$ are *upper semi-continuous and compact-valued multifunctions*, and $\partial_c f(x) = \mathrm{conv}[\partial_a f(x)]$, where 'conv' denotes convex hull.

When f is locally Lipschitz at x,

$$f^-(x; \cdot) \leq f^+(x; \cdot) \leq f^\diamond(x; \cdot) \leq f^0(x; \cdot),$$

always hold. We say that f is *regular* at x if $f^-(x; v) = f^0(x; v)$ for each $v \in \mathbb{R}^n$, and f is *pseudo-regular* at x if $f^+(x; v) = f^0(x; v)$ for each $v \in \mathbb{R}^n$. Whenever $\partial_c f(x) \neq \emptyset$, f is regular at x if and only if $\partial_- f(x) = \partial_c f(x)$.

For $f : U \subset \mathbb{R} \to \mathbb{R}$ and $x \in U$, we will frequently use the following *Dini derivatives*:

$$f^+(x) := \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h}, \qquad f_+(x) := \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h},$$

$$f^-(x) := \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h}, \qquad f_-(x) := \liminf_{h\to 0^-} \frac{f(x+h)-f(x)}{h},$$

$$\overline{f}'(x) := \max\{f^+(x), f^-(x)\} \qquad \underline{f}'(x) := \min\{f_-(x), f_+(x)\}.$$

By "differentiable" we will always mean "having a finite derivative". Different choices of w in Equation (1) provide inequalities linking the Michel-Penot subderivative with the Dini derivates of f at x:

$$f^\diamond(x;1) \geq \max\{f^+(x), f^-(x)\}, \text{ and } f^\diamond(x;-1) \geq \max\{-f_-(x), -f_+(x)\}. \tag{4}$$

In the sequel, if a property is valid for all points in a complete metric space (respectively a measure space) except for a subset of the first category (respectively a set of measure zero), we shall say that the property holds *typically* or *residually* (respectively *almost everywhere, abbreviated a.e.*). The complement of a first-category set is called a *residual set*. For a set $A \subset \mathbb{R}$, we will use $\mu(A)$ to denote its Lebesgue measure.

The paper is laid out as follows. Section 2 is a brief introduction on nowhere monotone functions while in Section 3 basic properties of nowhere monotone functions are given. The concrete constructions of nowhere monotone functions with large subdifferentials are given in Sections 4, 5. Utilizing nowhere monotone functions of second species, in Sections 6, 7, 8, 9 we show that typical continuous functions in the spaces of nondecreasing continuous functions, automorphisms, or continuous functions have large subdifferentials. In Section 10, we show that Lipschitz functions with large subdifferentials are also typical in the space of Lipschitz functions with controlled rank. In Section 11, we answer one question posed in Sciffer's thesis. At the end, we cite Rockafellar's result on convex functions for comparison, and give some open problems concerning $\partial_a f$, $\partial_c f$.

2 Nowhere Monotone Functions.

Definition 2.1. We say a finite real function f defined on $[0,1]$ is *nowhere monotone* if f is not monotone in any subinterval of $[0,1]$. A nowhere monotone function f is of the first species in $[0,1]$ if there exists a real number r such that the function $f(x) + r \cdot x$ becomes monotone in $[0,1]$, and is of the second species in $[0,1]$ provided that for every $r \in \mathbb{R}$ the function $f(x) + r \cdot x$ is also nowhere monotone.

From the definition, we see that if a nowhere monotone f is not the second species on $[0,1]$, then for some r the function $f(x)+rx$ is monotonic on some subinterval $I \subset [0,1]$. Thus the complement of the second species need not be the first species. Since every nondifferentiable function f is nowhere monotone and for every $r \in \mathbb{R}$ the function $f(x)+r\cdot x$ is also nowhere monotone in $[0,1]$, every nondifferentiable function f is a nowhere monotone function of the second species.

Definition 2.2. A continuous function f defined on $[0,1]$ is said to be nondecreasing at $x \in [0,1]$ if there exists a $\delta > 0$ such that $f(t) \leq f(x)$ on $(x-\delta, x)\cap[0,1]$ and $f(t) \geq f(x)$ on $(x, x+\delta)\cap[0,1]$; that is, $(f(t)-f(x))/(t-x) \geq 0$ for all $t \neq x$ in some neighborhood of x. The function f is nonincreasing at x if $-f$ is nondecreasing at x, and f is monotonic at x if it is either nondecreasing or nonincreasing at x. We shall say that f is of *monotonic type* at x if there exists $\nu \in \mathbb{R}$ such that $f_\nu(x) := f(x)+\nu\cdot x$ is monotonic at x. If f is not of monotonic type at any point of $[0,1]$, we say f is of *nonmonotonic type* [5].

Note that if f is not monotonic type at x, then f does not simply cross any line at $(x, f(x))$. Recall Corollary 4.3 [5, page 129]: Suppose f is continuous on $[a,b]$, $f^+ \geq 0$ almost everywhere and $f^+ > -\infty$ except, perhaps, on a countable set. Then f is nondecreasing.

Proposition 2.3. *Monotonic type at no point $\Rightarrow$ monotonic at no point $\Rightarrow$ nowhere monotone of second species $\Rightarrow$ nowhere monotone.*

PROOF. Only the second ' $\Rightarrow$' needs a proof. Let f be monotonic at no point. If f is not nowhere monotone of second species, then there exists m such that $f(x)-mx$ is monotone on some subinterval $[a,b]$. Without loss of generality we assume that $f(x)-mx$ is nondecreasing on $[a,b]$. Then $f(x)-mx$, and therefore f is differentiable almost everywhere on $[a,b]$. Since f is monotonic at no point, $f'(x)=0$ almost everywhere on $[a,b]$. Moreover, since $f(x)-mx$ is nondecreasing on $[a,b]$, $f_+(x) \geq m$ everywhere on $[a,b]$. Thus, f is nondecreasing on $[a,b]$, which is a contradiction. □

The following shows the nonreversibility of the implications in Proposition 2.3.

Example 2.4. (1) Every differentiable nowhere monotone function is not nowhere monotone of second species. Indeed, if f is nowhere monotone, then

$$D = \{x : f'(x)=0,\ f' \text{ is continuous at } x\},$$

is residual. Given $m>0$, for every $x \in D$ there exists a neighborhood N_x of x in which $|f'(y)| < m/2$. Since f' is continuous at x, $f(y)+my$ is increasing on N_x.

(2) Theorem 4.2 will give an absolutely continuous nowhere monotone function f of second species, but f is monotonic at each x with $f'(x) > 0$ or $f'(x) < 0$.

(3) Let $M \subset [0,1]$ be a first category F_σ set. Then there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that $f'(x) = 0$ for $x \in M$, f is nonmonotone at each $x \in M$ and f is of nonmonotonic type at each $x \in [0,1] \setminus M$ [4]. Thus, f is monotonic at no point, but f is monotonic type at each $x \in M$.

3 Properties of Nowhere Monotone Functions.

We now give some key properties of continuous and nowhere monotone functions used in the paper.

Definition 3.1. A function f is nondecreasing (nonincreasing) on the right of a point t if there exists a real number $h > 0$ such that $f(x) \geq f(t)$ ($f(x) \leq f(t)$) for $t < x < t + h$. If f is neither nondecreasing nor nonincreasing on the right of t, we say f is oscillating on the right of t or is O_+ at t. The property that f is oscillating on the left of t or is O_- at t is defined in a similar way.

The following lemma may be found in [10]. Here we supply a simpler proof.

Lemma 3.2. *If $f : [0,1] \to \mathbb{R}$ is continuous and nowhere monotone, then the set of points at which f is both O_+ and O_- is residual in $[0,1]$. In particular, there exists a residual set $G \subset [0,1]$ such that*

$$f_-(x) \leq 0 \leq f^-(x) \text{ and } f_+(x) \leq 0 \leq f^+(x) \text{ if } x \in G.$$

PROOF. We show that $E := \{x \in [0,1] : f \text{ is nondecreasing on the right of } x\}$ is first category. Let $E_n := \{x \in [0,1] : \ f(t) \geq f(x) \text{ for } x < t < x + 1/n\}$. Then $E = \bigcup_{n=1}^{\infty} E_n$.

First, E_n is closed. Indeed, assume $x_k \in E_n$ and $x_k \to x$. For $0 < t - x < 1/n$, when k is large we have $0 < t - x_k < 1/n$, so $f(t) \geq f(x_k)$. By the continuity of f we have $f(t) \geq f(x)$. Next, E_n is nondense in $[0,1]$. Let I' be an arbitrary interval contained in $[0,1]$, and $J = [a,b] \subset I'$ with $b - a < 1/n$. Since f is nowhere monotone in $[0,1]$, it is not nondecreasing in J, so there exist points $c, d \in J, c < d$ such that $f(c) > f(d)$. Let $m := \min\{f(t) : t \in [c,d]\}$. Since $f(c) > f(d) \geq m$, there exists $c' \in [c,d]$ such that $f(x) > m$ if $x \in [c,c']$. Choosing $t \in [c,d]$ with $f(t) = m$ we then have $t > c'$. If $x \in [c,c']$, then $0 < t - x < 1/n$ and $f(t) = m < f(x)$, so $x \notin E_n$. Therefore E_n is nondense in $[0,1]$. Similar arguments show that the set of points at which f is nonincreasing on the right is first category. Then f is O_+ at a residual subset $G_+ \subset [0,1]$. If $t \in G_+$, then $(t, t+h)$ contains two points t_1, t_2 with $f(t_1) < f(t) < f(t_2)$ for every $h > 0$. As f is continuous, there exists $x \in (t, t+h)$ such that

$f(x) = f(t)$. Therefore $f_+(t) \le 0 \le f^+(t)$. Similarly, we obtain a residual subset $G_- \subset [0,1]$ such that $f_-(t) \le 0 \le f^-(t)$ if $t \in G_-$. Then the claim holds on $G_- \cap G_+$. $\square$

Lemma 3.3. *If f is continuous and nowhere monotone on $[0,1]$, then the set of points at which f attains local minima is dense in $(0,1)$.*

PROOF. Take an arbitrary $x \in (0,1)$ and $h > 0$ such that $[x-h, x+h] \subset (0,1)$. We will show that f has a local minimum in $(x-h, x+h)$. Since f is nowhere monotone in $[x, x+h]$, it can not be non-increasing in $[x, x+h]$, and so there exist points $c, d \in [x, x+h]$ such that $c < d$ and $f(c) < f(d)$. There exists $\delta > 0$ such that $f(t) > f(c)$ on $[d-\delta, d]$ and $d - \delta > c$. On $[x-h, x]$ the same arguments show that there exist $c' > d'$ with $c', d' \in [x-h, x]$ such that $f(c') < f(d')$. There exists $\delta' > 0$ such that $f(t) > f(c')$ on $[d', d'+\delta']$ and $d' + \delta' < c'$. Hence the minimum of f on $[d', d]$ is attained in $(d'+\delta', d-\delta) \subset (x-h, x+h)$. $\square$

4 Rockafellar Type Functions.

In this section, we construct absolutely continuous functions on $\mathbb{R}$ such that $\partial_a f = \partial_c f \equiv \mathbb{R}$. We show that Rockafellar's function is Dini subdifferentiable only on a first category set. In the sequel, by a *thick Cantor set* we mean a nowhere dense perfect set of positive measure. The following classical result is well-known.

Lemma 4.1. *The interval $[0,1]$ can be expressed as a disjoint union of measurable sets, $[0,1] = \bigcup_{k=1}^{\infty} B_k$, each of which has positive measure in every subinterval of $[0,1]$.*

PROOF. We reproduce the simple proof given by Bruckner [6].

Let A_1 be a thick Cantor set contained in $[0,1]$. Let $A_2 := A_2^0 \cup A_2^1$ where, for $i = 0, 1$, A_2^i is a thick Cantor set contained in $(i/2, (i+1)/2)$ and such that $A_1 \cap A_2 = \emptyset$. Inductively we obtain a sequence of sets $\{A_k\}$ such that for each k,

(i) $A_k \cap (A_1 \cup A_2 \cup \cdots \cup A_{k-1}) = \emptyset$.

(ii) A_k is a union of thick Cantor sets, $A_k := A_k^0 \cup A_k^1 \cup \cdots \cup A_k^{k-1}$, with, for each $i = 0, 1, \cdots, k-1$, $A_k^i \subset (i/k, (i+1)/k)$.

Such a sequence can be defined because for every k, the set $A_1 \cup A_2 \cup \cdots \cup A_{k-1}$ is nowhere dense in $[0,1]$. Now let $A_0 := [0,1] \setminus (\bigcup_{k=1}^{\infty} A_k)$. Define a sequence of B_k by

$$B_1 := A_0 \bigcup (\bigcup_{n=0}^{\infty} A_{2n+1}), \quad \text{and} \quad B_{k+1} := \bigcup_{n=0}^{\infty} A_{2^k(2n+1)} \text{ for } k \geq 1.$$

By (i) the sequence $\{A_k\}$ and therefore the sequence $\{B_k\}$ is a disjoint sequence of sets. Clearly, $[0,1] = \bigcup_{k=1}^{\infty} B_k$. Let $I \subset [0,1]$ be a nondegenerate interval and let $|I|$ denote its length. Choose n_0 so that $2/n_0 < |I|$. For each $n \geq n_0$, there exists a nonnegative integer $i_n < n$ such that $(i_n/n, (i_n+1)/n)$ is contained in I. It follows that the set $A_n \cap I$ has positive measure for every $n \geq n_0$. Since for each k, the set B_k contains infinitely many of the sets A_n, we infer that the set $\mu(B_k \cap I) > 0$. $\square$

Theorem 4.2. *Let $A := \{a_1, a_2, \ldots\}$ be any sequence of real numbers. There exists an absolutely continuous function F such that for every interval $I \subset [0,1]$ and every k, the set $\{x : F'(x) = a_k\} \cap I$ has positive measure.*

PROOF. Let B_k be a sequence of sets satisfying the conclusion of Lemma 4.1. We may assume that $|a_k|\mu(B_k) < 1/k^2$ for each $k > 1$. It follows that the function f defined by $f(x) := a_k$ if $x \in B_k$ is Lebesgue integrable, since

$$\int_0^1 |f(x)|\, dx \leq |a_1|\mu(B_1) + \sum_{k=2}^{\infty} \frac{1}{k^2} < +\infty.$$

Let F be defined by $F(x) := \int_0^x f(t)\, dt$. Then F is absolutely continuous and $F'(x) = f(x)$ a.e. in $[0,1]$ [36, pages 107–110]. In particular for each k, F' takes on the value a_k at almost all points of B_k. The proof is completed since B_k has positive measure in I. $\square$

Theorem 4.2 is very useful in constructing pathological examples. In the sequel, by "infinitely many" we mean that the pairwise difference of these functions is not a constant.

Corollary 4.3. *There exist infinitely many strictly increasing and absolutely continuous functions F such that $\partial_a F = \partial_c F \equiv [0, \infty)$. For each such a function F, the inverse function F^{-1} satisfies $\partial_a F^{-1} = \partial_c F^{-1} \equiv [0, \infty)$ on the range of F, which is $F([0,1])$.*

PROOF. Let $A := \{r \in (0, \infty) : r \text{ is a rational number}\} = \{a_k\}_{k=1}^{\infty}$. Note that $F(x) := \int_0^x f(s)\, ds$ where $f(x) := a_k$ if $x \in B_k$. Let $x, y \in [0,1]$ and $x < y$. Taking any rational $a_k > 0$, we have

$$F(y) - F(x) = \int_x^y f(s)\, ds \geq \int_{(x,y) \cap B_k} f(s)\, ds \geq a_k \mu(B_k \cap (x,y)) > 0.$$

Thus F is strictly increasing. In particular, $\partial_a F(x) \subset [0,\infty)$. Theorem 4.2 and (2) imply $[0,\infty) \subset \partial_a F(x)$. Thus $\partial_a F(x) = [0,\infty) = \partial_c F(x)$ for every $x \in [0,1]$. We proceed to compute $\partial_a F^{-1}$ and $\partial_c F^{-1}$. Since F is absolutely continuous, F maps sets of zero measure onto sets of zero measure and $F(B_k)$ is measurable. Because F is strictly increasing on $[0,1]$ and $F'(x) = a_k$ at almost every $x \in B_k$, we have

$$\mu(F(B_k) \cap [F(x), F(y)]) = \mu(F(B_k \cap [x,y])) = a_k \mu(B_k \cap [x,y]) > 0,$$

for any $x < y \in [0,1]$. This shows that the range of F is a countable union of disjoint measurable sets $\{F(B_k)\}_{k=1}^{\infty}$, each with positive measure in every subinterval of the range of F. On $F(B_k)$ we have $(F^{-1})' = 1/a_k$ almost everywhere. The proof is completed by observing that $\{1/a_k\}_{k=1}^{\infty}$ is also dense in $[0,\infty)$ and that F^{-1} is strictly increasing. □

Corollary 4.4. *There are infinitely many absolutely continuous functions such that $\partial_a F = \partial_c F \equiv \mathbb{R}$ on $[0,1]$. For each such a function F, there is a residual set G such that $\partial_{mp} F(x) = \mathbb{R}$ if $x \in G$.*

Proof. Let $A := \{r \in \mathbb{R} : r \text{ is rational}\}$. Then for arbitrary rational $r \in A$, Theorem 4.2 and (2) imply $r \in \partial_a F(x)$. Thus $\mathbb{R} \subset \partial_a F(x) \subset \partial_c F(x) \subset \mathbb{R}$. We proceed to compute $\partial_{mp} F$. For every r, the function $F_r : [0,1] \to \mathbb{R}$ defined by $F_r(x) := F(x) - rx$ is continuous. In every subinterval of $[0,1]$, there are positive measure sets on which $F_r' > 0$ and some positive measure sets on which $F_r' < 0$, thus F_r is a nowhere monotone function. By Lemma 3.2 the sets

$$G_{-n} := \{x : F_-(x) \leq -n < n \leq F^-(x)\}$$

and

$$G_n := \{x : F_+(x) \leq -n < n \leq F^+(x)\},$$

are residuals. The set $G := \bigcap_{n=1}^{\infty} G_n$ is residual in $[0,1]$, and at $x \in G$ we have $F^+(x) = F^-(x) = +\infty$ and $F_+(x) = F_-(x) = -\infty$. It follows from (4) that $F^\diamond(x;1) \geq +\infty$ and $F^\diamond(x;-1) \geq +\infty$, and so $\partial^\diamond F(x) = \mathbb{R}$ if $x \in G$. □

Corollary 4.5. *There exist infinitely many Lipschitz functions F on $[0,1]$ such that $\partial_a F = \partial_c F \equiv [-1,1]$.*

Proof. Choose $A := \{r \in [-1,1] : r \text{ is a rational number}\}$. For every $x \in [0,1]$ and $r \in A$, Theorem 4.2 and (2) imply $r \in \partial_a F(x)$. Since r is arbitrary and $\partial_a F(x) \subset [-1,1]$ is closed, we have $\partial_a F(x) = [-1,1]$. □

When $A = \{-1, 1\}$ the function F is called *Rockafellar's function*. The computation both of the approximate subdifferential and the Michel-Penot subdifferential of Rockafellar's function is not immediately clear. One indirect way to compute its approximate subdifferential is to use the result given by Borwein and Fitzpatrick [2]. Below we give a direct approach by using nowhere monotone functions.

Theorem 4.6. *Let f be Rockafellar's function. Then:*

(i) $\partial_c f = \partial_a f \equiv [-1, 1]$ *on* $[0, 1]$.

(ii) The set $G := \{x : f^+(x) = f^-(x) = 1, f_-(x) = f_+(x) = -1\}$ *is a residual set in* $[0, 1]$*. Thus, f is Dini subdifferentiable at most on a first category subset.*

(iii) For $x \in G$, $\partial_{mp} f(x) = [-1, 1]$.

Proof. (i). Choose $-1 < r < 1$. Consider the function g defined by $g(x) := f(x) + rx$. Since both $\{x : g'(x) = 1 + r > 0\}$ and $\{x : g'(x) = -1 + r < 0\}$ are dense in $[0, 1]$, g is nowhere monotone and so g has local minimizers densely on $[0, 1]$. Let S_r denote those minimizers. If $x \in S_r$, we have $f(y) + ry \geq f(x) + rx$ for y near by x. Then $-r \in \partial_- f(x)$. Since $-1 < r < 1$ is arbitrary, we have $\partial_a f(x) = [-1, 1]$.

(ii) and (iii). For $n \geq 2$, both the functions given by $f(x) + (-1 + 1/n)x$ and $f(x) + (1 - 1/n)x$ are continuous and nowhere monotone in $[0, 1]$. Thus by Lemma 3.2

$$G_{-n} := \{x : f_-(x) \leq -1 + 1/n < 1 - 1/n \leq f^-(x)\},$$
$$G_n := \{x : f_+(x) \leq -1 + 1/n < 1 - 1/n \leq f^+(x)\}.$$

are residuals in $[0, 1]$. If $x \in G := \bigcap_{n=2}^{\infty}(G_n \cap G_{-n})$, we have $f_-(x) \leq -1, f^-(x) \geq 1, f_+(x) \leq -1, f^+(x) \geq 1$. Since f has Lipschitz constant 1, we deduce $f_-(x) = f_+(x) = -1$ and $f^-(x) = f^+(x) = 1$. Moreover, by (4), $1 \geq f^\diamond(x; 1) \geq \max\{f^+(x), f^-(x)\} = 1$, and

$$1 \geq f^\diamond(x; -1) \geq \max\{-f_-(x), -f_+(x)\} = 1.$$

Hence $\partial_{mp} f(x) = [-1, 1]$. $\square$

One may compare Corollary 4.5 and Theorem 4.6 to Theorems 10.3, 10.4. In many cases, Rockafellar's function is the beginning point for building more pathological Lipschitz functions. In the following, we give one of many such applications.

Lemma 4.7. *Let F be continuously differentiable around z and g locally Lipschitz around $F(z)$. Then $\partial_a(g \circ F)(z) = F'(z) \cdot \partial_a g(F(z))$.*

PROOF. By Corollary 5.4 [20] we have $\partial_a(g \circ F)(z) \subset F'(z) \cdot \partial_a g(w)$ where $w = F(z)$. To prove the reverse inclusion, we consider two cases: (1) if $F'(z) = 0$: since $g \circ F$ is locally Lipschitz, $\emptyset \neq \partial_a g \circ F(z) \subset \{0\}$. Thence $\partial_a(g \circ F)(z) = \{0\}$; (2) if $F'(z) \neq 0$: By the inverse function theorem, F is locally invertible around z. Write $g(w) = g(F \circ F^{-1}(w))$. Then

$$\partial_a g(w) \subset \partial_a(g \circ F)(F^{-1}(w)) \cdot (F^{-1})'(w) = \partial_a(g \circ F)(z) \cdot \frac{1}{F'(z)},$$

That is, $\partial_a g(w) \cdot F'(z) \subset \partial_a(g \circ F)(z)$. □

Theorem 4.8. *Suppose f_1 and f_2 are continuous on $\mathbb{R}$. There exists a locally Lipschitz $h : \mathbb{R} \to \mathbb{R}$ with $\partial_a h(x) = conv\{f_1(x), f_2(x)\}$ for every $x \in \mathbb{R}$.*

PROOF. Let f denote Rockafellar's function. Let $F(x) := \int_0^x (f_1(s) - f_2(s))\, ds$, $k(x) := (f \circ F(x) + F(x))/2$, and $h(x) := k(x) + \int_0^x f_2(s)\, ds$. By Lemma 4.7 we have $\partial_a k(x) = [0,1] \cdot (f_1(x) - f_2(x))$, and so $\partial_a h(x) = \mathrm{conv}\{f_1(x), f_2(x)\}$. □

5 The Michel-Penot Subdifferential on Null Sets.

We now show that given any null set there exists a Lipschitz function such that its Michel-Penot subdifferential is large on that set. We start with a lemma from [24, page 195].

Lemma 5.1. *Let $F \subset \mathbb{R}$ be closed, $T \subset \mathbb{R}$ be measurable, $F \cap T = \emptyset$, and let ω be any real, positive increasing function on $(0, +\infty)$. Then there is an open set U such that*

$$T \subset U \subset (\mathbb{R} \setminus F) \text{ and } \mu((x - r, x + r) \cap (U \setminus T)) \leq \omega(r),$$

whenever $x \in F$ and $r > 0$.

PROOF. Let d_F be the distance function associated with the closed set F. For $n \in \mathbb{N}$ we let $R_n := \{x \in \mathbb{R} : d_F(x) > 1/n\}$. For each $n \in \mathbb{N}$ there is an open set $U_n \subset R_n$ such that

$$T \cap R_n \subset U_n \text{ and } \mu(U_n \setminus (T \cap R_n)) = \mu(U_n \setminus T) < \epsilon_n,$$

where $\{\epsilon_n\}$ is a sequence of positive numbers satisfying $\sum_{j=k}^{\infty} \epsilon_j < \omega(1/k)$ for each $k \in \mathbb{N}$. We set $U := \bigcup_{n=1}^{\infty} U_n$. Obviously, $T \subset U \subset \bigcup_{n=1}^{\infty} R_n = \mathbb{R} \setminus F$. Let $x \in F$ and $r > 0$. There is a smallest $n \in \mathbb{N}$ for which $1 \leq nr$. Hence

$$\mu((x - r, x + r) \cap (U \setminus T)) \leq \sum_{k=n}^{\infty} \mu(U_k \setminus T) \leq \sum_{k=n}^{\infty} \epsilon_k < \omega(1/n) \leq \omega(r). \quad \square$$

Theorem 5.2. *Let $N \subset \mathbb{R}$ with $\mu(N) = 0$. Then there exists a Lipschitz function H on $\mathbb{R}$ such that $\partial_{mp}H(x) = [0,1]$ if $x \in N$.*

PROOF. The proof follows Lemma 1 [18]. Inductively, we define a sequence $\{G_n\}_{n=1}^{\infty}$ of open subsets of $\mathbb{R}$ in the following fashion.

(i) Choose an open set $G_1 \supset N$ such that $\mu(G_1) < 1$;

(ii) Once an open set $G_n \supset N$ is defined, we choose an open set $G_n \supset G_{n+1} \supset N$ such that $\mu(G_{n+1}) < 1/(n+1)$, and whenever $x \in \mathbb{R} \setminus G_n$ and $h > 0$ we have

$$\mu((x-h, x+h) \cap G_{n+1}) < h/(n+1).$$

The existence of G_{n+1} may be deduced as follows. After G_n has been defined, we set $F := \mathbb{R} \setminus G_n$, $T := N$ and $\omega(r) := r/(n+1)$. Applying Lemma 5.1, we obtain

$$N \subset G_{n+1} \subset G_n \text{ and } \mu((x-r, x+r) \cap G_{n+1}) \leq \frac{r}{n+1},$$

whenever $x \in \mathbb{R} \setminus G_n$ and $r > 0$. Moreover,

$$\mu(G_{n+1}) \leq \sum_{n=1}^{\infty} \mu(U_n) < \sum_{n=1}^{\infty} \epsilon_n < \omega(1) = \frac{1}{n+1}.$$

Now put $P := \bigcup_{n=1}^{\infty}(G_{2n-1} \setminus G_{2n})$ and $H(x) := \int_0^x \chi_P(t)\,dt$. Clearly H is 1-Lipschitz function. We show that $\partial_{mp}H(x) = [0,1]$ if $x \in N$. To this end, we consider a positive integer k. Let (a_k, b_k) be the component of G_k which contains x. By (ii), $b_k - a_k < \frac{1}{k}$ and

$$\mu(G_{k+1} \cap (x, b_k)) \leq \frac{1}{k+1}(b_k - x), \quad \mu(G_{k+1} \cap (a_k, x)) \leq \frac{1}{k+1}(x - a_k). \tag{5}$$

If k is odd, then $G_k \setminus G_{k+1} \subset P$ and therefore (5) gives

$$\frac{H(b_k) - H(x)}{b_k - x} = \frac{\mu(P \cap (x, b_k))}{b_k - x} \geq 1 - \frac{1}{k+1} \text{ and}$$
$$\frac{H(a_k) - H(x)}{a_k - x} = \frac{\mu(P \cap (a_k, x))}{x - a_k} \geq 1 - \frac{1}{k+1}$$

If k is even, then $P \cap (a_k, b_k) \subset G_{k+1}$ and therefore by (5) we have

$$\frac{H(b_k) - H(x)}{b_k - x} \leq \frac{1}{k+1} \text{ and } \frac{H(a_k) - H(x)}{a_k - x} \leq \frac{1}{k+1}.$$

Since H is nondecreasing and 1-Lipschitz, we have $H^+(x) \le 1, H^-(x) \le 1$, $H_+(x) \ge 0, H_-(x) \ge 0$, and so $H^+(x) = H^-(x) = 1$ and $H_+(x) = H_-(x) = 0$. By (4), $H^\diamond(x;1) = 1$ and $H^\diamond(x;-1) = 0$. Therefore $\partial_{mp}H(x) = [0,1]$ if $x \in N$. □

When $N \subset \mathbb{R}$ is an F_σ set with $\mu(N) = 0$, a generic result holds (See Lemma 10.2).

Corollary 5.3. *Let N be dense in $\mathbb{R}$ with $\mu(N) = 0$. Then there exists a Lipschitz function H on $\mathbb{R}$ such that $\partial_{mp}H(x) = [0,1]$ on a residual set containing N and $\partial_c H \equiv [0,1]$ on $\mathbb{R}$.*

Proof. Let H be the function given in Theorem 5.2. The mean-value theorem in Michel-Penot subdifferential form implies $\partial_c H(x) = \limsup_{y\to x} \partial_{mp}H(y)$ [3]. When N is dense in $\mathbb{R}$, we obtain $\partial_c H(x) \equiv [0,1]$ for every $x \in \mathbb{R}$. For every $n \ge 2$, both functions given by $H(x) - x/n$ and $H(x) - (1-1/n)x$ are nowhere monotone in $\mathbb{R}$. By Lemma 3.2 the sets

$$G_{-n} := \{x : H_-(x) \le 1/n \le 1 - 1/n \le H^-(x)\}$$

and

$$G_n := \{x : H_+(x) \le 1/n \le 1 - 1/n \le H^+(x)\},$$

are residuals. Since H is nondecreasing and has Lipschitz constant 1, the set

$$G := \bigcap_{n=2}^{\infty} (G_{-n} \cap G_n) = \{x \in \mathbb{R} : H_+(x) = H_-(x) = 0, H^+(x) = H^-(x) = 1\},$$

is residual in $\mathbb{R}$ and $N \subset G$. If $x \in G$, we have $\partial_{mp}H(x) = [0,1]$ by (4). □

6 The Space of Nondecreasing Continuous Functions.

Consider the complete metric space

$$X := \{f : f \text{ is continuous and nondecreasing on } [a,b]\}, \text{ with metric}$$

$$\rho(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)| \text{ for } f,g \in X.$$

For $\nu \in \mathbb{R}$, we define $f_{-\nu} : [a,b] \to \mathbb{R}$ by $f_{-\nu}(x) := f(x) - \nu \cdot x$.

Theorem 6.1. *In (X,ρ), the set $\{f \in X : \partial_c f = \partial_a f \equiv [0,+\infty)\}$ is residual.*

PROOF. Let I denote an open subinterval of $[a,b]$, and let

$$A_I^n := \{f \in X\colon \text{there exists } \nu \in \left[\tfrac{1}{n}, n\right] \text{ such that } f_{-\nu} \text{ is nondecreasing on } I\},$$

$$B_I^n := \{f \in X\colon \text{there exists } \nu \in \left[\tfrac{1}{n}, n\right] \text{ such that } f_{-\nu} \text{ is nonincreasing on } I\}.$$

(1). A_I^n is closed. Assume $\{f_m\} \subset A_I^n$ is Cauchy. Then $f_n \to f$ uniformly for some $f \in X$. For each k, there exists $\nu_k \in [1/n, n]$ such that $f_k(x) - \nu_k x \geq f_k(y) - \nu_k y$ for all $x \geq y$ with $x, y \in I$. There exists an increasing sequence $\{k_i\}$ such that $\{\nu_{k_i}\}$ converges to some $\nu \in [1/n, n]$. Taking the limits, we have $f(x) - \nu x \geq f(y) - \nu y$ for $x \geq y$ with $x, y \in I$. Similar arguments show that B_I^n is closed.

(2). To show that A_I^n is nowhere dense, with $f \in X$ we verify that every open ball $B_{2\epsilon}(f)$ contains points of $X \setminus A_I^n$. Fix $x_0 \in I$, and define a nondecreasing h by $h(x) := f(x_0) + \epsilon + \min\{x - x_0, 0\}$. Then $h_1 := \max\{f, h\}$ and $h_2 := \min\{f + 2\epsilon, h_1\}$ are continuous and nondecreasing. As $h_1 \geq f$, $f + 2\epsilon \geq f$, we have $f + 2\epsilon \geq h_2 \geq f$. For $\delta > 0$ sufficiently small, we have $f(x_0) - \epsilon \leq f(y) \leq f(x_0) + \epsilon$ for $|y - x_0| \leq \delta$. For $x_0 + \delta \geq y \geq x_0$, $h(y) = f(x_0) + \epsilon$. Thus $h_1(y) = f(x_0) + \epsilon$ for $x_0 \leq y \leq x_0 + \delta$. But $f(x_0) + \epsilon \leq f(y) + 2\epsilon \leq f(x_0) + 3\epsilon$ for $x_0 \leq y \leq x_0 + \delta$. Then $h_2(y) = f(x_0) + \epsilon$ for $x_0 \leq y \leq x_0 + \delta$. For every $\nu \in [1/n, n]$, on $[x_0, x_0 + \delta]$ we have $(h_2(y) - \nu \cdot y)' = -\nu < 0$ almost everywhere. Thus $h_2(y) - \nu y$ is decreasing on $[x_0, x_0 + \delta]$, and $h_2 \notin A_I^n$.

To show that B_I^n is nowhere dense, we use similar arguments. Define $h \in X$ by $h(x) := \max\{(n+1)(x - x_0), 0\} + f(x_0) - \epsilon$, $h_1 := \min\{f, h\}$, and $h_2 := \max\{f - 2\epsilon, h_1\}$. Then $h_2 \in X$ and $f - 2\epsilon \leq h_2 \leq f$. For $\delta > 0$ sufficiently small, $h_2(x) = (n+1)(x - x_0)$ on $[x_0, x_0 + \delta]$. For every $\nu \in [1/n, n]$, $(h_2(x) - \nu \cdot x)' = n + 1 - \nu > 0$ almost everywhere. Thus $h_2(x) - \nu \cdot x$ is increasing on $[x_0, x_0 + \delta]$, and $h_2 \notin B_I^n$.

(3). Thus both A_I^n and B_I^n are nowhere dense and closed. The sets $A_I := \bigcup_{n=1}^{\infty} A_I^n$ and $B_I := \bigcup_{n=1}^{\infty} B_I^n$ are first category of type F_σ in X. Let $\{I_k\}$ be all open subintervals of $[a,b]$ having rational endpoints. The sets $A := \bigcup_{k=1}^{\infty} A_{I_k}$ and $B := \bigcup_{k=1}^{\infty} B_{I_k}$ are first category of type F_σ. It follows that the set $X \setminus (A \cup B)$ is a residual set of type G_δ. If $f \in X \setminus (A \cup B)$, then for every $\nu > 0$, the function $f_{-\nu}$ is not monotonic on every I_k; thus nowhere monotonic on $[a,b]$. The set of points at which $f_{-\nu}$ attains local minimum is dense in $[a,b]$. We have $\nu \in \partial_a f(x)$ for every $x \in [a,b]$. Since $\nu \in (0, +\infty)$ is arbitrary, we have $[0, +\infty) \subset \partial_a f(x) \subset \partial_c f(x) \subset [0, +\infty)$, completing the proof of the theorem. $\square$

Combining Lemma 3.2 and Theorem 6.1 we see that a typical nondecreasing continuous real-valued function on $[a,b]$ has a finite derivative only on a first category set on $[a,b]$. Compare this with *Lebesgue's Differentiation Theorem* [36, page 100]: If f is an increasing real-valued function on the interval $[a,b]$, then f has a finite derivative almost everywhere.

7 The Space of Automorphisms.

A function of bounded variation is called *singular* if it has almost everywhere a zero derivative. As a singular function has almost everywhere a zero derivative, all of its variation is centered at points of the complementary set of measure zero. So it is the set of measure zero which contributes towards the entire structure of a singular function.

Definition 7.1. A homeomorphism h of an interval $[a,b]$ onto $[a,b]$ that satisfies $h(a)=a$ and $h(b)=b$ is called an *automorphism* on $[a,b]$.

Note that an automorphism from $[a,b]$ to $[a,b]$ is simply a continuous surjective and strictly increasing function. Let us recall that a metric space (X,ρ) is called *topologically complete* if X can be remetrized with a topologically equivalent metric so as to be complete. Alexanderoff's Theorem [7, page 458] asserts that a non-empty set of type G_δ contained in a complete metric space can be remetrized so as to be complete.

Let H denote the family of strictly increasing continuous functions on $[0,1]$ that fix the endpoints. Since a uniform limit of functions on H need not be strictly increasing, H is not closed in $C[0,1]$ (see page 153). But H is of type G_δ in the complete space $\overline{H}$ (the closure of H in $C[0,1]$) and therefore topologically complete. Consequently, Baire category arguments can still be applied. The following lemma is from [7, pages 468–471].

Lemma 7.2. *Let A be a first-category subset of $[0,1]$. Let $H_1 := \{h \in H : \mu(h(A)) = 0\}$. Then H_1 is residual in the topologically complete space H.*

Now let A be a first category subset of $[0,1]$ with $\mu(A)=1$. For $h \in H_1$, $\mu(h(A)) = 0$. Since h is differentiable almost everywhere, we have $h'(x)=0$ for almost every $x \in [0,1]$ [38, page 323]; so every $h \in H_1$ is a strictly increasing continuous singular function.

Lemma 7.3. *If a singular function f is continuous and strictly increasing, then, for every real number $r > 0$, the function $f(x)-rx$ is nowhere monotone.*

PROOF. Assume the derivates of f are bounded from above in some interval $J \subset [0,1]$. As f is increasing, its derivates are ≥ 0 throughout J. For a continuous function, the lower and upper bounds of each its derivates are the same as those of the difference quotient $(f(y)-f(x))/(y-x)$ with $x,y \in J$,

$x \neq y$ [5]. It follows that f is Lipschitz on J. Since f has zero derivative almost everywhere in J, f is constant in J. This contradicts the fact that f is strictly increasing in $[0,1]$. Hence the derivates of f are unbounded from above in every subinterval of $[0,1]$. Assume $r > 0$. Define the function F_r by $F_r(x) := f(x) - rx$. F_r has derivates > 0 at points everywhere dense in $[0,1]$. Moreover, since f is singular, the function F_r also has a derivative $-r < 0$ at an everywhere dense set of points in $[0,1]$. This shows F_r is nowhere monotone on $[0,1]$. □

Theorem 7.4. *The set* $H_1 := \{f \in H : \partial_c f = \partial_a f \equiv [0,+\infty)\}$ *is residual in the topologically complete space* H. *Moreover, for every* $f \in H_1$ *we have* $\partial_{mp} f(x) = [0,+\infty)$ *on a residual set of* $[0,1]$.

Proof. In Lemma 7.2, we chose A to be of first category and $\mu(A) = 1$. As indicated, each $f \in H_1$ is a continuous and strictly increasing singular function. For fixed $r > 0$, Lemma 7.3 shows F_r is nowhere monotone. Each nowhere monotone continuous function has everywhere dense sets of maxima and minima by Lemma 3.3. At each minimal point $x \in (0,1)$, we have $0 \in \partial_- F_r(x)$ and so the set $\{x \in [0,1] : r \in \partial_- f(x)\}$ is dense in $[0,1]$. This implies $r \in \partial_a f(x)$ for every $x \in [0,1]$. Since $r > 0$ is arbitrary and $\partial_a f(x) \subset [0,+\infty)$, we have $[0,+\infty) = \partial_a f(x) = \partial_c f(x)$.

Next, given an $f \in H_1$ and a natural number n, since the functions F_n and $F_{1/n}$ are both nowhere monotone in $[0,1]$, by Lemma 3.2 there exists a residual set G_n in $[0,1]$ such that when $x \in G_n$ we have

$$f_-(x) \leq \frac{1}{n} < n \leq f^-(x), \quad f_+(x) \leq \frac{1}{n} < n \leq f^+(x).$$

If $x \in G := \bigcap_{n=1}^{\infty} G_n$, then $f_-(x) \leq 0$, $f_+(x) \leq 0$, $f^+(x) = f^-(x) = +\infty$. Since f is increasing, its derivates are all non-negative. Then $f_-(x) = f_+(x) = 0$. The proof is complete by using (4) to obtain

$$\begin{aligned} &f^\diamond(x;1) \geq \{f^+(x), f^-(x)\} = +\infty \text{ and} \\ &\qquad 0 \geq f^\diamond(x;-1) \geq \max\{-f_+(x), -f_-(x)\} = 0. \end{aligned}$$ □

The Cantor function $f : [0,1] \to [0,1]$ is continuous and nondecreasing [38, pages 129–130]. Besides, almost everywhere on $[0,1]$, we have $f'(x) = 0$. The most usual strictly increasing continuous singular function on $[0,1]$ or $\mathbb{R}$ is constructed from Cantor's function [38, page 210]. It is interesting to compute $\partial_a f$ and $\partial_c f$ on the Cantor ternary set K associated with f. Because f is not strictly increasing, Lemma 7.3 does not apply.

Theorem 7.5. *Let* f *be the Cantor function* $[0,1] \to [0,1]$ *associated with the Cantor ternary set* K. *Then* $\partial_a f(x) = \partial_c f(x) = [0,+\infty)$ *if* $x \in K$.

PROOF. Fix $x \in K$ and $r > 0$. Assume $I \subset [0,1]$ is an arbitrary open subinterval with $x \in I$. Theorem 7.20 [7] shows that $\{x : f'(x) = +\infty, x \in I\}$ is uncountable. But it is not true that $f'(x) = +\infty$ at all two-sided limit points of K. By Morse's theorem [5] for every $\alpha > 0$ the set $\{x : f_+(x) = \alpha, x \in I\}$ has cardinality $\mathfrak{c}$. Choose $y \in I$ with $f'(y) = +\infty$. Consider F_r defined by $F_r(x) := f(x) - r \cdot x$. Then $F_r'(y) = +\infty$, and for sufficiently small $\delta > 0$ we have $F_r(z) > F_r(y)$ if $z \in (y, y+\delta)$. Since $F_r' = -r$ almost everywhere, we may choose $\hat{y} < y$ with $F_r'(\hat{y}) = -r$, and so there exists $\hat{\delta} > 0$ such that $F_r(z) > F_r(\hat{y})$ if $z \in (\hat{y} - \hat{\delta}, \hat{y})$. It follows that F_r has a local minimizer in $(\hat{y} - \hat{\delta}, y + \delta) \subset I$. Then $0 \in \partial_- F_r(z)$ for some $z \in (\hat{y} - \hat{\delta}, y + \delta)$; that is, $r \in \partial_- f(z)$. Because I is arbitrary, we have $r \in \partial_a f(x)$. But $r > 0$ is also arbitrary. Thus $[0, +\infty) \in \partial_a f(x)$. Since f is nondecreasing, $\partial_a f(x) \subset [0, +\infty)$. Hence $\partial_a f(x) = [0, +\infty)$. Now $\partial_a f(x) \subset \partial_c f(x) \subset [0, +\infty)$ implies $\partial_c f(x) = [0, +\infty)$. □

When $f'(x) = +\infty$, $\partial_- f(x) = \emptyset$. Every open interval $I \subset [0,1]$ containing points of the Cantor set K has uncountably many such points. Theorem 7.5 shows $\partial_a f(x) = \partial_c f(x) = [0, +\infty)$ at these points of K. We see that f is not regular at uncountably many points on every open interval containing points of the Cantor set.

8 The Space of Continuous Functions $C[0,1]$.

Let $C[0,1]$ denote the Banach space of real-valued continuous functions f defined on $[0,1]$ with the uniform norm $\|f\| := \sup_{0 \leq x \leq 1} |f(x)|$. We will show that a typical $f \in C[0,1]$ is an antiderivative of a constant Clarke, approximate and Michel-Penot subdifferential map; i.e., the set-valued map defined by $T(x) :\equiv \mathbb{R}$ for every $x \in \mathbb{R}$. Moreover for every such f, its Dini subdifferential is non-empty only on a set which is Lebesgue null and first category, and its minimal Jeyakumar's convexificator may be chosen as the empty set almost everywhere.

For a Lipschitz function, its Clarke subdifferential $\partial_c f$ has a closed graph, but for a continuous function f, this might fail. However, the following result helps when we compute the Clarke subdifferential for continuous functions.

Proposition 8.1. *Assume $\{x_k\}_{k=1}^{\infty}$ are local minimizers of g on a general Banach space X. If $x_k \to x$, $g(x_k) \to g(x)$, and g is lower semicontinuous around x, then $0 \in \partial_c g(x)$.*

PROOF. Suppose $0 \notin \partial_c g(x)$. We consider two cases: (1). If $\partial_c g(x) = \emptyset$, by Theorem 2.9.1[9], $g^{\uparrow}(x; 0) = -\infty$; (2). If $\partial_c g(x) \neq \emptyset$, by the strong separation theorem [17], there exists $h \in X$ such that

$$g^{\uparrow}(x;h) = \sup\{\langle x^*, h\rangle : x^* \in \partial_c g(x)\} < 0.$$

In either case, $g^{\uparrow}(x;h) < 0$ for some $h \in X$. Since

$$g^{\uparrow}(x;h) = \sup_{\epsilon>0} \limsup_{\substack{y\to x, g(y)\to g(x)\\ t\downarrow 0}} \inf_{\|w-h\|<\epsilon} \frac{g(y+tw)-g(y)}{t},$$

for every $\epsilon > 0$ and $t_k \downarrow 0$ we have

$$0 > \limsup_{\substack{t_k\to 0\\ x_k\to x}} \inf_{\|w-h\|<\epsilon} \frac{g(x_k+t_k w)-g(x_k)}{t_k}. \tag{6}$$

Since x_k is a local minimizer of g and $\|w\| \leq \epsilon + \|h\|$ (thus w is bounded), we may take $0 < t_k < 1/k$ such that $g(x_k + t_k w) \geq g(x_k)$ for every $\|w - h\| < \epsilon$. For such $(t_k)_{k\in\mathbb{N}}$ we have

$$\limsup_{\substack{t_k\to 0\\ x_k\to x}} \inf_{\|w-h\|<\epsilon} \frac{g(x_k+t_k w)-g(x_k)}{t_k} \geq 0.$$

But this contradicts equation (6). Hence $0 \in \partial_c g(x)$. □

Definition 8.2. The function f is said to have *Jeyakumar's convexificator*, $\partial^* f(x)$, at x if $\partial^* f(x)$ is closed and for each $v \in \mathbb{R}$ we have

$$f^-(x;v) \leq \sup_{x^*\in\partial^* f(x)} \langle x^*, v\rangle, \text{ and } f^+(x;v) \geq \inf_{x^*\in\partial^* f(x)} \langle x^*, v\rangle.$$

In term of classical Dini derivatives, a closed set $\partial^* f(x)$ is Jeyakumar's convexificator of f at x if

$$\max\{f_+(x), f_-(x)\} \leq \sup_{x^*\in\partial^* f(x)} x^* \text{ and } \min\{f^-(x), f^+(x)\} \geq \inf_{x^*\in\partial^* f(x)} x^*.$$

Obviously one can always choose $\partial^* f(x) = \mathbb{R}$. A convexificator, $\partial^* f(x)$, of f yields both an upper convex approximation and a lower concave approximation to f at x. The Clarke subdifferential and Michael-Penot subdifferential are convexificators when $f^{\uparrow}(x,\cdot)$ and $f^{\diamond}(x,\cdot)$ are lower semicontinuous. Moreover, if f is locally Lipschitz, then the approximate subdifferential and Treiman linear generalized subdifferential are convexificators [22]. The interesting thing is to find minimal convexificators.

Define

$$E(f) := \{x : f^+(x) = f^-(x) = +\infty, f_-(x) = f_+(x) = -\infty\} \text{ for } f \in C[0,1].$$

Our main result is the following.

Theorem 8.3. *There exists a residual set of functions $f \in C[0,1]$ for each of which:*

1) *The set $E(f)$ is residual in $(0,1)$ and $\mu(E(f)) = 1$.*

2) *If $x \in E(f)$, every closed set in $\mathbb{R}$, including the empty set, may be chosen as $\partial^* f(x)$.*

3) *For every $x \in E(f)$, $\partial_- f(x) = \emptyset$.*

4) *For every $x \in [0,1]$, we have $\partial_a f(x) = \partial_c f(x) = \mathbb{R}$.*

5) *For every $x \in (0,1)$, we have $\partial_{mp} f(x) = \mathbb{R}$ and $\partial_{mp} f(x) \neq \partial_- f(x)$.*

Proof. We prove Theorem 8.3 by piecing together results from [5, 16, 29]. Recall that a function f is called *nonangular* at x if $f_-(x) \leq f^+(x)$ and $f_+(x) \leq f^-(x)$.

Lemma 8.4. *The functions of nonmonotonic type form a dense subset, denoted by S_1, of type G_δ in $C[0,1]$.*

Lemma 8.5. *The nonangular functions form a dense set, denoted by S_2, of type G_δ in $C[0,1]$.*

The proofs of Lemma 8.4 and Lemma 8.5 may be found in [5, pages 212–213]. If $f \in S_1 \cap S_2$, then f is nowhere differentiable. Assume $\partial_- f(x) \neq \emptyset$ at $x \in (0,1)$. If $x^* \in \partial_- f(x)$, then $f^-(x) \leq x^* \leq f_+(x)$. Since f is nonangular at x, we have $f_+(x) \leq f^-(x)$. This means $\partial_- f(x) = \{x^* : x^* = f^-(x) = f_+(x)\}$. Hence every $f \in S_1 \cap S_2$ is nowhere differentiable and $\partial_- f(x)$ is either a singleton or empty at $x \in (0,1)$.

Lemma 8.6. *If $f \in S_1$, then $\overline{f}'(x) = +\infty$ and $\underline{f}'(x) = -\infty$ if $x \in (0,1)$.*

Proof. Fix $x \in (0,1)$ and $\nu \in (-\infty, \infty)$. Since f is of nonmonotonic type at x, for every n there exists $x_n \in (x - 1/n, x)$ and $y_n \in (x, x + 1/n)$ such that

$$\frac{f(y_n) - f(x)}{y_n - x} \geq \nu \text{ and } \frac{f(x_n) - f(x)}{x_n - x} < \nu, \quad \text{or}$$

$$\frac{f(y_n) - f(x)}{y_n - x} < \nu \quad \text{and} \quad \frac{f(x_n) - f(x)}{x_n - x} \geq \nu.$$

As $n \to \infty$, we have either $f^+(x) \geq \nu$ or $f^-(x) \geq \nu$, so that $\overline{f}'(x) \geq \nu$. Also, either $f_-(x) \leq \nu$ or $f_+(x) \leq \nu$, so that $\underline{f}'(x) \leq \nu$. Since ν is arbitrary, it follows that $\underline{f}'(x) = -\infty$, while $\overline{f}'(x) = +\infty$. □

Lemma 8.7. *There is a residual subset $S_3 \subset C[0,1]$ such that for every $f \in S_3$,*

$$\mu(\{x \in [0,1] : f^+(x) = f^-(x) = +\infty \text{ and } f_+(x) = f_-(x) = -\infty\}) = 1.$$

PROOF. See [16, page 453]. $\square$

Now we let

$$C_0 := S_1 \cap S_2 \cap S_3. \tag{7}$$

Lemma 8.8. *For every $f \in C_0$ the set $E(f)$ is residual in $(0,1)$.*

PROOF. Let f be a continuous nowhere monotone function of the second species in $[0,1]$. Given a positive integer n, as the functions $f(x) + n \cdot x$ and $f(x) - n \cdot x$ are both nowhere monotone in $[0,1]$, it follows from Lemma 3.2 that there exists a residual set $G_n \subset [0,1]$ such that for each $x \in G_n$, $f_+(x) = f_-(x) \leq -n < n \leq f^+(x) = f^-(x)$. Then set $G = \bigcap_{n=1}^{\infty} G_n$ is residual in $[0,1]$ and at each $x \in G$ we have $f_+(x) = f_-(x) = -\infty$, $f^+(x) = f^-(x) = +\infty$. $\square$

Lemma 8.9. *For each $r \in \mathbb{R}$, if $f \in C_0$, then $D_r = \{x \in (0,1) | \, \partial_- f(x) = \{r\}\}$ is dense in $(0,1)$. In particular, every $f \in C_0$ is Dini subdifferentiable at $\mathfrak{c}$-dense set of points (i.e., its cardinality is $\mathfrak{c}$ in each subinterval of $[0,1]$).*

PROOF. For every r we will show that

$$D_r := \{x : f_+(x) = r = f^-(x), f^+(x) = +\infty, f_-(x) = -\infty\},$$

is dense in $[0,1]$. Since f is nowhere monotone of second species, for every $r \in \mathbb{R}$ the function $g : [0,1] \to \mathbb{R}$ defined by $g(x) := f(x) - r \cdot x$ is continuous and nowhere monotone. By Lemma 3.3, g has minima at a set S being everywhere dense in $(0,1)$. At $x \in S$ we have $f^-(x) \leq r \leq f_+(x)$. Lemma 8.6 shows that $[f_-(x), f^-(x)] \cup [f_+(x), f^+(x)] = [-\infty, +\infty]$; whence $f_-(x) = -\infty$, $f^+(x) = +\infty$, and $f_+(x) \leq f^-(x)$. This shows $f^-(x) = r = f_+(x)$, so $S \subseteq D_r$. Fixing an arbitrary nondegenerate subinterval $I \subset [0,1]$, for every $r \in \mathbb{R}$ we have $D_r \cap I \neq \emptyset$ because D_r is dense. The $\mathfrak{c}$-dense result follows from the observation that $D_{r_1} \cap D_{r_2} = \emptyset$ if $r_1 \neq r_2$. $\square$

To finish the proof of Theorem 8.3, we observe that Lemma 8.7 and Lemma 8.8 give parts 1), 2), and 3). By Lemma 8.9 for every $r \in \mathbb{R}$ we have $r \in \partial_a f(x)$, this means $\mathbb{R} \subset \partial_a f(x) \subset \partial_c f(x) \subset \mathbb{R}$, which is part 4). By Lemma 8.6 and (4), $f^\diamond(x, 1) = +\infty$ and $f^\diamond(x, -1) = +\infty$ for every $x \in (0,1)$. Thus $\partial_{mp} f(x) = \mathbb{R}$, but $\partial_- f(x)$ is singleton whenever it exists, this gives 5). $\square$

Remark 8.10. Katriel has shown that for every lower semicontinuous function f defined on $\mathbb{R}$ the approximate subdifferential and the Clarke subdifferential agree on a G_δ set of $\mathbb{R}$ [23]. Our result shows that in $C[0,1]$ the functions which share the same trivial Clarke subdifferential and approximate subdifferential map form a dense G_δ subset of $C[0,1]$. There are many results on the integrability of subdifferentials of non-locally Lipschitz functions [31, 33, 39, 32]. Unless one assumes stringent conditions on the function or the subdifferential map, one can not recover the function from its subdifferential uniquely up to an additive constant.

Remark 8.11. In order to study the integration of proximal subdifferentials, Poliquin has introduced a class of "primal lower-nice" functions which can be uniquely determined, up to a constant, by their proximal subdifferentials. If f is primal lower-nice at x, then $\partial_p f(x) = \partial_c f(x)$ [31]. If $f \in C_0$ (see Equation (7)), we see that $\partial_p f(x)$ is either empty or a singleton, whereas $\partial_c f(x) \equiv \mathbb{R}$. Thus each function $f \in C_0$ is not primal lower-nice at any $x \in (0,1)$. If $f \in C_0$, then $\partial_- f(x)$ is either a singleton or empty, whereas $\partial_a f(x) = \partial_c f(x) = \mathbb{R}$. Hence each $f \in C_0$ is neither Clarke nor approximate subdifferentially regular at each point in $(0,1)$. Furthermore, every $f \in C_0$ is not directionally Lipschitz at each $x \in (0,1)$.

Example 8.12. Let S be a nonempty closed subset of $\mathbb{R}^n$ and $x \in S$. Then the distance function $d_S : \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_S(y) := \inf\{\|y - s\| : \ s \in S\}, \tag{8}$$

is regular at x if and only if S is regular at x [1]. If $f \in C_0$, then *epi* f is not regular at any $(x, f(x))$ for $x \in (0,1)$. With $S :=$ *epi* f, d_S is not regular at any point of its boundary.

Example 8.13. (1). The nowhere differentiable *Weierstrass function* $W : [0,1] \to \mathbb{R}$ is defined by $W(x) := \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x)$ where $0 < a < 1$, b is an odd positive integer, and $ab > 1 + 3\pi/2$. Set $E(W) := \{x : W^+(x) = W^-(x) = +\infty, W_+(x) = W_-(x) = -\infty\}$, and

$$\begin{aligned}
E_{c1} :=& \{x : W'_+(x) = +\infty, W'_-(x) = -\infty\}, \\
E_{c2} :=& \{x : W'_+(x) = -\infty, W'_-(x) = +\infty\}, \\
E_1 :=& \{x : W'_+(x) = +\infty, W^-(x) = +\infty, W_-(x) = -\infty\}, \\
E_2 :=& \{x : W'_+(x) = -\infty, W^-(x) = +\infty, W_-(x) = -\infty\}, \\
E_3 :=& \{x : W'_-(x) = +\infty, W^+(x) = +\infty, W_+(x) = -\infty\}, \\
E_4 :=& \{x : W'_-(x) = -\infty, W^+(x) = +\infty, W_+(x) = -\infty\}.
\end{aligned}$$

For $r \in \mathbb{R}$, we define

$$\begin{aligned}
E_{1r} &:= \{x : W^+(x) = r, W_+(x) = -\infty, W^-(x) = +\infty, W_-(x) = -\infty\},\\
E_{2r} &:= \{x : W^+(x) = +\infty, W_+(x) = r, W^-(x) = +\infty, W_-(x) = -\infty\},\\
E_{3r} &:= \{x : W^+(x) = +\infty, W_+(x) = -\infty, W^-(x) = r, W_-(x) = -\infty\},\\
E_{4r} &:= \{x : W^+(x) = +\infty, W_+(x) = -\infty, W^-(x) = +\infty, W_-(x) = r\}.
\end{aligned}$$

In [15] Garg has shown that the sets $E(W)$, E_{ci} $(i = 1, 2)$, E_i $(i = 1$ to $4)$, and E_{ir} $(i = 1$ to $4, r \in \mathbb{R})$ cover all the points of $(0, 1)$, and that the points of these sets are distributed in the interval in the following manner:

(i) $E(W)$ is residual in $(0, 1)$ with $\mu(E(W)) = 1$.

(ii) E_{ci} $(i = 1, 2)$ are both enumerable and everywhere dense in $(0, 1)$.

(iii) each of the sets E_i $(i = 1$ to $4)$ and E_{ir} $(i = 1$ to 4, $r \in \mathbb{R})$ is of the first category with measure equal to zero and has the power of the continuum in every subinterval of $(0, 1)$.

Then $\partial_- W(x) = \mathbb{R} = \partial_p W(x)$ if $x \in E_{c1}$ and $\partial_- W(x) = \emptyset = \partial_p W(x)$ if $x \in (0, 1)\setminus E_{c1}$, while $\partial_-(-W)(x) = \mathbb{R} = \partial_p(-W)(x)$ if $x \in E_{c2}$ and $\partial_-(-W)(x) = \emptyset = \partial_p(-W)(x)$ if $x \in (0, 1) \setminus E_{c2}$. This means both W and $-W$ are only countably Dini or proximally subdifferentiable on $(0, 1)$.

Now equation (2) shows $\partial_a W(x) = \mathbb{R} = \partial_c W(x)$ and $\partial_a(-W)(x) = \mathbb{R} = \partial_c(-W)(x)$ for every $x \in [0, 1]$. Since $\overline{W}'(x) = +\infty$ and $\underline{W}'(x) = -\infty$ for every $x \in (0, 1)$, $\partial_{mp} W(x) = \mathbb{R}$ for each $x \in (0, 1)$. Thus W is only subdifferentiably regular on E_{c1} and $-W$ is only subdifferentiably regular on E_{c2} and nowhere else. Let $\sharp$ stand for $-$ or p. For every $k > 0$,

$$\begin{aligned}
\partial_\sharp(kW)(x) &= \partial_\sharp W(x) = \mathbb{R} \text{ if } x \in E_{c1},\\
\partial_\sharp(kW)(x) &= \partial_\sharp W(x) = \emptyset \text{ if } x \in (0, 1) \setminus E_{c1},\\
\partial_\sharp(-kW)(x) &= \partial_\sharp(-W) = \mathbb{R} \text{ if } x \in E_{c2},\\
\partial_\sharp(-kW)(x) &= \partial_\sharp(-W) = \emptyset \text{ if } x \in (0, 1) \setminus E_{c2}.
\end{aligned}$$

But $kW - W = (k - 1)W$ is not constant if $k \neq 1$. This answers the following question negatively.

> *Let f and g both be continuous on $(0, 1)$. Assume $\partial_\sharp f(x) = \partial_\sharp g(x)$ and $\partial_\sharp(-f)(x) = \partial_\sharp(-g)(x)$ for every $x \in (0, 1)$. Is $f - g$ constant on $(0, 1)$?*

Let $\partial^*_m W$ denote the minimal convexificator map of W.

(i) If $x \in E(W)$, then every closed set, including the empty set, may be chosen as $\partial^* W(x)$. Thus $\partial^*_m W(x) = \emptyset$.

(ii) If $x \in E_{c1} \cup E_{c2}$, then every nonempty closed set, unbounded from above and below, may be chosen as $\partial^* W(x)$. Thus there is no minimal convexificator.

(iii) If $x \in E_1 \cup E_3$, then every nonempty closed set unbounded above may be chosen as $\partial^* W(x)$, so there is no minimal convexificator. If $x \in E_2 \cup E_4$, then every nonempty closed set unbounded below may be chosen as $\partial^* W(x)$, so there is no minimal convexificator.

(iv) Fix $r \in \mathbb{R}$, if $x \in E_{1r} \cup E_{3r}$, then every closed nonempty set with infimum less than or equal to r may be chosen as $\partial^* W(x)$, so $\partial^*_m W(x) = \{\hat{r}\}$ as long as $\hat{r} \leq r$; if $x \in E_{2r} \cup E_{4r}$, then every nonempty closed set with supremum greater than or equal to r may be chosen as $\partial^* W(x)$, so $\partial^*_m W(x) = \{\tilde{r}\}$ as long as $\tilde{r} \geq r$.

Observe that the minimal convexificator on E_{ir} ($i = 1$ to 4, $r \in \mathbb{R}$) is not unique.

(2). Let ϕ be the function on $\mathbb{R}$ defined by $\phi(x) =: |x|$ if $|x| \leq 2$ and $\phi(x + 4p) = \phi(x)$ if $x \in \mathbb{R}$ and $p \in Z$. ϕ is in fact the distance function $\phi(x) = d_A(x)$ where $A := \{4m | \; m \in Z\}$. Setting $f_n(x) := 4^{-n}\phi(4^n x)$, *the van derWaerden function* is defined by $f(x) := \sum_{n=1}^{\infty} f_n(x)$, and f is continuous and nowhere differentiable [38, pages 174–175]. Hence nowhere monotone of the second species. Therefore $\partial_a f(x) = \partial_c f(x) = \mathbb{R}$ for every $x \in \mathbb{R}$.

(3). Choosing any nondifferentiable function $f : \mathbb{R} \to \mathbb{R}$ we define $F(x, y) := f(x) + f(y)$. Then $\partial_a F(x, y) = \partial_a f(x) \times \partial_a f(y)$. Since $\partial_a f(x) = \mathbb{R}$ for every $x \in \mathbb{R}$, we have $\partial_a F(x, y) = \mathbb{R}^2 = \partial_c F(x, y)$ for every $(x, y) \in \mathbb{R}^2$.

9 Such Pathological Behavior is Actually Prevalent!

What happens measure theoretically if we consider the nondifferentiable functions in $C[0, 1]$ with supremum norm? The set of nowhere differentiable functions in the metric space $C[0, 1]$ forms a set that is co-analytic; that is, the complement of an analytic set, and not Borel, but universally measurable [7].

Definition 9.1. A function $f \in C[0, 1]$ is *M-Lipschitz at a point* $x \in [0, 1]$ if there exists a constant M such that for all $y \in [0, 1]$, $|f(y) - f(x)| \leq M|y - x|$. We say f is Lipschitz at x if it is M-Lipschitz for some M.

The concept of M-Lipschitz at x is called calmness at x in optimization. See [35, pages 322, 351] for characterizations and applications. One can prove that if $f : [0,1] \to \mathbb{R}$ is Lipschitz at every point in $[0,1]$, then f is densely locally Lipschitz on $[0,1]$.

Let

$$A_n := \{f \in C[0,1] : \ f \text{ is } n\text{-Lipschitz at some } x \in [0,1]\}.$$

Then A_n is closed and nowhere dense. The nowhere Lipschitz functions $A := \bigcap_{n=1}^{\infty} C[0,1] \setminus A_n$ are a dense G_δ in $C[0,1]$. Let $g(x) := \sum_{k=1}^{\infty} 1/k^2 \cos 2^k \pi x$, and $h(x) := \sum_{k=1}^{\infty} 1/k^2 \sin 2^k \pi x$. Hunt showed [19] the following.

Proposition 9.2. *For all $f \in C[0,1]$, $\{(\alpha,\beta) : (\alpha g + \beta h) \in f + \bigcup_{n=1}^{\infty} A_n\}$ has Lebesgue measure zero in $\mathbb{R}^2$.*

From this we see that $\bigcup_{n=1}^{\infty} A_n$ is Haar null. Since the set of nowhere differentiable functions B contains A, we have $C[0,1] \setminus B \subset \bigcup_{n=1}^{\infty} A_n$, so $C[0,1] \setminus B$ is Haar null. One may now say almost every function in $C[0,1]$ has trivial Clarke and approximate subdifferentials. A self contained arguments, using nowhere monotone functions, come as follows:

If f is not nowhere monotone of the second species on $[0,1]$, then for some r we have $f(x) + rx$ monotone on some subinterval $I \subset [0,1]$. Let $r \in \mathbb{R}$ and define f_r by $f_r(x) := f(x) + rx$. Let I be a subinterval of $[0,1]$. Define

$$A_I := \{f \in C[0,1] : \text{there exists a } r \in \mathbb{R} \text{ with } f_r \text{ being nondecreasing on } I\}.$$

For each $n \in N$, let A_n denote those functions $f \in C[0,1]$ for which there exists $r \in [-n,n]$ such that f_r is nondecreasing on I. Then $A_I = \bigcup_{n=1}^{\infty} A_n$. We show that for each $n \in N$ the set A_n is closed and $C[0,1] \setminus A_n$ is dense.

To verify that A_n is closed, let f_k be a sequence of functions in A_n such that $f_k \to f$ uniformly. Then $f \in C[0,1]$, and we must show that $f \in A_n$. For each k, there exists $r_k \in [-n,n]$ such that $f_k(x) + r_k x \geq f_k(y) + r_k y$ if $x \geq y$ and $x, y \in I$. There exists an increasing sequence k_i from $\mathbb{N}$ such that $\{r_{k_i}\}$ converges to some $r \in [-n,n]$. Then $f(x) + rx \geq f(y) + ry$. Thus $f \in A_n$, and A_n is closed in $C[0,1]$. To show that A_n is nowhere dense, we verify that A_n has no interior. Take a continuous nowhere differentiable function g defined on $[0,1]$. For every $\epsilon > 0$, we claim $f + \epsilon g \notin A_n$ if $f \in A_n$. Suppose $f + \epsilon g \in A_n$. Then for some r_1 we have $h(x) := f(x) + \epsilon g + r_1 x$ being monotone on I. Since $f \in A_n$, there exists another r_2 with $f(x) + r_2 x$ being monotone on I. But

$$h(x) - r_1 x + r_2 x = (f(x) + r_2 x) + \epsilon g(x),$$

implies $g(x) = [h(x) - (f(x) + r_2 x) - r_1 x + r_2 x]/\epsilon$. Hence g is differentiable almost everywhere on I, a contradiction. Thus A_n is nowhere dense and closed.

Now we show that A_n is Haar null. Let g be a nowhere differentiable function. Define a Borel probability measure by $\lambda(E) = \mu\{t \in [0,1] : tg \in E\}$.

We will verify $\lambda(f + A_n) = 0$ for every $f \in C[0,1]$. In fact, the set $\{t \in [0,1] : tg \in A_n + f\}$ is either empty or a singleton. If not, we may find $t_1 \neq t_2$ such that $t_1 g \in f + A_n$ and $t_2 g \in f + A_n$. Then there exists $r_1, r_2 \in [-n, n]$ such that $h_1(x) := t_1 g(x) - f(x) + r_1 x$ and $h_2(x) := t_2 g(x) - f(x) + r_2 x$ are nondecreasing on I. It follows that $g(x) = [h_1(x) - h_2(x) - (r_1 - r_2)x]/(t_1 - t_2)$ is differentiable almost everywhere on I, a contradiction.

Since $A_I = \bigcup_{n=1}^{\infty} A_n$, A_I is Haar null and a countable union of nowhere dense closed sets. The same is true of the set $B_I := \{f \in C[0,1] : -f \in A_I\}$.

Let $\{I_k\}$ be all the subintervals of $[0,1]$ with rational endpoints. Define $A := \bigcup_k A_{I_k}$ and $B := \bigcup_k B_{I_k}$. It follows that each of A and B is Haar null and a countable union of nowhere dense closed subsets in $C[0,1]$. Then $C[0,1] \setminus (A \cup B)$ is a residual set of type G_δ and $A \cup B$ is Haar null. If $f \in C[0,1] \setminus (A \cup B)$, then for every $r \in \mathbb{R}$ the function f_r is not monotonic at any subinterval of $[0,1]$. Thus it is nowhere monotonic of the second species. Each nowhere monotonic function of the second species f has $\partial_a f = \partial_c f \equiv \mathbb{R}$, and $\partial_- f$ exists only on a first category set of $[0,1]$. Hence, we have proved the following theorem.

Theorem 9.3. *The set*

$$D := \{f \in C[0,1] : \partial_a f = \partial_c f \equiv \mathbb{R} \text{ and } \partial_- f \text{ exists only on a first category set}\}.$$

is prevalent and residual in $C[0,1]$.

This follows from that $C[0,1] \setminus (A \cup B) \subset D$. Nondifferentiable functions constitute a proper subclass of the class of continuous nowhere monotone functions of the second species.

10 Typical Lipschitz Functions Have Constant Subdifferentials.

How should we consider the subdifferentials of Lipschitz functions instead of nowhere monotone functions of the second species? Three spaces come into mind right away:

(1). The space of all Lipschitz functions with supremum norm. Because nowhere differentiable functions are uniform limits of polynomials, the space is not complete.

(2). The space of all Lipschitz functions with the norm given by

$$\|f\| := |f(0)| + \sup\{|f(y) - f(x)|/|y - x| : x, y \in [0,1], x \neq y\},$$

is a Banach space [25]. It is too big in the following sense: (i) the differentiable functions are not dense. Under the Lipschitz norm, if $f_n \to f$, for every $\epsilon > 0$

we have

$$\partial f_n \subset \partial(f_n - f) + \partial f \subset \epsilon B + \partial f, \quad \text{and} \quad \partial f \subset \partial(f - f_n) + \partial f_n \subset \epsilon B + \partial f_n.$$

Let f be the Rockafellar function. Then $\partial f_n \subset \epsilon B + [-1, 1]$, $[-1, 1] \subset \epsilon B + \partial f_n$. If f_n is differentiable, we may take x_0 such that $\partial f_n(x_0) = \{f_n'(x_0)\}$. Then $1 = \epsilon + f_n'(x_0)$ and $-1 = f_n'(x_0) - \epsilon$. If $\epsilon < 1$, we obtain a contradiction; (ii) Lipschitz functions with constant subdifferential maps are not dense. To see this, we define $f(x) = 0$ if $0 \leq x \leq 1/2$, and $f(x) = x - 1/2$ if $1/2 \leq x \leq 1$. If $f_n \to f$ in Lipschitz norm and $\partial_c f_n \equiv [a_n, b_n]$ on $[0, 1]$, then $\|f_n\| \to 0$ on $[0, 1/2]$ and $\|f_n\| \to 1$ on $[1/2, 1]$, a contradiction.

(3). It is in the space of Lipschitz functions with uniformly Lipschitz constant in the topology of uniform convergence that we show typical functions have constant Clarke and approximate subdifferential map. More precisely, we consider the space

$$Lip_M := \{f : [0, 1] \to \mathbb{R} : |f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in [0, 1]\},$$

with the metric

$$\rho(f, g) := \max_{x \in [0,1]} |f(x) - g(x)| \quad \text{for } f, g \in Lip_M.$$

The following lemma may be found in [38, page 165].

Lemma 10.1. *Suppose the metric space Y is complete and that $(f_n)_{n=1}^\infty$ is an equicontinuous sequence in $C(X, Y)$ that converges at each point of a dense subset D of the topological space X. Then there is a function $f \in C(X, Y)$ such that $(f_n)_{n=1}^\infty$ converges to f uniformly on each compact subset K of X.*

As functions in Lip_M are equicontinuous, Lemma 10.1 shows in Lip_M the topology of pointwise convergence and the topology of uniform convergence are the same.

Lemma 10.2. *Let $E \subset [0, 1]$ be an F_σ set of measure 0. Then there is a residual set $S \subset Lip_M$ such that for every $f \in S$ and $x \in E$ we have*

$$\limsup_{y \to x} \frac{f(y) - f(x)}{y - x} = M \text{ and } \liminf_{y \to x} \frac{f(y) - f(x)}{y - x} = -M.$$

In particular, $\partial_{mp} f(x) = [-M, M]$ whenever $f \in S$ and $x \in E$.

PROOF. (1). Let E be a nonempty closed set of measure zero. Let G_k be the set of those $f \in Lip_M$ for which one can find $\delta > 0$ with the property that for every $x \in E$ there is $y \in [0, 1]$ such that $\delta < |y - x| < 1/k$ and

$$\frac{f(y) - f(x)}{y - x} > M - \frac{1}{k} + \delta.$$

We will show that G_k is open in Lip_M. Assume $f_0 \in G_k$. By definition, for some $\delta > 0$, for each $x \in E$, there exists $1/k > |y - x| > \delta$ such that

$$\frac{f_0(y) - f_0(x)}{y - x} > M - \frac{1}{k} + \delta.$$

For this y, there exists $0 < \eta_x < \delta$ and $|y - x| + \eta_x < 1/k$ such that for each $z \in [x - \eta_x, x + \eta_x]$ we have

$$\frac{f_0(y) - f_0(z)}{y - z} > M - \frac{1}{k} + \delta.$$

Then $\{(x - \eta_x, x + \eta_x) : x \in E\}$ covers E. By compactness, we may take a finite number of them to cover E, say $\{(x_i - \eta_{x_i}, x_i + \eta_{x_i})\}_{i=1}^m$. Set $\eta := \max\{\eta_{x_i}\}$. For every $x \in E$, there exists x_i with $x \in [x_i - \eta_{x_i}, x_i + \eta_{x_i}]$ such that

$$1/k > \eta_{x_i} + |y_i - x_i| > |y_i - x_i| + |x - x_i| \geq |y_i - x| \geq |y_i - x_i| - |x_i - x| > \delta - \eta,$$

$$\frac{f_0(y_i) - f_0(x)}{y_i - x} > M - \frac{1}{k} + \delta.$$

Since $(f_0(y_i) - f_0(z))/(y_i - z)$ is continuous on $[x_i - \eta_{x_i}, x_i + \eta_{x_i}]$, its minimum exists denoted by $m_i > M - 1/k + \delta$. Setting $m := \min\{m_i\}$, we have $m > M - 1/k + \delta$. Now, assuming $\rho(f, f_0) \leq \epsilon$, for $x \in (x_i - \eta_{x_i}, x_i + \eta_{x_i})$ with $y = y_i$ we have $1/k > |y - x| > \delta - \eta$ and

$$\begin{aligned}\frac{f(y) - f(x)}{y - x} &= \frac{f(y) - f_0(y) + f_0(x) - f(x)}{y - x} + \frac{f_0(y) - f_0(x)}{y - x} \\ &> \frac{-2\epsilon}{\delta} + \frac{f_0(y) - f_0(x)}{y - x} > -\frac{2\epsilon}{\delta} + m.\end{aligned}$$

If ϵ is sufficiently small, then $-2\epsilon/\delta + m > M - 1/k + \delta > M - 1/k + \delta - \eta$. For this ϵ, we have $B_\epsilon(f_0) \subset G_k$. Thus G_k is an open set.

To prove $G := \bigcap_{k=1}^\infty G_k$ is a residual subset of Lip_M, it suffices to show that it is dense. Whenever $f \in Lip_M$, let $f_j(x) := f(0) + \int_0^x \phi_j(t)\,dt$, where $\phi_j(t) = f'(t)$ if $d_E(t) > 1/j$ and $\phi_j(t) = M$ if $d_E(t) \leq 1/j$ (See (8) for the definition of d_E). Since E is a closed subset of $[0, 1]$ with measure 0, $\bigcap_{j=1}^\infty E_j = E$ and $E_j \subset E_{j-1}$, we have $\lim_{j\to\infty} \mu(\{t \in [0, 1] : d_E(t) \leq 1/j\}) = \mu(E) = 0$. Now

$$|f_j(x) - f(x)| = |\int_0^x \phi_j(t) - f'(t)\,dt| \leq 2M\mu(\{t \in [0, 1] : d_E(t) \leq 1/j\}),$$

shows f_j uniformly converges to f. For fixed j, $f_j \in G_k$ for every k because if $k < j$, we set $\delta_k = 1/(2j)$; if $k \geq j$, we set $\delta_k = 1/(2k)$. Thus $f_j \in G$, and G is residual in Lip_M. Then the set $S := G \cap \{f \in Lip_M : -f \in G\}$ is also residual in Lip_M. If $f \in S$ and $x \in E$, then for every k there exists $\delta_k < |y_k - x| < 1/k$ such that $(f(y_k) - f(x))/(y_k - x) > M - 1/k + \delta_k$. Letting $k \to \infty$, together with $f \in Lip_M$, we have $\limsup_{y\to x}(f(y) - f(x))/(y - x) = M$. Applying the same arguments to $-f$, we obtain $\liminf_{y\to x}(f(y) - f(x))/(y - x) = -M$.

(2). Let $E = \bigcup_{n=1}^{\infty} E_n$ with E_n being closed sets measure 0. We may apply (1) on each E_n to get a residual set S_n. Then $\bigcap_{n=1}^{\infty} S_n$ is the desired residual set.

By (4), we have $f^\diamond(x, 1) = f^\diamond(x, -1) = M$ for every $x \in E$. Then $\partial_{mp} f(x) = [-M, M]$ at $x \in E$. □

Define $E := \{r : r \in (0, 1) \cap \mathbb{Q}\}$. Then E is countable and dense in $[0, 1]$, in particular, of measure zero and F_σ. Thus $\partial_c f = \partial_a f \equiv [-M, M]$. We have proved the following.

Theorem 10.3. *The typical $f \in Lip_M$ has the following property:*

(1) $\partial_c f = \partial_a f \equiv [-M, M]$ on $[0, 1]$.

(2) $\partial_{mp} f(x) = [-M, M]$ for every $x \in (0, 1) \cap \mathbb{Q}$.

Clearly, the same arguments apply on $\mathbb{R}$. One may also deduce Theorem 10.3 via nowhere monotone functions:

Theorem 10.4. *In Lip_1, the set*

$\{f : f(x) - r \cdot x$ is nowhere monotone on $[0, 1]$ for every $|r| < 1\}$ is residual.

PROOF. Let I denote an open subinterval of $[a, b]$, and let

$$A_I^n := \{f \in Lip_1 : \text{there exists some } r \in [-1 + 1/n, 1 - 1/n] \text{ with } f(x) - r \cdot x \text{ being nondecreasing on } I\}.$$

To verify that A_I^n is closed, let $\{f_k\}$ be a sequence of functions in A_I^n such that $f_k \to f$ uniformly. Then $f \in Lip_1$, and we must show that $f \in A_I^n$. For each $k \in N$, there exists $r_k \in [-1+1/n, 1-1/n]$ such that $f_k(x)-r_k x \geq f_k(y)-r_k y$ for $x \geq y \in I$. There exists an increasing sequence $\{k_i\}$ from $\mathbb{N}$ such that $\{r_{k_i}\}$ converges to some $r \in [-1 + 1/n, 1 - 1/n]$, then $f(x) - rx \geq f(y) - ry$ for $x \geq y \in I$. Then $f \in A_I^n$ and A_I^n is closed in Lip_1.

To show that A_I^n is nowhere dense, we verify that every ball in Lip_1 contains points of $Lip_1 \setminus A_I^n$. Let $B_\epsilon(f)$ be an open ball in Lip_1. If $f \notin A_I^n$, there is nothing to prove, so assume $f \in A_I^n$. Let $(x_0 - \epsilon, x_0 + \epsilon) \subset I$. We define

$$\phi_\epsilon(t) := \begin{cases} -1 & \text{if } t \in (x_0 - \epsilon, x_0], \\ 1 & \text{if } t \in (x_0, x_0 + \epsilon], \\ f'(t) & \text{otherwise and provided } f'(t) \text{ exists.} \end{cases}$$

Let $f_\epsilon(x) := f(0) + \int_0^x \phi_\epsilon(t)\,dt$. Then $f_\epsilon \in Lip_1$ and

$$|f(x) - f_\epsilon(x)| = \Big| \int_0^x f'(t) - \phi_\epsilon(t)\,dt \Big| \le \int_0^1 |f'(t) - \phi_\epsilon(t)|\,dt = 4\epsilon.$$

On I, for every $r \in [-1+1/n, 1-1/n]$, the function $f_\epsilon(x) - rx$ is not nondecreasing on I because on $(x_0-\epsilon, x_0)$ the function f_r has derivative $-1-r \le -1/n$. Thus A_I^n is nowhere dense, and so $A_I := \bigcup_{n=2}^\infty A_I^n$ is of first category. Now let $\{I_k\}$ be an enumeration of those open subintervals of $[0,1]$ having rational endpoints. Set $A := \bigcup_{k=1}^\infty A_{I_k}$. Then A is a first category set. Similarly, we show that

$$B := \{f \in Lip_1 : f(x) - rx \text{ is nonincreasing on some open subinterval of } [0,1] \text{ for some } r \in (-1,1)\},$$

is of first category in Lip_1. If $f \in Lip_1 \setminus (A \cup B)$, then for every $r \in (-1,1)$ the function $f(x) - r\cdot x$ is nowhere monotone on $[0,1]$. □

This naive result shows that a typical $f \in Lip_1$ has $\partial_a f = \partial_c f = [-1,1]$. For every such f, $\partial_- f$ exists only on a first category set by Lemma 3.2. Hence, a typical function in Lip_1 is only differentiable on a first category set. This generalizes the classical known fact (exercise 7.9.4 [7]): There exists a Lipschitz function for which the set of points of differentiability is first category.

Now we consider

$$X := \{f : |f(x) - f(y)| \le |x-y| \quad \text{for } x, y \in [a,b] \text{ and } f \text{ is nondecreasing}\},$$

endowed with the supremum metric ρ.

Theorem 10.5. *In (X,ρ), the set*

$$\{f \in X : \partial_a f = \partial_c f \equiv [0,1] \text{ and } f \text{ is strictly increasing}\},$$

is residual.

PROOF. Fix $x \in (a,b)$. Consider

$$G_k := \{f \in X : \frac{f(x+t_1) - f(x)}{t_1} - 1 > -\frac{1}{k} \text{ and } \frac{f(x+t_2) - f(x)}{t_2} < \frac{1}{k} \text{ for some } 0 < t_1, t_2 < \tfrac{1}{k}\}.$$

(1). G_k is open. Let $f_0 \in G_k$. If $\epsilon > 0$ is sufficiently small, for every $f \in X$ satisfying $\rho(f, f_0) < \epsilon$, we have

$$\frac{f(x+t_1) - f(x)}{t_1} - 1 > \frac{-2\epsilon}{t_1} + \frac{f_0(x+t_1) - f_0(x)}{t_1} - 1 > -\frac{1}{k},$$

$$\frac{f(x+t_2)-f(x)}{t_2} < \frac{2\epsilon}{t_2} + \frac{f_0(x+t_2)-f_0(x)}{t_2} < \frac{1}{k},$$

for the same t_1, t_2 associated with f_0.
(2). G_k is dense. Given $f \in X$ and $\epsilon > 0$. Define $\tilde{f}(x) := f(0) + \int_0^x \phi_\delta(t)\,dt$ with

$$\phi_\delta(t) := \begin{cases} f'(t) & \text{if } t \notin [x, x+\delta] \\ 0 & \text{if } t \in (x, x+\tilde{\delta}) \\ 1 & \text{if } t \in (x+\tilde{\delta}, x+\delta), \end{cases}$$

where $\min\{\epsilon/2, 1/k\} > \delta > \tilde{\delta} > 0$ such that $\delta^{-1}[\tilde{f}(x+\delta) - \tilde{f}(x)] = 1 - 1/k^2$ and $\tilde{\delta}^{-1}[\tilde{f}(x+\tilde{\delta}) - \tilde{f}(x)] = 0$. Then $\tilde{f}$ is nondecreasing, 1-Lipschitz, $\tilde{f} \in G_k$ and

$$|f(x) - \tilde{f}(x)| = \Big| \int_0^x f'(s) - \phi_\delta(s)\,ds \Big| \leq 2\delta < \epsilon.$$

Then $G_x := \bigcap_{k=1}^\infty G_k$ is a dense G_δ set in X. If $f \in G_x$, then $f^+(x) = 1, f_+(x) = 0$, so

$$1 \geq f^0(x,1) \geq f^\diamond(x,1) \geq f^+(x) = 1$$

and

$$0 \leq -f^0(x,-1) \leq -f^\diamond(x,-1) \leq f_+(x) = 0.$$

Thus $\partial_{mp} f(x) = \partial_c f(x) = [0,1]$. Let $\{x_k\}$ be dense in $[a,b]$ and set $G := \bigcap_{k=1}^\infty G_{x_k}$. Then G is a dense G_δ in X. If $f \in G$, we have $\partial_{mp} f(x_k) = \partial_c f(x_k) = [0,1]$ for every x_k, so $\partial_c f(x) = \partial_a f(x) = [0,1]$ for each $x \in [a,b]$. Moreover, every $f \in G$ must be strictly increasing, otherwise f would be constant on some subinterval I. Hence $\partial_c f = \{0\}$ on I, a contradiction. $\square$

Of course, as in Theorem 10.4, we can deduce Theorem 10.5 via nowhere monotone functions. The advantage of the above proof is that it can be extended to $\mathbb{R}^n$ or separable Banach spaces.

11 Can the Pseudo-Regular Points Generate the Subdifferential?

One of the open problems in Sciffer's thesis [37] is: "For a locally Lipschitz function ϕ on a separable Banach space, do the pseudo-regular points generate the subdifferential?". See page 139 for the definition of pseudo-regularity. The answer is seen to be 'no' by using nowhere monotone differentiable functions. Observe that a Gâteaux differentiable function ϕ is pseudo-regular at x if and only if $\partial_c \phi$ is a singleton.

A function $f : [0,1] \to \mathbb{R}$ is said to be a bounded derivative function if f is bounded on $[0,1]$ and there exists $F : [0,1] \to \mathbb{R}$ such that $F'(x) = f(x)$ for every $x \in [0,1]$. The space of bounded derivative functions on $[0,1]$, denoted by $M\triangle'$, with metric

$$\rho(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|$$

is complete. Let $M\triangle'_o = \{f \in M\triangle' : f = 0 \text{ on a dense set}\}$.

Lemma 11.1 (Weil). *The set of functions in $M\triangle'_o$ which are positive on one dense subset of $[0,1]$ and negative on another dense subset of $[0,1]$ forms a residual subset of $(M\triangle'_o, \rho)$.*

For the proof of Lemma 11.1, see [38]. For $f \in M\triangle'_o$, we define $F(x) := \int_0^x f(s)\,ds$. Then F is globally Lipschitz and $f' = f$ on $[0,1]$. Lemma 11.1 shows the following.

Proposition 11.2. *Let $\triangle_o$ denote the set of differentiable functions F on $[0,1]$ such that $F(0) = 0$ and $F' \in M\triangle'_o$. For F, $G \in \triangle_o$, let $\rho(F,G) = \sup_{x\in[0,1]} |F'(x) - G'(x)|$. Then*

(i) $(\triangle_o, \rho)$ *is a complete metric space.*

(ii) *If $F \in \triangle_0$, then $0 \in \partial_a F(x) = \partial_c F(x)$ for every $x \in [0,1]$.*

(iii) *A typical $F \in \triangle_o$ has a $\partial_c F$ which is not a singleton on a positive measure subset of each nondegenerate subinterval $I \subseteq [0,1]$.*

Choose $F \in \triangle_o$ satisfying (iii). As $0 \in \partial_c F(x)$ for each $x \in [0,1]$, F is pseudo-regular at x if and only if $\partial_c F(x) = \{0\}$. The cusco generated by pseudo-regular points is identically $\{0\}$. Since $\partial_c F \not\equiv \{0\}$, $\partial_c F$ can not be generated by the pseudo-regular points.

12 A Comparison to Convex Analysis.

For a sequence of convex functions $\{f_i\}$ defined on $A \subset \mathbb{R}^n$, if $\sup\{f_i(x) : i \in \mathbb{N}\} < +\infty$ for every $x \in A$, then $\{f_i\}$ are locally equi-Lipschitz. Thus f_i converges uniformly to f on each compact convex subset of A when $\{f_i\}$ converges to f pointwise on A [34, page 90]. Our typical results may be compared with the following result in convex analysis [34, page 233].

Proposition 12.1 (Rockafellar). *Let f be a convex function on $\mathbb{R}^n$, and let A be an open convex set on which f is finite. Let f_1, $f_2, \ldots,$ be a sequence of convex functions finite on A and converging pointwise to f on A. Let*

$x \in A$, and let $x_1, x_2, \ldots,$ be a sequence of points in A converging to x. Then, for any $y \in \mathbb{R}^n$ and any sequence $y_1, y_2, \ldots,$ converging to y, one has $\limsup_{i\to\infty} f_i'(x_i; y_i) \leq f'(x; y)$. Moreover, given any $\epsilon > 0$, there exists an index i_0 such that $\partial f_i(x_i) \subset \partial f(x) + \epsilon\mathbb{B}_{\mathbb{R}^n}$ for all $i \geq i_0$.

Because every C^1 function is a uniform limit of nondifferentiable functions from A, Theorems 6.1, 8.3, and 10.3 show that Proposition 12.1 fails dramatically for nonconvex continuous functions and Lipschitz functions. In order to pose open questions, we recall [5, page 47].

Theorem 12.2 (Denjoy-Young-Saks). *Let f be an arbitrary finite function defined on $[a, b]$. Then almost every $x \in [a, b]$ is in one of the following four sets:*

(i) A_1 on which f has a finite derivative;

(ii) A_2 on which $f^+ = f_-$ (finite), $f^- = \infty, f_+ = -\infty$;

(iii) A_3 on which $f^- = f_+$ (finite), $f^+ = \infty$, $f_- = -\infty$;

(iv) A_4 on which $f^- = f^+ = \infty$, $f_- = f_+ = -\infty$.

From (i) to (iv), we see that $\partial_- f$ must be either empty-valued or single-valued a.e.. In fact, on the line, for any real function $f : \mathbb{R} \to \mathbb{R}$, the set of points at which $\partial_- f(x)$ is a non-degenerated interval is countable [5, page 45].

Problem 12.3. *What is the analogue of the Denjoy-Young-Saks theorem in terms of $\partial_a f$ or $\partial_c f$ in nonsmooth analysis?*

Problem 12.4. *Let A be an open subset of $\mathbb{R}^n$ with $n > 1$. For each continuous or locally Lipschitz function $f : A \to \mathbb{R}$, is $\{x \in A : \partial_a f(x) = \partial_c f(x)\}$ residual in A?*

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