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A LEBESGUE TYPE DIFFERENTIATION THEOREM FOR BEST APPROXIMATIONS BY CONSTANTS IN ORLICZ SPACES

Abstract

The best approximation operator by constants is extended from an Orlicz space $L^{\varphi}(\mathbb{R}^m)$ to the space $L^{\varphi'}(\mathbb{R}^m)$, and some properties of this extended operator are established. Let $f_\varepsilon(x)$ be any best approximation of $f \in L^{\varphi'}(\mathbb{R}^m)$ on a suitable set $B_\varepsilon(x) \subset \mathbb{R}^m$. Weak and strong inequalities are proved for the maximal function associated with the family $\{f_\varepsilon(x)\}$ which are used in the study of pointwise convergence of $f_\varepsilon(x)$ to $f(x)$.

1 Introduction and Results.

Let Φ be the set of all non negative convex functions $\varphi \in C^1[0,\infty)$ such that $\varphi(0) = 0$ and $\varphi \not\equiv 0$. Let Ω be a bounded measurable set in $\mathbb{R}^m$ and as usual, we denote by $L^{\varphi}(\Omega)$ the class of all Lebesgue measurable functions defined on $\mathbb{R}^m$ such that the integral $\int_\Omega \varphi(\lambda|f(x)|)\,dx$ is finite for some $\lambda > 0$, where dx is Lebesgue measure on $\mathbb{R}^m$. We write $|E|$ for the Lebesgue measure of a measurable set E in $\mathbb{R}^m$.

The space $L^{\varphi'}(\Omega)$ is analogously defined, where φ' is the derivative of the function φ. Observe that for $\varphi \in \Phi$, we have $\varphi(x) \le x\varphi'(x) \le \varphi(2x),\ x \ge 0$. Therefore $L^{\varphi}(\Omega) \subseteq L^{\varphi'}(\Omega)$ for any bounded set Ω.

Given a function $f \in L^{\varphi}(\Omega)$ we denote by $\mu_\varphi(f)$ the set of all real constants c which minimizes the integral $\int_\Omega \varphi(|f(x) - c|)\,dx$. It is easy to prove that $\mu_\varphi(f) \neq \emptyset$ for $f \in L^{\varphi}(\Omega)$. The mapping which assigns to each $f \in L^{\varphi}(\Omega)$ the

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set $\mu_\varphi(f)$ is monotone; that is, if $f_1 \le f_2$ pointwise, $c_1 \in \mu_\varphi(f_1)$ and $c_2 \in \mu_\varphi(f_2)$, then $\min(c_1, c_2) \in \mu_\varphi(f_1)$ and $\max(c_1, c_2) \in \mu_\varphi(f_2)$, (see [4]). Since the set $\mu_\varphi(f)$ of the best approximations of the function f is a closed bounded interval, we can restate the above definition of monotonicity as follows. If $f \le g$, then $\min \mu_\varphi(f) \le \min \mu_\varphi(g)$ and $\max \mu_\varphi(f) \le \max \mu_\varphi(f)$. We will use this characterization of monotony.

A function φ satisfies the Δ_2 condition if there exists $k > 0$ such that $\varphi(2t) \le k\varphi(t)$, for all $t > 0$, and in this case we write $\varphi \in \Delta_2$. It is straightforward to prove that $\mu_\varphi(f) \ne \emptyset$, if $\varphi \in \Phi \cap \Delta_2$.

Let Φ^* be the set of all functions $\varphi \in \Phi$ such that $\varphi'(x+1) \le c\varphi'(x)$ for $x \ge 1$ and some fixed $c > 0$.

Let $\varphi \in \Phi$. We will write $\widetilde{L}^\varphi(\Omega)$ for the set of all measurable functions f such that $\int_\Omega \varphi(|f|)\, dx < \infty$. Similarly we define $\widetilde{L}^{\varphi'}(\Omega)$.

The next characterization of $\mu_\varphi(f)$ is well known when $\varphi(t) = t^p,\ p \ge 1$.

Lemma 1. *Let $\varphi \in \Phi^*$. Then for each $f \in \widetilde{L}^\varphi(\Omega)$ the following are equivalent:*

1. $c \in \mu_\varphi(f)$.
2. $|\int_{\{f \ne c\}\cap\Omega} \varphi'(|f-c|)\operatorname{sgn}(f-c)\, dx| \le \varphi'(0)\, |\{f = c\} \cap \Omega|$.

We denoted by $\varphi'(0)$ the right derivative of φ at 0. Here we have used, for example, the notation $|\{f = c\} \cap \Omega|$ to emphasizes the set Ω, since sometimes the function f will be defined on a larger set than Ω.

Other characterizations of the best approximations may be found in [5], for the case of $\varphi(t) = t^p,\ p > 1$, and a rather exhaustive set of characterizations has recently appeared in [6], for the case $p = 1$. In both papers the approximation class is a σ lattice of functions. We state the next theorem for the particular case when the approximation class is the set of real constants.

Theorem 2. *Let $\varphi \in \Phi^*$ and $f \in \widetilde{L}^\varphi(\Omega)$. Then any of the following three statements are equivalent to $c \in \mu_\varphi(f)$:*

(1) (a) $\int_{\{f>c\}\cap\Omega} \varphi'(|f-c|)\, dx \le \int_{\{f\le c\}\cap\Omega} \varphi'(|f-c|)\, dx$.

(b) $\int_{\{f<c\}\cap\Omega} \varphi'(|f-c|)\, dx \le \int_{\{f\ge c\}\cap\Omega} \varphi'(|f-c|)\, dx$.

(2) (a) $\int_\Omega \varphi'(|f-c|)\, dx \le 2\int_{\{f\le c\}\cap\Omega} \varphi'(|f-c|)\, dx$.

(b) $\int_\Omega \varphi'(|f-c|)\, dx \le 2\int_{\{f\ge c\}\cap\Omega} \varphi'(|f-c|)\, dx$.

(3) (a) For any $\alpha > c$ we have $\int_\Omega \varphi'(|f-c|)\, dx \le 2\int_{\{f<\alpha\}\cap\Omega} \varphi'(|f-c|)\, dx$.

(b) For any $\alpha < c$ we have $\int_\Omega \varphi'(|f-c|)\, dx \le 2\int_{\{f>\alpha\}\cap\Omega} \varphi'(|f-c|)\, dx$.

From the proof of Theorem 2 it is easy to see the next remark.

Remark. The conditions (1), (2) and (3) of Theorem 2 are equivalent even for $f \in \widetilde{L}^{\varphi'}(\Omega)$, when $\varphi \in \Phi^*$.

Note that $\Phi \cap \Delta_2 \subsetneq \Phi^*$. For example $\varphi(x) = e^{\sqrt{x+1}} - \sqrt{x+1} - e + 1$ is a function in Φ^* which is not a Δ_2 function. For the remain results we will not analyze spaces generated by functions in $\Phi^* - \Delta_2$. Also observe that $\widetilde{L}^{\varphi}(\Omega) = L^{\varphi}(\Omega)$ if $\varphi \in \Delta_2$.

Definition 1. Let $\varphi \in \Phi \cap \Delta_2$. We say that a constant c is a best approximation of f for $f \in L^{\varphi'}(\Omega)$ if the real number c satisfies any of the three conditions in Theorem 2, and we denote by $\mu_{\varphi'}(f)$ the set of the best approximations of f.

Given $\varphi \in \Phi$ and assume φ' be a strictly increasing function. Then the set $\mu_{\varphi}(f)$ has only one element. If $\varphi \in \Phi \cap \Delta_2$ and $\varphi'(0) = 0$ the characterization of $\mu_{\varphi}(f)$ is particularly simple, in this case a constant c is a best approximation of the function $f \in L^{\varphi}(\Omega)$ if and only if

$$\int_{\Omega} \varphi'(|f-c|)\operatorname{sgn}(f-c)\, dx = 0. \tag{1.1}$$

Definition 2. Let Φ_0 be the set of all functions $\varphi \in \Phi \cap \Delta_2$ such that $\varphi'(0) = 0$.

In order to get sharper estimates for the best approximation operator which is originally defined in some function space it is suitable to extend the operator to a wider space. For example in [5] the authors extend the best approximation operator from L^p to L^{p-1} when $p > 1$ and the approximation class is a σ lattice of functions. In [6] it is considered an extension from L^1 to the set of all measurable functions which are finite almost everywhere and the approximation class is again a σ lattice of functions. The same authors in [7] extend the best approximation operator by constants from L^p to L^{p-1}, for $p \geq 1$, where L^0 means the set of measurable functions which are finite a. e..

Now for a function $f \in L^{\varphi'}(\Omega),\ \varphi \in \Phi_0$ it is easy to see that there exists an unique solution for (1.1), provided φ' is a strictly increasing function, which will be called the best approximation of f. For the case $\varphi \in \Phi \cap \Delta_2$ and $f \in L^{\varphi'}(\Omega)$ to show that $\mu_{\varphi'}(f) \neq \emptyset$ requires more work; see Lemma 10.

For f locally in $L^{\varphi'}(\mathbb{R}^m)$ we write $f \in L^{\varphi'}_{loc}(\mathbb{R}^m)$ and for any $x \in \mathbb{R}^m$ we consider a family $\{B_\varepsilon(x)\}$ of bounded measurable sets with $0 < |B_\varepsilon(x)|$. Also we set $\mu^{\varepsilon}_{\varphi'}(f)$ for the set $\mu_{\varphi'}(f)$ of the best approximations of f by constants on the set $B_\varepsilon(x)$.

Theorem 3. *Let $\varphi \in \Phi_0,\ f \in L^{\varphi'}(\mathbb{R}^m)$, $\varepsilon > 0$ and $f_\varepsilon(x) \in \mu^{\varepsilon}_{\varphi'}(f)$. Then we have the following estimates.*

1. $\varphi'(|f_\varepsilon(x)|) \le \frac{3k^2}{2}|B_\varepsilon(x)|^{-1}\int_{B_\varepsilon(x)}\varphi'(|f(y)|)\,dy.$

2. $\varphi'(|f_\varepsilon(x)-f(x)|) \le \frac{3k^2}{2}|B_\varepsilon(x)|^{-1}\int_{B_\varepsilon(x)}\varphi'(|f(y)-f(x)|)\,dy,$

where the constant k is the one given by the Δ_2 condition on φ.

We say that a family $\{B_\varepsilon(x)\}$ differentiates $L^{\varphi'}(\mathbb{R}^m)$ if for every $f \in L^{\varphi'}_{loc}(\mathbb{R}^m)$,

$$\lim_{\varepsilon\to 0}\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)}\varphi'(|f(y)-f(x)|)\,dy = 0,$$

for almost every $x \in \mathbb{R}^m$. Many families can be found in the literature which differentiate specific examples of $L^{\varphi'}$. The classical example among others is the family of balls centered at x and radius ε or cubes containing x with side ε and where the space of functions is $L^1_{loc}(\mathbb{R}^m)$, (see [1]). It is wort noting that in our set up sometimes the family $\{B_\varepsilon(x)\}$ is asked to differentiate a bigger space than $L^1_{loc}(\mathbb{R}^m)$. As a corollary of (2) in Theorem 3 we obtain the next result.

Theorem 4. *Let $\phi \in \Phi_0$ with $\varphi'(t) > 0,\ t > 0$ and $\{B_\varepsilon(x)\}$ be a family that differentiates $L^{\varphi'}(\mathbb{R}^m)$. Then for every $f \in L^{\varphi'}_{loc}(\mathbb{R}^m)$ and for almost every $x \in \mathbb{R}^m$ we have*

$$\lim_{\varepsilon\to 0}(\sup\{|f_\varepsilon(x)-f(x)| : f_\varepsilon(x) \in \mu^\varepsilon_{\varphi'}(f)\}) = 0.$$

Given a function $f \in L^1_{loc}(\mathbb{R}^m)$ we denote by $M_H(f)$ the Hardy-Littlewood maximal function $\sup_{\varepsilon>0}\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)}|f(y)|\,dy$, and for $f \in L^{\varphi'}_{loc}(\mathbb{R}^m)$ set

$$Mf(x) = \sup_{\varepsilon>0}\{|f_\varepsilon(x)| : f_\varepsilon(x) \in \mu^\varepsilon_{\varphi'}(f)\}.$$

The next theorem is a direct consequence of part 1 in Theorem 3.

Theorem 5. *Let $\varphi \in \Phi_0$ and $f \in L^{\varphi'}_{loc}(\mathbb{R}^m)$. Then we have*

$$\varphi'(Mf(x)) \le \frac{3k^2}{2}M_H(\varphi'\circ f)(x),$$

where the constant k corresponds to the Δ_2 condition on φ.

According to [3] we say that a function φ satisfies the ∇_2 condition, and we write $\varphi \in \nabla_2$, if there exists a constant $r > 1$ such that $\varphi(t) < \frac{1}{2r}\varphi(rt)$, for all $t \ge 0$. From now on the family $\{B_\varepsilon(x)\},\ \varepsilon > 0,\ x \in \mathbb{R}^m$ should be more specific, for example we will take the standard balls centered at x with radius ε, however other different families of sets can be used.

Let Ψ denote the set of all increasing functions ψ from $[0,\infty)$ into itself such that $\psi(0^+) = 0$, $\lim_{t\to\infty}\psi(t) = \infty$. The following result is proved in [2]. Given $\psi \in \Psi$. Then the ∇_2 condition on ψ is equivalent to the strong inequality

$$\int_{\mathbb{R}^m} \psi(M_H(f)(x))\,dx \le C\int_{\mathbb{R}^m} \psi(Cf(x))\,dx,$$

for all $f \in L^1_{loc}(\mathbb{R})$, and where the constant C depends only on ψ. The next theorem is consequence of the above result and Theorem 5.

Theorem 6. *Let $\varphi \in \Phi_0,\ f \in L^{\varphi'}_{loc}(\mathbb{R}^m)$. Then for any $\psi \in \Psi \cap \nabla_2$ we have*

$$\int_{\mathbb{R}^m} \psi(\varphi'(Mf(x)))\,dx \le C\int_{\mathbb{R}^m} \psi(C\varphi'(f(x)))\,dx,$$

where the constant C depends on φ and ψ.

Corollary 7. *For $\varphi \in \Phi_0 \cap \nabla_2$ and $\varphi(t)/t \to \infty$, as $t \to \infty$, let $f \in L^\varphi(\mathbb{R}^m)$. Then*

$$\int_{\mathbb{R}^m} \varphi(Mf(x))\,dx \le C\int_{\mathbb{R}^m} \varphi(f(x))\,dx,$$

where C is independent of f.

Note that the statements of Theorem 6 and Corollary 7 have a meaning even if the maximal function Mf is not measurable. It is easy to prove that if $\varphi \in \Phi_0$ and φ' is a strictly increasing function, then there exists an unique constant satisfying (1.1); i. e., $\mu^\varepsilon_{\varphi'}(f)$ is a singleton. Besides, given a sequence x_n tending to x it follows that the sequence $f_\varepsilon(x_n)$ is bounded. By the uniqueness of the set $\mu^\varepsilon_{\varphi'}(f)$, a standard argument shows that $f_\varepsilon(x_n) \to f_\varepsilon(x)$. Thus $f_\varepsilon(x)$ is a continuous function of x and therefore Mf is a measurable function. The measurability of Mf for $\varphi \in \Phi$ is an open problem.

Given $f \in L^\varphi(\mathbb{R}^m)$ we have

$$\varphi(|f(x) - f_\varepsilon(x)|)|B_\varepsilon(x)| = \int_{B_\varepsilon(x)} \varphi(|f(x) - f(y) + f(y) - f_\varepsilon(x)|)\,dy.$$

Since $f_\varepsilon(x)$ is a best constant approximation to f and taking into account the Δ_2 condition on φ, we obtain

$$\varphi(|f(x) - f_\varepsilon(x)|) \le 2C\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} \varphi(|f(y) - f(x)|)\,dy.$$

Thus $f_\varepsilon(x) \to f(x)$, as $\varepsilon \to 0$, for a. e. x. By Corollary 7 we also have $f_\varepsilon \to f$ in the norm of $L^\varphi(\mathbb{R}^m)$.

In the next theorem $|E|^*$ will denote the outer Lebesgue measure of a set $E \subseteq \mathbb{R}^m$.

Theorem 8. *Let $\varphi \in \Phi \cap \Delta_2$ and $\varphi'(0) > 0$. Then there exists $C > 0$ such that*

$$|\{x \in \mathbb{R}^m : Mf(x) > t\}|^* \le \frac{C}{\varphi'(0)} \int_{\{|f|>t\}} \varphi'(|f(y)|)\, dy,$$

for every $f \in L^{\varphi'}(\mathbb{R}^m)$ and $t > 0$.

Theorem 9. *Let $\phi \in \Phi \cap \Delta_2$ with $\varphi'(0) > 0$, and $\{B_\varepsilon(x)\}$ be the family of balls centered at x with radius ε. Then for every $f \in L^{\varphi'}_{loc}(\mathbb{R}^m)$ we have*

$$\lim_{\varepsilon\to 0}(\sup\{|f_\varepsilon(x) - f(x)| : f_\varepsilon(x) \in \mu^\varepsilon_{\varphi'}(f)\}) = 0,$$

almost everywhere $x \in \mathbb{R}^m$.

2 Proof of the Results.

For completeness reasons we will sketch a proof of Lemma 1.

PROOF OF LEMMA 1. Set $h(t) = \int_\Omega \varphi(|f(x) - t|)\, dx$. Since h is a convex function, it has a minimum at c if and only if $h'(c^+) \ge 0$ and $h'(c^-) \le 0$. Now a direct calculation gives

$$0 \le h'(c^+) = \varphi'(0)|\{f = c\} \cap \Omega| - \int_{\{f\ne c\}\cap\Omega} \varphi'(|f(x) - c|)\, sgn(f(x) - c)\, dx,$$

and

$$0 \ge h'(c^-) = -\varphi'(0)|\{f = c\} \cap \Omega| - \int_{\{f\ne c\}\cap\Omega} \varphi'(|f(x) - c|)\, sgn(f(x) - c)\, dx.$$

Thus the above inequalities give part 2. □

Theorem 2 is known for the case that $\varphi(t) = t$ and its proof is quite involved when the approximant class of the real constants is replaced by a σ-lattice of functions, as it is done in [6]. In our case the proof is simpler.

PROOF OF THEOREM 2. Observe that the inequalities in (1) are a restatement of those in Lemma 1. Moreover the statement (2) is clearly equivalent to (1). Since (2) implies (3), we will prove that (2) is a consequence of (3). In fact set $\alpha = c + 1/n$ in (3) (a), and a straightforward limit procedure gives (2) (a). By setting $\alpha = c - 1/n$ in (3) (b) we obtain (2) (b), and the proof is completed. □

The next lemma allows us to extend the best approximation operator from L^{φ} to $L^{\varphi'}$.

Lemma 10. *Let $\varphi \in \Phi \cap \Delta_2$ and $f \in L^{\varphi'}(\Omega)$. Then there exits a constant c which satisfies (3) of Theorem 2.*

PROOF. Let f be in $L^{\varphi'}(\Omega)$ and set $f_{mn} = \min(\max(f, -m), n)$. Since f_{mn} is bounded, by Theorem 2 (2) (b),

$$\int_{\Omega} \varphi'(|f_{mn} - c_{mn}|)\, dx \le 2 \int_{\{f_{mn} \ge c_{mn}\} \cap \Omega} \varphi'(|f_{mn} - c_{mn}|)\, dx, \tag{2.1}$$

where $c_{mn} \in \mu_{\varphi}(f_{mn})$ are selected in such a way that the sequence $(c_{mn})_n$ is increasing, for example take $c_{mn} = \min \mu_{\varphi}(f_{mn})$.

For a fix m we have $f_{mn} \nearrow f_m = \max(f, -m)$, as $n \to \infty$ and set $c_m = \lim_{n\to\infty} c_{mn}$. We will first proof that c_m is finite. From (2.1) and the fact that φ' is monotone we have

$$\int_{\Omega} \varphi'(|f_{mn} - c_{mn}|)\, dx \le 2 \int_{\{f_{mn} > \alpha\} \cap \Omega} \varphi'(|f_{mn} - \alpha|)\, dx,$$

for every $\alpha < c_{mn}$. Now, using the Fatou's Lemma on the left hand side of the above inequality and the Lebesgue Theorem on the right hand side we get

$$\int_{\Omega} \varphi'(|f_m - c_m|)\, dx \le 2 \int_{\{f_m > \alpha\} \cap \Omega} \varphi'(|f_m - \alpha|)\, dx. \tag{2.2}$$

Therefore $c_m \in \mathbb{R}$.

Since Theorem 2 (3) (b) holds for the pair f_{mn} and c_{mn}, by taking n tending to infinity we have

$$\int_{\Omega} \varphi'(|f_m - c_m|)\, dx \le 2 \int_{\{f_m > \alpha\} \cap \Omega} \varphi'(|f_m - c_m|)\, dx, \tag{2.3}$$

for any $\alpha < c_m$. Now, since $(c_{mn})_m$ is a decreasing sequence, the sequence $(c_m)_m$ is also decreasing. Let $c = \lim_{m\to\infty} c_m$ and taking the limit in (2.2) we get that $c \in \mathbb{R}$. Given $\alpha < c$ choose an integer k such that $\alpha < \alpha + 1/k < c$, and since $c_m \ge c$, by (2.3) we have that

$$\int_{\Omega} \varphi'(|f_m - c_m|)\, dx \le 2 \int_{\{f_m > \alpha + 1/k\} \cap \Omega} \varphi'(|f_m - c_m|)\, dx.$$

Then, if $B_k = \cap_{m=1}^{\infty} \{f_m > \alpha + 1/k\}$, we have

$$\int_{\Omega} \varphi'(|f - c|)\, dx \le 2 \int_{B_k \cap \Omega} \varphi'(|f - c|)\, dx. \tag{2.4}$$

Since $\{f > \alpha + 1/k\} \subseteq B_k \subseteq \{f \geq \alpha + 1/k\}$, by taking k tending to infinity in (2.4) we get

$$\int_\Omega \varphi'(|f - c|)\,dx \leq 2 \int_{\{f>\alpha\}\cap\Omega} \varphi'(|f - c|)\,dx;$$

that is, we have proved condition (3) (b) of Theorem 2 for the pair f and c. It remains to prove condition (3) (a) for f and c. For the function f_{mn} and its best approximant c_{mn} we know that

$$\int_\Omega \varphi'(|f_{mn} - c_{mn}|)\,dx \leq 2 \int_{\{f_{mn}<\alpha\}\cap\Omega} \varphi'(|f_{mn} - c_{mn}|)\,dx, \tag{2.5}$$

for any $\alpha > c_{mn}$.

Given $\alpha > c_m$ take an integer k such that $\alpha - 1/k > c_m \geq c_{mn}$ and by (2.5) we have

$$\int_\Omega \varphi'(|f_{mn} - c_{mn}|)\,dx \leq 2 \int_{\{f_{mn}<\alpha-1/k\}\cap\Omega} \varphi'(|f_{mn} - c_{mn}|)\,dx. \tag{2.6}$$

Now set $A_{mk} = \cap_{n=1}^{\infty}\{f_{mn} < \alpha - 1/k\}$. Then $\{f_m < \alpha - 1/k\} \subseteq A_{mk} \subseteq \{f_m \leq \alpha - 1/k\}$. Thus $A_{mk} \to \{f_m < \alpha\}$, and taking $n \to \infty$ in (2.6) and further letting $k \to \infty$ we get

$$\int_\Omega \varphi'(|f_m - c_m|)\,dx \leq 2 \int_{\{f_m<\alpha\}\cap\Omega} \varphi'(|f_m - c_m|)\,dx, \tag{2.7}$$

for $\alpha > c_m$. Now if $\alpha > c$ and recalling that f_m is a decreasing sequence, by taking limit in (2.7) we get

$$\int_\Omega \varphi'(|f - c|)\,dx \leq 2 \int_{\{f<\alpha\}\cap\Omega} \varphi'(|f - c|)\,dx. \qquad \square$$

Lemma 11. *Let $\varphi \in \Phi \cap \Delta_2$ and $f \in L^{\varphi'}(\Omega)$. Then the set $\mu_{\varphi'}(f)$ is a closed bounded interval.*

PROOF. Given $f \in L^{\varphi'}(\Omega)$ we will see that if a constant c_1 satisfies (1) (a) of Theorem (2), so does any constant $c \geq c_1$. In fact

$$\begin{aligned}\int_{\{f>c\}} \varphi'(|f - c|)\,dx &\leq \int_{\{f>c_1\}} \varphi'(|f - c_1|)\,dx \leq \int_{\{f\leq c_1\}} \varphi'(|f - c_1|)\,dx \\ &\leq \int_{\{f\leq c\}} \varphi'(|f - c|)\,dx\end{aligned}$$

Similarly if a constant c_2 satisfies (1) (b) of Theorem 2 so does any constant $c \leq c_2$. Thus $\mu_{\varphi'}(f)$ is an interval, and using (3) of Theorem 2 we see that it is closed. To see that it is bounded from above we use (2) (b) of Theorem 2 and we have, for any $\alpha < c,\ c \in \mu_{\varphi'}(f)$,

$$\int_{\Omega} \varphi'(|f - c|)\, dx \leq 2 \int_{\{f \geq c\} \cap \Omega} \varphi'(|f - c|)\, dx.$$

Thus

$$\int_{\Omega} \varphi'(|f - c|)\, dx \leq 2 \int_{\{f \geq \alpha\} \cap \Omega} \varphi'(|f - \alpha|)\, dx,$$

which shows that the set $\mu_{\varphi'}(f)$ has an upper bound.

Similarly condition (2) (a) implies that the set $\mu_{\varphi'}(f)$ is bounded from below. □

Now we prove that the best approximation operator extended to the space $L^{\varphi'}(\Omega)$ is a monotone operator.

Lemma 12. *If $\varphi \in \Phi \cap \Delta_2$, then the multivalued operator $\mu_{\varphi'}(f)$ is monotone on $L^{\varphi'}(\Omega)$.*

Proof. Let $c_1 \in \mu_{\varphi}(f_1),\ c_2 \in \mu_{\varphi}(f_2)$ and be $f_1 \leq f_2$. We will assume that $\min(c_1, c_2) = c_2$. Then

$$\begin{aligned}\int_{\{f_1 > c_2\} \cap \Omega} \varphi'(|f_1 - c_2|)\, dx &\leq \int_{\{f_2 > c_2\} \cap \Omega} \varphi'(|f_2 - c_2|)\, dx \\ &\leq \int_{\{f_2 \leq c_2\} \cap \Omega} \varphi'(|f_2 - c_2|)\, dx \\ &\leq \int_{\{f_1 \leq c_2\} \cap \Omega} \varphi'(|f_1 - c_2|)\, dx.\end{aligned}$$

Thus

$$\begin{aligned}&\int_{\{f_1 > \min(c_1, c_2)\} \cap \Omega} \varphi'(|f_1 - \min(c_1, c_2)|)\, dx \\ &\qquad\qquad \leq \int_{\{f_1 \leq \min(c_1, c_2)\} \cap \Omega} \varphi'(|f_1 - \min(c_1, c_2)|)\, dx.\end{aligned}$$

Therefore we have proved condition (1) (a) of Theorem 2, for the constant $\min(c_1, c_2)$. Similarly it follows (1) (b) for $\min(c_1, c_2)$ which implies $\min(c_1, c_2)$ belongs to $\mu_{\varphi'}(f_1)$. It is shown analogously that $\max(c_1, c_2) \in \mu_{\varphi'}(f_2)$. □

Lemma 13. *Let $\varphi \in \Phi$ such that $\varphi(2t) \leq k\varphi(t)$, $t > 0$. Then $\varphi'(a+b) \leq \frac{k^2}{2}(\varphi'(a)+\varphi'(b))$, $a, b > 0$.*

PROOF. Since φ' is an increasing function, we have $\varphi(x) \leq x\varphi'(x) \leq \varphi(2x)$. Moreover from the convexity and the Δ_2 condition on φ we have $\varphi(a+b) \leq \frac{k}{2}(\varphi(a)+\varphi(b))$. Now

$$\begin{aligned}(a+b)\varphi'(a+b) &\leq \varphi(2(a+b)) \leq k\varphi(a+b) \leq \frac{k^2}{2}(\varphi(a)+\varphi(b)) \\ &\leq \frac{k^2}{2}(a\varphi'(a)+b\varphi'(b)) \leq (a+b)\frac{k^2}{2}(\varphi'(a)+\varphi'(b)). \qquad \square\end{aligned}$$

PROOF OF THEOREM 3. Given $f_\varepsilon(x) \in \mu^\varepsilon_{\varphi'}(f)$ we may assume $f_\varepsilon(x) \geq 0$. In fact, for a general $f_\varepsilon(x)$ there exist best approximations $0 \leq c_{1,\varepsilon} \in \mu^\varepsilon_{\varphi'}(|f|)$, and a non positive constant $c_{2,\varepsilon} \in \mu^\varepsilon_{\varphi'}(-|f|)$ such that $c_{2,\varepsilon} \leq f_\varepsilon(x) \leq c_{1,\varepsilon}$. Set $c_\varepsilon(|f|) = \max(c_{1,\varepsilon}, -c_{2,\varepsilon})$, and taking into account that $c \in \mu_{\varphi'}(f)$ if and only if $-c \in \mu_{\varphi'}(-f)$, we have $c_\varepsilon(|f|) \in \mu^\varepsilon_{\varphi'}(|f|)$ and $|f_\varepsilon(x)| \leq c_\varepsilon(|f|)$.

Now $\varphi'(f_\varepsilon(x))$ can be written as

$$\begin{aligned}\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} \varphi'(f_\varepsilon(x))\,dy =& \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)>f\}} \varphi'((f_\varepsilon(x)-f(y))+f(y))\,dy \\ &+ \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)\leq f\}} \varphi'(f_\varepsilon(x))\,dy,\end{aligned}$$

and using Lemma 13 the above expression can be estimated by

$$\begin{aligned}&\frac{k^2}{2}\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)>f\}} \varphi'(f_\varepsilon(x)-f(y))\,dy \\ &+ \frac{k^2}{2}\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)>f\}} \varphi'(f(y))\,dy + \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)\leq f\}} \varphi'(f_\varepsilon(x))\,dy.\end{aligned} \tag{2.8}$$

By (1) (b) of Theorem 2 we have

$$\begin{aligned}&\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)>f\}} \varphi'(f_\varepsilon(x)-f(y))\,dy \\ &\qquad\leq \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)\leq f\}} \varphi'(f(y)-f_\varepsilon(x))\,dy.\end{aligned}$$

Then we can estimate (2.8) by

$$k^2\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)\cap\{f_\varepsilon(x)\leq f\}} \varphi'(f(y))\,dy + \frac{k^2}{2}\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} \varphi'(f(y))\,dy \leq$$

$$\frac{3k^2}{2}\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)}\varphi'(f(y))\,dy.$$

To prove part (2), observe that if $f_\varepsilon(x)\in\mu^\varepsilon_{\varphi'}(f)$, then

$$f_\varepsilon(x)-f(x)\in\mu^\varepsilon_{\varphi'}(f-f(x)).$$

Thus applying part (1) to the function $f-f(x)$, the proof is completed. □

Given $\varphi\in\Psi$ we set $\widetilde{\varphi}(s)=\sup\{st-\varphi(t);t>0\}$ for the complementary function of φ. Observe that $\widetilde{\varphi}(0)=0$, and if $\varphi(t)/t\to\infty$, as $t\to\infty$ we have $\widetilde{\varphi}\in\Psi$, see [1] or [8].

PROOF OF COROLLARY 7. For $f\in L^\varphi(\mathbb{R}^m)$ we get

$$\int_{\mathbb{R}^m}\varphi(Mf(x))\,dx\le\int_{\mathbb{R}^m}Mf(x)\;\;\varphi'(Mf(x))\;dx.$$

Now for $0<\varepsilon<1$ and by the Young inequality the last integral is bounded by

$$\varepsilon\int_{\mathbb{R}^m}\varphi(Mf(x))\,dx+\int_{\mathbb{R}^m}\widetilde{\varphi}\Big(\frac{1}{\varepsilon}\varphi'(Mf(x))\Big)\,dx.$$

Recalling that $\widetilde{\varphi}\in\Delta_2$ if and only if $\varphi\in\nabla_2$ we get

$$(1-\varepsilon)\int_{\mathbb{R}^m}\varphi(Mf(x))\,dx\le C\int_{\mathbb{R}^m}\widetilde{\varphi}(\varphi'(Mf(x)))\,dx,$$

and applying Theorem 6 we get

$$\int_{\mathbb{R}^m}\varphi(Mf(x))\,dx\le C\int_{\mathbb{R}^m}\widetilde{\varphi}(C\varphi'(f(x)))\,dx\le C\int_{\mathbb{R}^m}\varphi(f(x))\,dx,$$

where the last inequality follows from $\widetilde{\varphi}(\varphi'(x))\le C\varphi(x)$, with $C>1$. In fact, since $\varphi\in\Delta_2$ there exists $\alpha>1$ such that $x\varphi'(x)\le\alpha\varphi(x)$, (see Theorem 4.1, pg. 24 in [3]). Moreover we always have $x\varphi'(x)=\varphi(x)+\widetilde{\varphi}(\varphi'(x))$. Then $\widetilde{\varphi}(\varphi'(x))\le(\alpha-1)\varphi(x)$. □

Lemma 14. *Given $\varphi\in\Phi\cap\Delta_2$, $\varphi'(0)>0$ and a nonnegative $f\in L^{\varphi'}(\mathbb{R}^m)$. Then every $c\in\mu_{\varphi'}(f)$ is a nonnegative constant.*

PROOF. By (1) (a) of Theorem 2 for every $c\in\mu_{\varphi'}(f)$,

$$\int_{\{f>c\}\cap\Omega}\varphi'(|f-c|)\,dx\le\int_{\{f\le c\}\cap\Omega}\varphi'(|f-c|)\,dx.$$

Now if some $c\in\mu_{\varphi'}(f)$ is less than 0, the inequality above gives us a contradiction. □

PROOF OF THEOREM 8. By Lemma 14 and Lemma 12 $|f_\varepsilon(x)| \leq \max \mu^\varepsilon_{\varphi'}(|f|)$. So we can assume $f \geq 0$ and every $f_\varepsilon(x) \geq 0$. Set $E = \{x \in \mathbb{R}^m : Mf(x) > t\}$. For $x \in E$ choose $f_\varepsilon(x) \in \mu^\varepsilon_{\varphi'}(f)$ such that $f_\varepsilon(x) > t$. Then, by (2) (b) of Theorem 2 we have

$$|B_\varepsilon(x)|\varphi'(0) \leq \int_{B_\varepsilon(x)} \varphi'(|f - f_\varepsilon(x)|)\, dy \leq 2 \int_{\{f \geq f_\epsilon\} \cap B_\varepsilon(x)} \varphi'(|f - f_\varepsilon(x)|)\, dy$$
$$\leq 2 \int_{\{f > t\} \cap B_\varepsilon(x)} \varphi'(f)\, dy.$$

Now by a standard covering lemma there exists a constant C, and a disjoint family of balls $\{B_{\varepsilon_n}(x_n)\}$ such that

$$|E|^* \leq C \sum |B_{\varepsilon_n}(x_n)|, \tag{2.9}$$

and for every n we have

$$|B_{\varepsilon_n}(x_n)| \leq \frac{2}{\varphi'(0)} \int_{\{f > t\} \cap B_{\varepsilon_n}(x_n)} \varphi'(2f(y))\, dy. \tag{2.10}$$

Now from (2.9) and (2.10) we get the theorem. □

PROOF OF THEOREM 9. Let $f \in L^{\varphi'}(\mathbb{R}^m)$, $f_\varepsilon(x) \in \mu^\varepsilon_{\varphi'}(f)$ and s be a step function. Then, for almost every x, there exists an $\varepsilon(x)$ such that for every ε, $0 < \varepsilon < \varepsilon(x)$, we have $f_\varepsilon(x) = (f - s)_\varepsilon(x) + s(x)$. Here we have used that for a constant c we have $(f + c)_\varepsilon(x) = f_\varepsilon(x) + c$.

The remainder of the proof follows the same patterns as the proof of the Lebesgue Differentiation Theorem using the Hardy-Littlewood maximal function. Set

$$\Gamma f(x) = \limsup_{\varepsilon \to 0} (\sup\{|f_\varepsilon(x) - f(x)| : f_\varepsilon(x) \in \mu^\varepsilon_{\varphi'}\}).$$

Then clearly $\Gamma f(x) = \Gamma(f - s)(x)$, for a. e. $x \in \mathbb{R}^m$ and consequently $\Gamma f(x) \leq M(f - s)(x) + |f(x) - s(x)|$. Then we have

$$|\{\Gamma f > t\}|^* \leq |\{M(f - s) > t/2\}|^* + |\{|f - s| > t/2\}|.$$

By Theorem 8 and the Tchebyshev inequality we have

$$|\{\Gamma f > t\}|^* \leq \frac{C}{\varphi'(0)} \int_{\{|f - s| > t/2\}} \varphi'(|f - s|)\, dy + \frac{1}{\varphi'(t/2)} \int_{\mathbb{R}^m} \varphi'(|f - s|)\, dy.$$

Since $\varphi' \in \Delta_2$, the step functions are dense in $L^{\varphi'}(\mathbb{R}^m)$. Therefore $\Gamma f(x) = 0$, for a. e. $x \in \mathbb{R}^m$. □

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