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SEPARATELY CONTINUOUS FUNCTIONS WITH CLOSED GRAPHS

Abstract

In this paper we prove that if $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ has a closed graph and all of its x-sections are continuous, and at least one y-section is continuous, then f is continuous. It was already proved by Piotrowski and Wingler [PW] that if $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ has a closed graph and is separately continuous, then f is continuous. Our result is stronger.

In this paper the focus is on the situation where f is a function defined on the Cartesian product of two spaces: $f\colon X\times Y\to Z$. A global property of such a function is one that treats the product $X\times Y$ as a whole and the function is seen as $f\colon S\to Z$. A sectionwise property is one that is expressed in terms of the x-sections and y-sections. Usually a global property implies the analogous sectionwise property. For example, joint continuity implies separate continuity (sectionwise continuity), and the same goes for differentiability, measurability, and closedness of the graph. There are examples that show that a global property cannot be derived from the analogous sectionwise property. For example, the function

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & \text{if } (x,y)\in\mathbb{R}^2\setminus\{(0,0)\},\\ 0 & \text{if } (x,y)=(0,0),\end{cases}$$

is sectionwise continuous but discontinuous at $(0,0)$; also its sections have closed graphs but the function does not.

The problem of deriving global properties from sectionwise properties is a hard research problem. The classical theorem of this nature is the celebrated

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Baire-Lebesgue-Kuratowski-Montgomery theorem which says that if X and Y are metric and if $f\colon X \times Y \to \mathbb{R}$ is continuous in x and is of class α in y, then f is of class $\alpha + 1$. ([Ku] Ch. II, Sec. 31, Th. 2, p. 378). This subject has a rich bibliography. For other interesting results and further references see for instance [LP], [PW], or [P1].

Let us focus on the question of joint versus separate continuity for functions $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. It is already known that the condition "separate continuity" guarantees for such functions that they have a dense G_δ set of continuity points ([P1] sec. 3 and 5 and [MC] Th. 9.2). But, as our little example shows, if one is to hope for deriving continuity everywhere, one must make additional assumptions. One such result is by Piotrowski and Wingler [PW] where they proved that if $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has a closed graph and is separately continuous, then f is continuous. Notice that "closed graph" is a global property.

In our paper we achieve a similar but stronger result. Our main theorem says that if $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has a closed graph and all of its x-sections are continuous, and at least one y-section is continuous, then f is continuous. We state and prove our theorem in the general topological setting. Our paper is self-contained in the sense that the proofs start from scratch and can be followed without external reference by anyone who is familiar with textbook knowledge of elementary theory of topological spaces. In particular, we do not use the result of Piotrowski and Wingler, and, in fact, our Corollary 6 is a statement of their result.

We use only standard symbols and terminology, and we do not introduce any definitions, but for the sake of clarity we recall the following definitions. A Darboux function maps connected sets onto connected sets. The following definitions refer to the situation $f\colon X \times Y \to Z$. When $x_0 \in X$ is fixed, then the function $Y \ni y \mapsto f(x_0, y) \in Z$ is called an x-section. When $y_0 \in Y$ is fixed, then the function $X \ni x \mapsto f(x, y_0) \in Z$ is called a y-section. A separately continuous function $f : X \times Y \to Z$ has all x-sections and all y-sections continuous (separately continuous = sectionwise continuous).

The first three theorems are known folklore results which we include with proofs for the sake of completeness.

Theorem 1. ***If*** *X, Y are topological spaces, $f\colon X \to Y$, $Gr(f)$ is closed, $E \subset Y$, and E is compact,* ***then*** *$f^{-1}(E)$ is closed.*

PROOF. Take any $x_0 \in \overline{f^{-1}(E)}$. We have a net $(x_t)_{t\in\Pi}$ in $f^{-1}(E)$ which converges to x_0. Notice that $(f(x_t))_{t\in\Pi}$ is a net in the compact set E. Hence we get a subnet $(f(x_{\alpha(s)}))_{s\in\Pi_0}$ which converges to some $y_0 \in E$. Notice that $(x_{\alpha(s)}, f(x_{\alpha(s)})) \to (x_0, y_0)$. Since $Gr(f)$ is closed, $f(x_0) = y_0$. So $f(x_0) \in E$, and $x_0 \in f^{-1}(E)$. We showed that $\overline{f^{-1}(E)} \subset f^{-1}(E)$, so $f^{-1}(E)$ is closed. $\square$

Theorem 2. ***If*** *X is a topological space, Y is a compact space, $f\colon X \to Y$, and $Gr(f)$ is closed,* ***then*** *f is continuous.*

PROOF. Take any closed set $E \subset Y$. Since Y is a compact space, E is compact. By Theorem 1, $f^{-1}(E)$ is closed. We showed that f is continuous. □

Theorem 3. ***If*** *X is a topological space, Y is a locally compact space, $f\colon X \to Y$, and $Gr(f)$ is closed,* ***then*** *$W = \{x \in X : f$ is continuous at $x\}$ is open.*

PROOF. Take any $x_0 \in W$. Since Y is locally compact, we have an open set $U \subset Y$ such that $f(x_0) \in U$ and $\overline{U}$ is compact. Since f is continuous at x_0, we have an open $G \subset X$ such that $x_0 \in G$ and $f(G) \subset U$. Notice that $f|_G\colon G \to \overline{U}$ and $Gr(f|_G)$ is closed in $G \times \overline{U}$. By Theorem 2, $f|_G$ is continuous. Since G is open, f is continuous on G. Hence $G \subset W$, and so $x_0 \in \mathrm{Int}(W)$. We showed that $W \subset Int(W)$, so W is open. □

The following two lemmas (Lemma 4, Lemma 5) are extensively used in the proof of the main theorem (Theorem 7).

Lemma 4. *If X is a topological space, Y is a locally compact space, $f\colon X \to Y$, $Gr(f)$ is closed, $A \subset X$, $x_0 \in A$, $\Pi = \{U \subset X : x_0 \in U$ and U is open$\}$,*
(1) $P = \{E \subset X : f(E)$ is connected$\}$,
(2) $f|_A$ is continuous at x_0,
(3)$\forall_{U\in\Pi}\exists_{G\in\Pi}\forall_{y\in G}\exists_{E\in P}\ y \in E \wedge E \subset U \wedge E \cap A \neq \emptyset$
then f is continuous at x_0.

PROOF. Since Y is locally compact, we have an open set $V_0 \subset Y$ such that $f(x_0) \in V_0$ and $\overline{V_0}$ is compact. Now we have two possibilities:

$$(a)\ \exists_{G\in\Pi} f(G) \subset \overline{V_0}$$

$$(b)\ \forall_{G\in\Pi} f(G) \not\subset \overline{V_0}$$

If (a), then by Theorem 2, the function $f|_G\colon G \to \overline{V_0}$ is continuous. Since G is open, f is continuous at x_0, as desired.

We will show that (b) leads to a contradiction. We will show $(*)$.

$$(*)\ \forall_{U\in\Pi}\exists_{y\in U}\exists_{E\in P}\exists_{a\in U} f(y) \notin \overline{V_0} \wedge y \in E \wedge E \subset U \wedge a \in A \cap E$$

Take any $U \in \Pi$. We can choose a set $G \in \Pi$ as in (3). By (b), we have a $y \in G$ such that $f(y) \notin \overline{V_0}$. Considering how G was chosen, we have that $G \subset U$. So $y \in U$ and there exists an $E \in P$ such that $y \in E \wedge E \subset U \wedge E \cap A \neq \emptyset$. So we showed $(*)$. By the Axiom of Choice, we get nets $(y_U)_{U\in\Pi}$, $(a_U)_{U\in\Pi}$, and $(E_U)_{U\in\Pi}$ such that

$$\forall_{U\in\Pi} y_U \in E_U \subset U \wedge E_U \in P \wedge f(y_U) \notin \overline{V_0} \wedge a_U \in A \cap E_U \subset U.$$

Notice that $a_U \to x_0$ $(U \in \Pi)$. Since $a_U \in A$, by (2) we have that $f(a_U) \to f(x_0)$. So we get a $U_0 \in \Pi$ such that $f(a_U) \in V_0$ whenever $U \in \Pi$ and $U \subset U_0$. Let $\Pi_0 = \{U \in \Pi : U \subset U_0\}$. We will construct a net $(z_U)_{U \in \Pi_0}$ in X. Take any $U \in \Pi_0$. We have $f(a_U) \in V_0$. We will show that $f(E_U) \cap (\overline{V_0} \setminus V_0) \neq \emptyset$.

Suppose that $f(E_U) \cap (\overline{V_0} \setminus V_0) = \emptyset$. Then $f(E_U) \subset V_0 \cup (Y \setminus \overline{V_0})$, and $f(E_U) \cap V_0 \cap (Y \setminus \overline{V_0}) = \emptyset$; $f(a_U) \in f(E_U) \cap V_0 \neq \emptyset$, and $f(y_U) \in f(E_U) \cap (Y \setminus \overline{V_0}) \neq \emptyset$. Hence $f(E_U)$ is not connected contrary to (1). So we can choose a $z_U \in E_U$ with $f(z_U) \in \overline{V_0} \setminus V_0$. We have a net $(z_U)_{U \in \Pi_0}$ such that $z_U \to x_0$ and $f(z_U) \in \overline{V_0} \setminus V_0$ for all $U \in \Pi_0$. We have that $z_U \in f^{-1}(\overline{V_0} \setminus V_0)$ for all $U \in \Pi_0$. The set $\overline{V_0} \setminus V_0$ is compact. By Theorem 1, the set $f^{-1}(\overline{V_0} \setminus V_0)$ is closed. So $x_0 \in f^{-1}(\overline{V_0} \setminus V_0)$, and $f(x_0) \in \overline{V_0} \setminus V_0$. But $f(x_0) \in V_0$. Contradiction. The proof is complete. □

Lemma 5. *If X is a topological space, Y is a locally connected space, Z is a locally compact space, $f\colon X \times Y \to Z$, $Gr(f)$ is closed, $y_0 \in Y$,*
(1) the mapping $Y \ni y \mapsto f(x, y) \in Z$ is Darboux for all $x \in X$,
(2) the mapping $X \ni x \mapsto f(x, y_0) \in Z$ is continuous,
then f is continuous at (x, y_0) for all $x \in X$.

PROOF. Take any $x_0 \in X$. We are preparing to apply Lemma 4. Let $\Pi = \{U \subset X \times Y : (x_0, y_0) \in U \text{ and } U \text{ is open}\}$. Let $P = \{E \subset X \times Y : f(E) \text{ is connected}\}$. Let $A = X \times \{y_0\}$. By (2), $f|_A$ is continuous at (x_0, y_0). Take any $U \in \Pi$. We have a G_X open in X and a G_Y open in Y with $(x_0, y_0) \in G_X \times G_Y \subset U$. Since Y is locally connected, we have a connected set $K \subset G_Y$ such that $y_0 \in \mathrm{Int}(K)$. Let $G = G_X \times \mathrm{Int}(K)$. Now, $(x_0, y_0) \in G$ and G is open in $X \times Y$. Take any $v = (x, y) \in G$. Let $E = \{x\} \times K$. By (1), $E \in P$. Notice that $v \in E$ and $E \subset U$. Notice that $(x, y_0) \in E$ and $(x, y_0) \in A$, so $E \cap A \neq \emptyset$. We showed that

$$\forall_{U \in \Pi} \exists_{G \in \Pi} \forall_{v \in G} \exists_{E \in P} v \in E \wedge E \subset U \wedge E \cap A \neq \emptyset.$$

Now, by Lemma 4, f is continuous at (x_0, y_0). Since $x_0 \in X$ was arbitrary, the proof is complete. □

The following corollary is the previously known result by Piotrowski and Wingler [PW].

Corollary 6. *If X is a topological space, Y is a locally connected space, Z is a locally compact space, $f\colon X \times Y \to Z$, $Gr(f)$ is closed,*
(1) the mapping $Y \ni y \mapsto f(x, y) \in Z$ is Darboux for all $x \in X$,
(2) the mapping $X \ni x \mapsto f(x, y) \in Z$ is continuous for all $y \in Y$
***then** f is continuous.*

The main theorem follows.

Theorem 7. *If X is a locally connected space, Y is a connected and locally connected space, Z is a locally compact space, $f\colon X\times Y\to Z$, $Gr(f)$ is closed,*
(1) *the mapping $Y\ni y\mapsto f(x,y)\in Z$ is continuous for all $x\in X$,*
(2) *the mapping $X\ni x\mapsto f(x,y)\in Z$ is continuous for some $y\in Y$*
then f is continuous.

PROOF. Let $W=\{(x,y)\in X\times Y : f \text{ is continuous at } (x,y)\}$. By Theorem 3, W is open. Take any $x_0\in X$. Let $D=\{y\in Y : f \text{ is continuous at } (x_0,y)\}$. Notice that D is open in Y because W is open in $X\times Y$. We will show that D is closed in Y. Take any $y_0\in\overline{D}$.Let $\Pi=\{U\subset X\times Y : (x_0,y_0)\in U \text{ and } U \text{ is open}\}$. Let $P=\{E\subset X\times Y : f(E) \text{ is connected}\}$. Let $A=\{(x_0,y) : y\in Y\}$. Take any $U\in\Pi$. We have open sets $G_X\subset X$, $G_Y\subset Y$ such that $(x_0,y_0)\in G_X\times G_Y\subset U$. Since Y is locally connected, we have a connected set $K\subset G_Y$ such that $y_0\in\operatorname{Int}(K)$. Since $y_0\in\overline{D}$, we can choose a $y'\in\operatorname{Int}(K)\cap D$. Since $y'\in D$, $(x_0,y')\in W$. Since W is open, we have an open set $V\subset X\times Y$ such that $(x_0,y')\in V$ and f is continuous on V. Now, we have open sets $V_X\subset X$, $V_Y\subset Y$ such that $(x_0,y')\in V_X\times V_Y\subset V\cap(G_X\times\operatorname{Int}(K))$. Since X is locally connected, we have a connected set $T_X\subset V_X$ such that $x_0\in\operatorname{Int}(T_X)$. Since Y is locally connected, we have a connected set $T_Y\subset V_Y$ such that $y'\in\operatorname{Int}(T_Y)$. Let $G=\operatorname{Int}(T_X)\times\operatorname{Int}(K)$. $G\in\Pi$ because $x_0\in\operatorname{Int}(T_X)$ and $y_0\in\operatorname{Int}(K)$. Take any $g=(v,z)\in G$. Let $E=T_X\times T_Y\cup\{v\}\times K$. We will show that $E\in P$. Notice that $T_Y\subset K$ and $v\in T_X$. So $T_X\times T_Y\cap\{v\}\times K\neq\emptyset$. Hence $f(T_X\times T_Y)\cap f(\{v\}\times K)\neq\emptyset$. Now, $T_X\times T_Y$ is connected and contained in V. Since f is continuous on V, $f(T_X\times T_Y)$ is connected. By (1), $f(\{v\}\times K)$ is connected. Notice that $f(E)=f(T_X\times T_Y)\cup f(\{v\}\times K)$. Hence $f(E)$ is connected. So $E\in P$. Notice that $(v,z)\in E$. Notice that $E\subset U$. We have $(x_0,y')\in T_X\times T_Y\subset E$ and $(x_0,y')\in A$. So $E\cap A\neq\emptyset$. We showed that

$$\forall_{U\in\Pi}\exists_{G\in\Pi}\forall_{g\in G}\exists_{E\in P}\, g\in E\wedge E\subset U\wedge E\cap A\neq\emptyset.$$

By (1), $f|_A$ is continuous at (x_0,y_0). By Lemma 4, f is continuous at (x_0,y_0). So $y_0\in D$. We showed that $\overline{D}\subset D$. So D is closed in Y. So D is open and closed in Y. By (2), we have a $y\in Y$ such that the mapping $X\ni x\mapsto f(x,y)\in Z$ is continuous. By Lemma 5, we conclude that f is continuous at (x_0,y). So $y\in D$ and $D\neq\emptyset$. Since Y is connected, $D=Y$. Hence f is continuous at (x_0,y) for all $y\in Y$. But $x_0\in X$ was arbitrary. Thus f is continuous, and the proof is complete. □

Piotrowski and Wingler in [PW] give an example (Example 2) which shows that the condition "Z is locally compact" is not redundant. It can be shown that every function $f\colon\mathbb{R}\to\mathbb{R}$ with a connected graph is Darboux. Then the next assertion follows from Corollary 6.

Theorem 8. *If $f\colon \mathbb{R} \to \mathbb{R}$ has a closed and connected graph, then f is continuous.*

Notice that the combination of "closed graph" and "connected graph" is a characterization of continuity for functions $f\colon \mathbb{R} \to \mathbb{R}$. For other such characterizations see [GG].

We wonder if this one-dimensional result can be extended to the following.

Hypothesis 9. *If $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has a closed and connected graph, then f is continuous.*

There is a lot of charm in Hypothesis 9 because if it was true, the following corollary could be proved without applying Theorem 7.

Corollary 10. *If $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ has a closed graph, the graphs of all x-sections are connected, and the graph of at least one y-section is connected, then f is continuous.*

PROOF. By Theorem 8, the conditions of Theorem 7 are satisfied. So we conclude that f is continuous, and the proof is finished. □

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