Real Analysis Exchange
Vol. 30(1), 2004/2005, pp. 105–114

Zbigniew Grande and Ewa Strońska,* Institute of Mathematics, Bydgoszcz Academy, Plac Weyssenhoffa 11, 85-072 Bydgoszcz, Poland.
email: grande@ab.byd.edu.pl and strewa@pro.onet.pl

SOME OBSERVATIONS ON REGULAR DEPENDENCE OF TOTAL VARIATION ON PARAMETERS

Abstract

Let I be a nondegenerate interval and let $X \neq \emptyset$ be a set. For a function $f : X \times I \to \mathbb{R}$ and $x \in X$ define $v(x)$ as the total variation of the section f_x on I. We investigate the regular dependence (measurability, Baire property, etc.) of v on the regularity of the sections f^t.

Let $\mathbb{R}$ be the set of all reals and let I be a nondegenerate interval (open, closed, half-closed, bounded or not). For a function $g : I \to \mathbb{R}$ we define the total variation of g on I as

$$V(g, I) = \sup_{\pi} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|,$$

where the supremum is taken over all partitions $\pi = \{x_0, x_1, \dots, x_n\}$ of I (i.e., $n \in N$, $x_0 < x_1 < \cdots < x_n$ and $x_i \in I$, $i = 0, 1, \dots, n$). We say that g is of bounded variation on I if $V(g, I) < \infty$.

Let X be the set of parameters. For a mapping $f : X \times I \to \mathbb{R}$ define the total variation of the sections $f_x(t) = f(x, t)$, $x \in X$ and $t \in I$ by

$$v(x) = V(f_x, I) \text{ for } x \in X.$$

In [1] the authors investigate some sufficient conditions for regular dependence (different measurability or continuity properties) of a mapping $v : X \to [0, \infty]$ on X. In this article we give some constructions concerning these results.

Key Words: Measurability, density topology, total variation, Baire property, uniform convergence
Mathematical Reviews subject classification: 26A45, 26A15, 26B99
Received by the editors October 28, 2003
Communicated by: B. S. Thomson
*Partially supported by Bydgoszcz Academy grant 2003

In [1] the authors observe that if $I = X = [0,1]$ and $A \subset [0,1]$ is a nonmeasurable subset in the Lebesgue sense (without Baire property), then the function

$$f(x,t) = \begin{cases} 1 & \text{if } x = t \text{ and } x \in A \\ 0 & \text{otherwise on } [0,1]^2 \end{cases}$$

is Lebesgue measurable and has the Baire property, but the function $v(x) = V(f_x, I)$ is not measurable in the Lebesgue sense (does not have the Baire property). In connection with these examples we present some constructions.

Let (Z, T_Z) be a topological space. Recall (e.g., in [4]) that a function $g : Z \to \mathbb{R}$ is said to be quasicontinuous at a point $z \in Z$ if for each real $r > 0$ and each T_Z-open neighborhood $U \subset Z$ of z there is an T_Z-open nonempty subset $V \subset U$ such that $g(V) \subset (g(z) - r, g(z) + r)$.

Theorem 1. *There is a Lebesgue measurable function $f : [0,1]^2 \to [0,1]$ having Darboux Baire 1 quasicontinuous sections $f^t(x) = f(x,t)$, $x, t \in [0,1]$, such that the function $v(x) = V(f_x, [0,1])$, $x \in [0,1]$, is not measurable in the sense of Lebesgue.*

PROOF. Let $C \subset (0,1)$ be a Cantor set of positive Lebesgue measure and let (I_n) be a sequence of all open intervals with rational endpoints. Then the set $A = C \setminus \bigcup\{(I_n \cap C);\ \mu(I_n \cap C) = 0\}$ is a nowhere dense perfect set of positive Lebesgue measure such that for each open interval J with $J \cap A \neq \emptyset$ the intersection $J \cap A$ is of positive Lebesgue measure. Enumerate the set of all components of the set $[0,1] \setminus A$ in a sequence (J_n) such that $J_n \neq J_m$ for $n \neq m$. Let a_n and b_n be the endpoints of J_n with $a_n < b_n$. In the interiors $int(J_n)$ of J_n find nondegenerate closed intervals $K_n = [c_n, d_n] \subset int(J_n)$. Define

$$g(x) = \begin{cases} 1 & \text{for} & x \in K_n, \quad n \geq 1 \\ 0 & \text{if} & x \in A \\ \textit{linear} & \text{on the intervals} & [a_n, c_n] \text{ and } [d_n, b_n] \ \ n \geq 1. \end{cases}$$

There is a Lebesgue nonmeasurable set $B \subset A$. For $(x,t) \in [0,1]^2$ let

$$f(x,t) = \begin{cases} 1 & \text{if} & x \in B, \text{ and } x = t \\ g(x) & \text{otherwise on} & [0,1]^2. \end{cases}$$

Then, evidently, the function f is Lebesgue measurable and the sections f^t, $t \in [0,1]$, are quasicontinuous functions of Baire class 1 that have the Darboux property. Moreover

$$v(x) = 0 \ \text{ for } \ x \in [0,1] \setminus B, \ \text{ and } \ v(x) = 2 \ \text{ for } \ x \in B,$$

so the variation $v : [0,1] \to [0,\infty]$ is a function nonmeasurable in the sense of Lebesgue. $\square$

The function f from the last theorem has the Baire property (it is quasi-continuous even) and the corresponding function v has also the Baire property (it vanishes on the set $[0,1]\setminus B$).

Theorem 2. *Assume the Continuum Hypothesis CH. Then there exists a function $f:[0,1]^2\to[0,1]$ with the Baire property such that the sections f^t, $t\in[0,1]$, have the Darboux property and the corresponding variation function $v(x)=V(f_x,[0,1])$ does not have the Baire property.*

Proof. Let $A\subset[0,1]$ be a G_δ-set of Lebesgue measure zero containing all rationals from $[0,1]$. Then the set A is residual in $[0,1]$ and for each open interval $I\subset[0,1]$ the equality $\mu(I)=\mu(I\setminus A)$ is true. There is a set $B\subset A$ without the Baire property. Moreover, there is a family $\{A_\alpha;\alpha<\omega_1\}$ (ω_1 denotes the first ordinal of the continuum cardinality) of pairwise disjoint subsets $\mathfrak{c}$-dense in $[0,1]$ such that $[0,1]\setminus A=\bigcup_{\alpha<\omega_c}A_\alpha$.

Indeed, let $A_{0,0}\subset[0,1]\setminus A$ be a countable set dense in $[0,1]$. Let $\alpha>0$ be a countable ordinal and assume that we have defined pairwise disjoint countable dense in $[0,1]$ sets $A_{\beta,\gamma}\subset[0,1]\setminus A$, where $\beta,\gamma<\alpha$. Observe that $E=\bigcup_{\beta,\gamma<\alpha}A_{\beta,\gamma}$ is countable set. Then let $A_{0,\alpha}\subset([0,1]\setminus A)\setminus E$ be a countable set dense in $[0,1]$ and for $\beta<\alpha$ let

$$A_{\beta,\alpha}\subset(([0,1]\setminus A)\setminus E)\setminus\bigcup_{\gamma<\beta}A_{\gamma,\alpha}$$

be a countable set dense in $[0,1]$. Moreover, for $\beta\le\alpha$ let

$$A_{\alpha,\beta}\subset\Big((([0,1]\setminus A)\setminus E)\setminus\bigcup_{\gamma<\alpha}A_{\gamma,\alpha}\Big)\setminus\bigcup_{\gamma<\beta}A_{\alpha,\gamma}$$

be a countable set dense in $[0,1]$. Next, for $1\le\alpha<\omega_1$ let $A_\alpha=\bigcup_{\beta<\omega_1}A_{\alpha,\beta}$ and let $A_0=([0,1]\setminus A)\setminus\bigcup_{1\le\alpha<\omega_1}A_\alpha$. Observe that

$$A_0=\bigcup_{\beta<\omega_1}A_{0,\beta}\cup\Big(([0,1]\setminus A)\setminus\bigcup_{\alpha,\beta<\omega_1}A_{\alpha,\beta}\Big).$$

Then the sets $A_\alpha,\ \alpha<\omega_1$, are pairwise disjoint and for each $\alpha<\omega_1$ and each open interval $I\subset[0,1]$ the intersection $A_\alpha\cap I$ is uncountable; i.e., A_α is $\mathfrak{c}$-dense in $[0,1]$. Moreover, $[0,1]\setminus A=\bigcup_{\alpha<\omega_1}A_\alpha$.

Now, fix $\alpha<\omega_c$ and enumerate all open subintervals of $[0,1]$ with rational endpoints in a sequence (I_n). By induction for each positive integer n there is a nowhere dense nonempty perfect set $B_{n,\alpha}\subset(I_n\cap A_\alpha)\setminus\bigcup_{k<n}B_{k,\alpha}$. For $n\ge1$ define a function $g_{n,\alpha}:B_{n,\alpha}\to[0,1]$ such that $g_{n,\alpha}(B_{n,\alpha})=[0,1]$. Let

$$g_\alpha(x) = \begin{cases} g_{n,\alpha}(x) & \text{if } x \in B_{n,\alpha}, \quad n \geq 1 \\ 0 & \text{otherwise on } [0,1]. \end{cases}$$

Now enumerate all elements of the set B in a transfinite sequence b_α, where $\alpha < \omega_c$ such that $b_\alpha \neq b_\beta$ for $\alpha \neq \beta$. For $(x,t) \in [0,1]^2$ put

$$f(x,t) = \begin{cases} 1 & \text{if } x = t \in B \\ g_\alpha(x) & \text{if } x \neq t = b_\alpha, \quad \alpha < \omega_c \\ 0 & \text{otherwise on } [0,1]^2. \end{cases}$$

Since $v(x) = 0$ for $x \in A \setminus B$ and $v(x) = 2$ for $x \in B$, the restricted function v/A does not have the Baire property. But the set A is residual in $[0,1]$, so $v : [0,1] \to [0,\infty]$ is without the Baire property.

We will prove that the sections f^t, $t \in [0,1]$, have the Darboux property. If $t \in A \setminus B$ or $t \in [0,1] \setminus A$, then $f^t(x) = 0$ for $x \in [0,1]$, and consequently the section f^t has the Darboux property. If $t \in B$, then there is an ordinal $\alpha < \omega_1$ with $t = b_\alpha$. So $f^t(x) = g_\alpha(x)$ for $x \neq t$ and $f^t(t) = 1$. Since $g_\alpha(J) = [0,1]$ for each open interval $J \subset [0,1]$, the section f^t has the Darboux property. $\square$

For the next theorems we recall the following definitions and prove some lemmas.

We will say that a family of functions $g_s : Z \to \mathbb{R}$, where $s \in S$ and S is a set of indices, is quasi-equicontinuous at a point $x \in Z$ if for each positive real r and each set $U \in T_Z$ containing x there is a nonempty set $V \subset U$ belonging to T_Z such that $g_s(V) \subset (g_s(x) - r, g_s(x) + r)$ for all indices $s \in S$.

Observe that the sections f^t, $t \in [0,1]$, of the function f constructed in the proof of Theorem 1 are quasi-equicontinuous at each point $x \in [0,1]$.

A function $g : Z \to [-\infty, \infty]$ is said to be lower (resp. upper) semi-quasicontinuous at a point $x \in Z$ if for each real a with $g(x) > a$ (resp. $g(x) < a$) and each set $U \in T_Z$ containing x there is a nonempty set $V \subset U$ belonging to T_Z such that $g(u) > a$ (resp. $g(u) < a$) for all points $u \in V$.

Lemma 1. *If a function $g : Z \to [-\infty, \infty]$ is lower (resp. upper) semi-quasicontinuous at each point $x \in Z$, then g has the Baire property.*

Proof. Fix a real a and observe that the set $A_a = \{x \in Z; f(x) > a\}$ is the union of its interior $\operatorname{int}(A_a)$ and of a set contained in the frontier $\operatorname{fr}(\operatorname{int}(A_a))$ of the $\operatorname{int}(A_a)$, which is nowhere dense. So the set A_a has the Baire property and the proof is completed. $\square$

Lemma 2. *If Φ is a family of lower semi-quasicontinuous functions $g_s : Z \to \mathbb{R}$, where $s \in S$ and S is a set of indices, then the pointwise supremum $h(x) = \sup\{g_s(x); s \in S\}$ is lower semi-quasicontinuous at each point $x \in X$.*

PROOF. Evidently $h(x) > -\infty$ for each point $x \in Z$. Fix a positive real r, a point $x \in Z$ and a set $U \in T_Z$ containing x. First we suppose that $h(x) < \infty$. Since $h(x) = \sup\{g_s(x); s \in S\}$, there is an index $s_1 \in S$ such that $g_{s_1}(x) > h(x) - r$. But the function g_{s_1} is lower semi-quasicontinuous at x, so there is a nonempty set $V \subset U$ belonging to T_Z such that $g_{s_1}(V) \subset (h(x) - r, \infty)$. For each point $u \in V$ we have $h(u) \geq g_{s_1}(u) > h(x) - r$.

If $h(x) = \infty$ then for each real a there is an index $s_2 \in S$ such that $g_{s_2}(x) > a$. The same as above we find a nonempty set $V \subset U$ belonging to T_Z such that $g_{s_2}(V) \subset (a, \infty)$ and observe that $h(V) \subset g_{s_2}(V) \subset (a, \infty]$. $\square$

Theorem 3. *Let (X, T_X) be a topological space and let $f : X \times I \to \mathbb{R}$ be a function such that for each nonempty finite set $S \subset I$ and for each point $x \in X$ the family of the sections f^t, where $t \in S$ is quasi-equicontinuous at x. Then the corresponding total variation $v(x) = V(f_x, I)$ is lower semi-quasicontinuous.*

PROOF. Fix a partition $\pi = \{t_0, \ldots, t_n\}$ of the interval I and observe that the function

$$X \ni x \to \sum_{i=1}^{n} |f(x, t_i) - f(x, t_{i-1})|$$

is quasicontinuous. Of course, for a fixed point $u \in X$ and a set $U \in T_X$ containing u and a positive real r by the quasi-equicontinuity of functions f^{t_i}, where $i = 0, 1, \ldots, n$, at u, there is a nonempty set $V \subset U$ belonging to T_X and such that

$$f^{t_i}(V) \subset \left(f(u, t_i) - \frac{r}{2n+2}, f(u, t_i) + \frac{r}{2n+2} \right) \text{ for } i = 0, 1, \ldots, n.$$

Consequently, for $x \in V$ and each $i \in \{0, 1, \ldots, n\}$ we obtain

$$\begin{aligned}
&|f(x, t_i) - f(x, t_{i-1})| \\
&\qquad \leq |f(x, t_i) - f(u, t_i)| + |f(u, t_i) - f(u, t_{i-1})| + |f(u, t_{i-1}) - f(x, t_{i-1})| \\
&\qquad < |f(u, t_i) - f(u, t_{i-1})| + \frac{r}{2n+2} + \frac{r}{2n+2} \\
&\qquad = |f(u, t_i) - f(u, t_{i-1})| + \frac{r}{n+1}.
\end{aligned}$$

So, for $x \in V$ the inequality

$$\left| \sum_{i=1}^{n} |f(x, t_i) - f(x, t_{i-1})| - \sum_{i=1}^{n} |f(u, t_i) - f(u, t_{i-1})| \right| \leq$$

$$\sum_{i=1}^{n} |(|f(x,t_i) - f(x,t_{i-1})| - |f(u,t_i) - f(u,t_{i-1})|)| < (n+1)\frac{r}{n+1} = r$$

is true and for each partition $\pi = \{t_0, \dots, t_n\}$ the function

$$X \ni x \to \sum_{i=1}^{n} |f(x,t_i) - f(x,t_{i-1})|$$

is quasicontinuous on X. So, by Lemma 2 the total variation v is lower semi-quasicontinuous, as the pointwise supremum of a family of quasicontinuous functions. □

In particular we consider the case, where $X = \mathbb{R}$ and T_X is the density topology. For this we recall some necessary notions.

Denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer density $D_u(A,x)$ ($D_l(A,x)$) of the set A at the point x as

$$\limsup_{h\to 0^+} \frac{\mu_e(A \cap [x-h, x+h])}{2h}$$

$$\left(\liminf_{h\to 0^+} \frac{\mu_e(A \cap [x-h, x+h])}{2h} \text{ respectively}\right).$$

A point x is said to be an outer density point (a density point) of a set A if $D_l(A,x) = 1$ (if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B,x) = 1$).

The family T_d of all sets A for which the implication

$$x \in A \Longrightarrow x \text{ is a density point of } A$$

holds, is a topology called the density topology ([2, 6]). The sets $A \in T_d$ are measurable ([2]).

Let T_e be the Euclidean topology in $\mathbb{R}$. Continuity (quasicontinuity) of functions $g : \mathbb{R} \to \mathbb{R}$ treated as mappings from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$ are said to be approximate continuity (approximate quasicontinuity) (see, e.g., [2, 5] or [3]).

Since a set $A \subset \mathbb{R}$ has the Baire property with respect to the density topology T_d if and only if it is Lebesgue measurable ([5]), as an obvious corollary from the last theorem we obtain the following.

Theorem 4. *If each finite family of sections f^t of functions $f : \mathbb{R} \times I \to \mathbb{R}$ is quasi-equicontinuous with respect to T_d at every point $x \in \mathbb{R}$, then the corresponding total variation $v(x) = V(f_x, I)$ is Lebesgue measurable.*

As an particular case of Theorem 3.2 from [1] we have:

Theorem 5. *Let (X, d_X) be a compact metric space and let $f : X \times [a, b] \to \mathbb{R}$ be a function. Assume that the sections f^t, $t \in [a, b]$, are continuous and the sections f_x, $x \in X$, are continuous of bounded variation on $[a, b]$. Then the corresponding total variation $v(x) = V(f_x, [a, b])$ is continuous on X if and only if the sequence (ϕ_n) of functions $\phi_n : X \to \mathbb{R}$ given by*

$$\phi_n(x) = \sum_{k=1}^{2^n} \left| f\left(x, a + k\frac{b-a}{2^n}\right) - f\left(x, a + (k-1)\frac{b-a}{2^n}\right) \right| \text{ for } x \in X,$$

is uniformly convergent on X.

Observe that the following theorem is true.

Theorem 6. *Assume that X is a nonempty set and Φ is a linear space (over $\mathbb{R}$) of functions from X to $\mathbb{R}$ which is uniformly closed (i.e., it is closed with respect to uniform convergence). Moreover suppose that if $f \in \Phi$, then also $|f| \in \Phi$. Let $f : X \times [a, b] \to \mathbb{R}$ be a function. Assume that the sections $f^t \in \Phi$ for $t \in [a, b]$ and the sections f_x, $x \in X$, are continuous of bounded variation on $[a, b]$. If the sequence (ϕ_n) of functions $\phi_n : X \to \mathbb{R}$ given by*

$$\phi_n(x) = \sum_{k=1}^{2^n} \left| f\left(x, a + k\frac{b-a}{2^n}\right) - f\left(x, a + (k-1)\frac{b-a}{2^n}\right) \right| \text{ for } x \in X,$$

is uniformly convergent on X, then the corresponding total variation $v(x) = V(f_x, [a, b])$ belongs to Φ.

Proof. We repeat the proof of Theorem 3.2 from [1]. Since for a fixed $x \in X$ the section f_x is continuous on $[a, b]$, we obtain $v(x) = \lim_{n\to\infty} \phi_n(x)$. But the sections $f^t \in \Phi$ for $t \in [a, b]$, so $\phi_n \in \Phi$ for $n \geq 1$. From the uniform convergence of the sequence (ϕ_n) follows that $v \in \Phi$. □

As some examples of Φ we can take the families of continuous real functions on arbitrary topological spaces or the family of real cliquish functions on topological spaces ([4]).

Theorem 7. *There is a function $f : [0, 1]^2 \to \mathbb{R}$ such that the sections f_x, $x \in [0, 1]$ are continuous of bounded variation, the sections f^t, $t \in [0, 1]$ are approximately continuous, the corresponding total variation $v(x) = V(f_x, [0, 1])$ is approximately continuous on $[0, 1]$ and the sequence (ϕ_n) of functions $\phi_n : X \to \mathbb{R}$ given by*

$$\phi_n(x) = \sum_{k=1}^{2^n} \left| f\left(x, a + k\frac{b-a}{2^n}\right) - f\left(x, a + (k-1)\frac{b-a}{2^n}\right) \right| \text{ for } x \in [0, 1],$$

is not uniformly convergent on $[0,1]$.

PROOF. Find closed intervals $I_n = [a_n, b_n]$ such that

$$0 < b_{n+1} < a_n < b_n < 1 \text{ for } n \geq 1 \text{ and } \lim_{n\to\infty} a_n = 0,$$

and

$$D_u\Big(\bigcup_{n\geq 1}[a_n, b_n], 0\Big) = 0. \tag{$*$}$$

In each interval I_n, $n \geq 1$, find a closed interval $J_n = [c_n, d_n] \subset (a_n, b_n)$. Next, for each $n \geq 1$ we choose a closed interval $K_n = [u_n, v_n]$, whose the center is of the form $\frac{k_n}{2^{n+1}}$, where $k_n \in \{1, \ldots, 2^n - 1\}$ and the length is less than $\frac{1}{2^{n+3}}$. Moreover, we assume that $K_n \cap K_m = \emptyset$ if $n \neq m$.

For $n \geq 1$ define the functions $g_n : [a_n, b_n] \to [0,1]$, $h_n : [u_n, v_n] \to [0,1]$ and $f_n : I_n \times K_n \to [0,1]$ by

$$g_n(x) = \begin{cases} 0 & \text{if } x \in \{a_n, b_n\} \\ 1 & \text{if } x \in J_n \\ \text{linear} & \text{on the intervals } [a_n, c_n] \text{ and } [d_n, b_n], \end{cases}$$

and

$$h_n(t) = \begin{cases} 0 & \text{if } t \in \{u_n, v_n\} \\ 1 & \text{if } t = \frac{u_n+v_n}{2} \\ \text{linear} & \text{on the intervals } [u_n, \frac{u_n+v_n}{2}] \text{ and } [\frac{u_n+v_n}{2}, v_n], \end{cases}$$

and

$$f_n(x,t) = g_n(x)h_n(t) \text{ for } (x,t) \in I_n \times K_n.$$

Now for $(x,t) \in [0,1]^2$ let

$$f(x,t) = \begin{cases} f_n(x,t) & \text{if } (x,t) \in I_n \times K_n, \quad n \geq 1 \\ 0 & \text{otherwise on } [0,1]^2. \end{cases}$$

Then evidently the sections f_x and f^t, x, $t \in [0,1]$, are continuous, the sections f_x have bounded variation and by $(*)$ the total variation $v(x) = V(f_x, [0,1])$ is approximately continuous. Since v is discontinuous at 0 and the functions ϕ_n are continuous for $n \geq 1$, the convergence of (ϕ_n) to v is not uniform. $\square$

Problem 1.
Let (X, T_X) be a topological space and let $f : X \times [a,b] \to \mathbb{R}$ be a function. Assume that the sections f^t, $t \in [a,b]$, are quasicontinuous. Must the corresponding total variation $v(x) = V(f_x, [a,b])$ have the Baire property?

Problem 2.

Let (X, T_X) be a topological space and let $f : X \times [a, b] \to \mathbb{R}$ be a function. Assume that the sections f^t, $t \in [a, b]$, are quasicontinuous and the sections f_x, $x \in X$, have bounded variation on $[a, b]$. Must the corresponding total variation $v(x) = V(f_x, [a, b])$ have the Baire property?

Problem 3.

Assume that the sections f^t, $t \in [a, b]$, of a function $f : [a, b]^2 \to \mathbb{R}$ are derivatives. Must the corresponding total variation $v(x) = V(f_x, [a, b])$ be Lebesgue measurable?

Problem 4.

Assume that the sections f^t, $t \in [a, b]$, of a function $f : [a, b]^2 \to \mathbb{R}$ are bounded derivatives. Must the corresponding total variation $v(x) = V(f_x, [a, b])$ be Lebesgue measurable?

References

[1] M. Balcerzak, A. Kucia, A. Nowak, *Regular dependence of total variation on parameters*, Real Analysis Exchange, **29** (2003-04), 419–428.

[2] A. M. Bruckner, *Differentiation of real functions*, Lectures Notes in Math., 659, Springer-Verlag, Berlin, 1978.

[3] Z. Grande, *Sur les functions approximativement quasi-continues*, Rev. Roumaine Math. Pures Appl., **34** (1989), 17–22.

[4] T. Neubrunn, *Quasi-continuity*, Real Analysis Exchange, **14, No. 2** (1988-89), 259–306.

[5] J. C. Oxtoby, *Measure and category*, Springer-Verlag, New York, 1996.

[6] F. D. Tall, *The density topology*, Pacific J. Math., **62** (1976), 275–284.