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MIXED NORM ESTIMATES FOR THE RIESZ TRANSFORMS ON $SU(2)$

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Abstract: In this paper we prove mixed norm estimates for Riesz transforms on the group $SU(2)$. From these results vector valued inequalities for sequences of Riesz transforms associated to Jacobi differential operators of different types are deduced.

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1. Introduction

In a recent article Ciaurri et al. [**1**] have studied mixed norm estimates for Riesz transforms on compact rank one symmetric spaces. Let M be such a space with Δ_M and ∇_M standing for the Laplace–Beltrami operator and the Riemannian gradient. Consider the shifted Laplacian $-\widetilde{\Delta}_M = -\Delta_M + \lambda_M$ where $\lambda_M \geq 0$ and the associated Riesz transform $\mathcal{R}_M f = |\nabla_M(-\widetilde{\Delta}_M)^{-\frac{1}{2}} f|$. Then it is well known that $\mathcal{R}_M$ is bounded on $L^p(M)$, $1 < p < \infty$. In the above mentioned article [**1**] the authors have studied mixed norm estimates for the Riesz transform $\mathcal{R}_M$.

On M there is a natural geodesic polar coordinate system that identifies M with $(0,\pi)\times S_M$, where S_M is a Euclidean unit sphere whose dimension depends on the symmetric space. Let (θ,ω) stand for this coordinate system. Then the Laplace–Beltrami operator $-\Delta_M$ can be written as $-\Delta_M = \mathcal{J}_{(\alpha,\beta)} - \rho_M(\theta)\Delta_{S_M}$ where $\rho_M(\theta)$ is a non-negative function and $\mathcal{J}_{(\alpha,\beta)}$ stands for the Jacobi differential operator

$$\mathcal{J}_{(\alpha,\beta)} = -\frac{d^2}{d\theta^2} - \frac{(\alpha-\beta)+(\alpha+\beta+1)\cos\theta}{\sin\theta}\frac{d}{d\theta} + \left(\frac{\alpha+\beta+1}{2}\right)^2. \tag{1.1}$$

Let $d\mu_{\alpha,\beta}(\theta) = (\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\beta+1}\,d\theta$ stand for the Jacobi measure on $(0,\pi)$. We define the mixed norm space $L^{p,2}(M)$ to be the space of all functions $f(\theta,\omega)$ for which

$$\|f\|_{L^{p,2}(M)} = \left(\int_0^\pi \left(\int_{\mathbf{S}_M} |f(\theta,\omega)|^2 \, d\sigma(\omega)\right)^{\frac{p}{2}} d\mu_{\alpha,\beta}\right)^{\frac{1}{p}} < \infty.$$

Then the main result proved in [**1**] (see Theorem 1.1) states that for $1 < p < \infty$

$$\|\mathcal{R}_M f\|_{L^{p,2}(M)} \le C \|f\|_{L^{p,2}(M)}. \tag{1.2}$$

This estimate is proved by means of vector valued inequalities for a sequence of Jacobi–Riesz transforms which we proceed to define now.

There is an orthonormal basis for $L^2((0,\pi), d\mu_{\alpha,\beta}(\theta))$ consisting of Jacobi polynomials $\mathcal{P}_k^{(\alpha,\beta)}(\theta)$ which are eigenfunctions of the Jacobi operator $\mathcal{J}_{(\alpha,\beta)}$, see Subsection 2.2. Let $\mathcal{R}_{(\alpha,\beta)} f = \frac{d}{d\theta} \mathcal{J}_{(\alpha,\beta)}^{-\frac{1}{2}} f$ be the Riesz transform associated to the Jacobi operator $\mathcal{J}_{(\alpha,\beta)}$. For $j = 0, 1, 2, \dots$ let $u_j(\theta) = (\sin\frac{\theta}{2})^{a\,j} (\cos\frac{\theta}{2})^{b\,j}$ where $a \ge 1$, $b = 0$, or $b \ge 1$. In [**1**] the authors have proved the following result.

Theorem 1.1. *Let $\alpha, \beta > -\frac{1}{2}$, $1 < p, r < \infty$ and let u_j be as above. Then*

$$\begin{aligned}\Big\| \Big(\sum_{j,k} |u_j \mathcal{R}_{(\alpha+a\,j,\beta+b\,j)} (u_j^{-1} f_{j,k})|^r \Big)^{\frac{1}{r}} \Big\|_{L^p(w,d\mu_{\alpha,\beta})} \\ \le C \Big\| \Big(\sum_{j,k} |f_{j,k})|^r \Big)^{\frac{1}{r}} \Big\|_{L^p(w,d\mu_{\alpha,\beta})}\end{aligned}$$

for all $f_{j,k} \in L^p(w, d\mu_{\alpha,\beta})$, $w \in A_p^{\alpha,\beta}$.

In Theorem 1.1, $A_p^{\alpha,\beta}$ stands for the Muckenhoupt's A_p-class of weights functions defined with respect to the measure $d\mu_{\alpha,\beta}$. This theorem is proved using sharp estimates for the Jacobi–Riesz kernels (see Theorem 1.2 in [**1**]) and an extrapolation theorem of Rubio de Francia [**5**]. Estimating the Jacobi–Riesz kernels is a difficult problem as can be seen from the work of Nowak–Sjögren [**4**] where the authors have proved that the kernels satisfy Calderón–Zygmund estimates. For the proof of Theorem 1.1 stated above one needs estimates for the Jacobi–Riesz kernels $\mathcal{R}_{(\alpha+a\,j,\beta+b\,j)}$ which are uniform in j. In [**1**] the authors have succeeded in obtaining such estimates. In order to prove the mixed norm estimates (1.2) one also needs vector valued inequalities for the operators $T_M^{(\alpha,\beta)} = \sqrt{\rho_M(\theta)} \mathcal{J}_{(\alpha,\beta)}^{-\frac{1}{2}}$.

The point of departure of this paper is the observation that the roles of the mixed norm estimates (1.2) for the Riesz transform $\mathcal{R}_M$ and the vector-valued inequalities for $\mathcal{R}_{(\alpha,\beta)}$ and $T_M^{(\alpha,\beta)}$ can be reversed. Indeed, Riesz transforms associated to compact rank one symmetric spaces have been well studied in the literature and their L^p boundedness are well known. Using a lemma of Herz and Rivière along with an idea of Rubio de Francia it is not difficult to prove mixed norm estimates for the Riesz transforms from which vector valued inequalities for Jacobi–Riesz transforms can be deduced. In principle, this procedure can be carried out for any compact rank one symmetric space. In [**10**] Strichartz has studied the boundedness of the Riesz transforms on complete Riemannian manifolds which includes compact Riemannian symmetric spaces. It should be possible to prove mixed norm estimates in the general setting but the computations involved may not be simple. We plan to return to this problem in the future.

For the sake of simplicity, in this article we only treat the case of $\mathbf{S}^3$ which already demonstrates the main ideas involved. The choice of the symmetric space $\mathbf{S}^3$ has the added advantage of being identified with the compact Lie group $SU(2)$. Riesz transforms on compact Lie groups have been studied by means of the very elegant Littlewood–Paley–Stein theory (see Stein [**9**]) and hence there is a painless proof of mixed norm estimates which does not require detailed estimates on Riesz kernels.

Let $SU(2)$ be the special unitary group i.e., the group of 2×2 unitary matrices having determinant one. Then any $g \in SU(2)$ can be written as $g = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a}\end{smallmatrix}\right)$ with $|a|^2 + |b|^2 = 1$. If $a = x_1 + i\,x_2$, $b = x_3 + i\,x_4$ with $x_j \in \mathbb{R}$ then $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a}\end{smallmatrix}\right) \mapsto (x_1, x_2, x_3, x_4)$ gives a one to one correspondence between $SU(2)$ and the unit sphere $\mathbf{S}^3$ in $\mathbb{R}^4$. Note that $\mathbf{S}^3 = SO(4)/SO(3)$ and hence $\mathbf{S}^3$ is a rank one symmetric space of compact type. Here $SO(n)$ stands for the special orthogonal group of $n\times n$ real matrices and $SO(n-1)$ is identified with the isotropy group of the vector $e_1 = (1, 0, \dots, 0)$.

If $su(2)$ stands for the Lie algebra of $SU(2)$ then it consists of all complex, skew-adjoint matrices of trace zero. A basis for $su(2)$ is provided by the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

We also denote the left-invariant vector fields corresponding to the matrices above by the same symbol. The Laplace–Beltrami operator Δ

on $SU(2)$ is then given by

$$\Delta = -(X^2 + Y^2 + Z^2).$$

It turns out that Δ coincides with the spherical Laplacian $\Delta_{\mathbf{S}^3}$ on the sphere $\mathbf{S}^3$. The operators $X\Delta^{-\frac{1}{2}}$, $Y\Delta^{-\frac{1}{2}}$, and $Z\Delta^{-\frac{1}{2}}$ are the Riesz transforms on $SU(2)$ and by Littlewood–Paley–Stein theory they are all bounded on $L^p(SU(2))$, $1 < p < \infty$. Here $L^p(SU(2))$ is the L^p-space taken with respect to the Haar measure dg on $SU(2)$.

The identification of $SU(2)$ with $\mathbf{S}^3$ allows us to define geodesic polar coordinates on $SU(2)$. Write any $(x_1, x_2, x_3, x_4) \in \mathbf{S}^3$ as $(\cos\theta, \omega_1 \sin\theta, \omega_2 \sin\theta, \omega_3 \sin\theta)$ where $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbf{S}^2$. Then the matrix g associated to this $x \in \mathbf{S}^3$ belongs to $SU(2)$. In other words the map

$$\Phi\colon (0, \pi) \times \mathbf{S}^2 \to SU(2)$$

defined by

$$\Phi(\theta, \omega) = \begin{pmatrix} \cos\theta + i\,\omega_1 \sin\theta & (\omega_2 + i\,\omega_3)\sin\theta \\ -(\omega_2 - i\,\omega_3)\sin\theta & \cos\theta - i\,\omega_1 \sin\theta \end{pmatrix} \tag{1.3}$$

gives us a coordinate system on $SU(2)$. In this coordinate system the Haar measure dg on $SU(2)$ is given by $\sin^2\theta\, d\theta\, d\sigma(\omega)$ where $d\sigma(\omega)$ is the surface measure on $\mathbf{S}^2$. Thus for a function f on $SU(2)$,

$$\int_{SU(2)} f(g)\, dg = \int_0^\pi \int_{\mathbf{S}^2} (f \circ \Phi)(\theta, \omega) \sin^2\theta\, d\sigma(\omega)\, d\theta.$$

In terms of these coordinates we introduce the mixed norm spaces $L^{p,2}(SU(2))$ as the space of all functions f on $SU(2)$ for which the norms

$$\|f\|_{p,2} = \left(\int_0^\pi \left(\int_{\mathbf{S}^2} |f \circ \Phi(\theta, \omega)|^2\, d\sigma(\omega) \right)^{\frac{p}{2}} \sin^2\theta\, d\theta \right)^{\frac{1}{p}} \tag{1.4}$$

are finite.

Let $\mathcal{R}f = (Z, X, Y)(\Delta + 1)^{-\frac{1}{2}} f$ be the vector of Riesz transforms on $SU(2)$. Here we have considered $(\Delta + 1)$ instead of Δ which simply has the effect of shifting the spectrum by 1 and hence making the eigenvalues of certain associated Jacobi operators into perfect squares. For the vector $\mathcal{R}f$ of Riesz transforms we have the following result. In what follows we use the notation $\langle \mathcal{R}f, \mathcal{R}f \rangle$ to denote

$$|X(\Delta + 1)^{-\frac{1}{2}} f|^2 + |Y(\Delta + 1)^{-\frac{1}{2}} f|^2 + |Z(\Delta + 1)^{-\frac{1}{2}} f|^2. \tag{1.5}$$

Theorem 1.2. *For any $1 < p < \infty$, there exists a constant $C = C_p$ such that*

$$\|\langle \mathcal{R}f, \mathcal{R}f \rangle^{\frac{1}{2}}\|_{p,2} \le C_p \|f\|_{p,2}$$

for all $f \in L^{p,2}(SU(2))$.

By expanding f in terms of eigenfunctions of Δ which involve Jacobi polynomials and spherical harmonics on $\mathbf{S}^2$, we can deduce a vector valued inequality for a sequence of Jacobi–Riesz transforms. The relevant parameters appearing in the expansion are $(n+\frac{1}{2}, n+\frac{1}{2})$ and hence for the sake of simplicity of notation we write $\mathcal{R}_n$ instead of $\mathcal{R}_{(n+\frac{1}{2},n+\frac{1}{2})}$ for the Riesz transforms associated to the Jacobi operator $\mathcal{J}_{(n+\frac{1}{2},n+\frac{1}{2})}$.

Theorem 1.3. *Let $u_n(\theta) = (\sin\theta)^n$. For any $1 < p < \infty$ there exists a constant C depending only on p such that*

$$\int_0^\pi \left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} |u_n \mathcal{R}_n(u_n^{-1} f_{n,j})|^2 \right)^{\frac{p}{2}} \sin^2\theta \, d\theta \leq C \int_0^\pi \left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} |f_{n,j}|^2 \right)^{\frac{p}{2}} \sin^2\theta \, d\theta$$

for any sequence of functions $f_{n,j}$, $j = 1, 2, \ldots, (2n+1)$, $n = 0, 1, 2, \ldots$ from $L^p((0,\pi), \sin^2\theta \, d\theta)$.

Note that Theorem 1.1 of Ciaurri et al. [**1**] is more general than the result above. However, by treating all possible compact rank one symmetric spaces we can cover some of the parameters (α, β). We use the notation $(\cdot,\cdot)_H$ for the inner product on a Hilbert space H.

2. Preliminaries

2.1. The Laplace–Beltrami operators on $\mathbf{S^3}$ and $SU(2)$. In this subsection we calculate the spherical gradient $\nabla_{\mathbf{S}^3}$ and the Laplacian $\Delta_{\mathbf{S}^3}$ in the coordinate system (θ, ω), $\theta \in (0,\pi)$, and $\omega \in \mathbf{S}^2$. Note that every element of $\mathbf{S}^3$ is of the form $(\cos\theta, \omega\sin\theta)$. In the same way, we can write ω as $(\cos\varphi, \sin\varphi\cos\psi, \sin\varphi\sin\psi)$. Let $\nabla_{\mathbf{S}^{d-1}}$ stand for the spherical part of the gradient ∇ on $\mathbb{R}^d$ and $\nabla^j_{\mathbf{S}^{d-1}}$ stand for the components. A simple calculation shows that

$$\begin{aligned}
\nabla^1_{\mathbf{S}^3} &= -\sin\theta \frac{\partial}{\partial\theta}, \\
\nabla^2_{\mathbf{S}^3} &= \cos\theta\cos\varphi \frac{\partial}{\partial\theta} + \frac{1}{\sin\theta} \nabla^1_{\mathbf{S}^2}, \\
\nabla^3_{\mathbf{S}^3} &= \cos\theta\sin\varphi\cos\psi \frac{\partial}{\partial\theta} + \frac{1}{\sin\theta} \nabla^2_{\mathbf{S}^2}, \\
\nabla^4_{\mathbf{S}^3} &= \cos\theta\sin\varphi\sin\psi \frac{\partial}{\partial\theta} + \frac{1}{\sin\theta} \nabla^3_{\mathbf{S}^2}.
\end{aligned}$$

From these expressions it is easy to check that

$$\Delta_{\mathbf{S}^3} = -\left(\frac{\partial^2}{\partial\theta^2} + 2\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Delta_{\mathbf{S}^2}\right).$$

Let $g \in SU(2)$ be given by $x = (x_1, x_2, x_3, x_4) \in \mathbf{S}^3$. Then the left invariant vector fields X, Y, and Z corresponding to the Pauli matrices can be calculated. We get

$$Xf(g) = \left[\left(x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1}\right) + \left(x_2\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_2}\right)\right] f(g),$$

$$Yf(g) = \left[\left(x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}\right) + \left(x_1\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_1}\right)\right] f(g),$$

$$Zf(g) = \left[\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) + \left(x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}\right)\right] f(g).$$

If we convert these expressions into geodesic polar coordinates we obtain

$$X = \omega_2\frac{\partial}{\partial\theta} + \cot\theta\nabla^2_{\mathbf{S}^2} + \omega_1\nabla^3_{\mathbf{S}^2} - \omega_3\nabla^1_{\mathbf{S}^2},$$

$$Y = \omega_3\frac{\partial}{\partial\theta} + \cot\theta\nabla^3_{\mathbf{S}^2} + \omega_2\nabla^1_{\mathbf{S}^2} - \omega_1\nabla^2_{\mathbf{S}^2},$$

$$Z = \omega_1\frac{\partial}{\partial\theta} + \cot\theta\nabla^1_{\mathbf{S}^2} + \omega_3\nabla^2_{\mathbf{S}^2} - \omega_2\nabla^3_{\mathbf{S}^2}.$$

The operator $\Delta = -(X^2 + Y^2 + Z^2)$ turns out to be simply $\Delta_{\mathbf{S}^3}$ as can be easily checked. Thus

$$\Delta = -\left(\frac{\partial^2}{\partial\theta^2} + 2\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Delta_{\mathbf{S}^2}\right) = \Delta_{\mathbf{S}^3}. \tag{2.1}$$

The calculations leading to the formulas in this subsection are cumbersome but can be done. We refer to Chapter 11 of [**7**] and Chapter 1 of [**2**] for these calculations.

2.2. Jacobi expansions and Jacobi–Riesz transforms. In this subsection we introduce the Jacobi polynomials and Jacobi trigonometric polynomials. We also give the expression for the Jacobi–Riesz transform (Riesz transform associated to Jacobi operator $\mathcal{J}_{(\alpha,\beta)}$ as in (1.1)) for functions in $L^2((0,\pi), d\mu_{\alpha,\beta})$. The Jacobi polynomials of degree $k \geq 0$ and type $\alpha, \beta > -1$ are given by

$$P_k^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^k}{2^k k!}\left(\frac{d}{dx}\right)^k\{(1-x)^{k+\alpha}(1+x)^{k+\beta}\}, \tag{2.2}$$

for $x \in (-1,1)$. The system $\{P_k^{(\alpha,\beta)}\}_{k\geq 0}$ forms an orthogonal basis for the Hilbert space $L^2((-1,1),(1-x)^\alpha(1+x)^\beta\,dx)$. After making the change of variable $x=\cos\theta$, we obtain the normalised Jacobi trigonometric polynomials which are given by

$$\mathcal{P}_k^{(\alpha,\beta)}(\theta)=d_k^{\alpha,\beta}P_k^{(\alpha,\beta)}(\cos\theta), \tag{2.3}$$

where the normalizing factor is

$$\begin{aligned} d_k^{\alpha,\beta} &= 2^{\frac{\alpha+\beta+1}{2}}\|P_k^{(\alpha,\beta)}\|^{-1}_{L^2((-1,1),\,(1-x)^\alpha(1+x)^\beta\,dx)} \\ &= \left(\frac{(2k+\alpha+\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}\right)^{\frac{1}{2}}. \end{aligned}$$

The trigonometric polynomials $\mathcal{P}_k^{(\alpha,\beta)}$ are eigenfunctions of the Jacobi differential operator $\mathcal{J}_{(\alpha,\beta)}$ in (1.1), with eigenvalue $\lambda_k^{\alpha,\beta}=(k+\frac{\alpha+\beta+1}{2})^2$. That is to say

$$\mathcal{J}_{(\alpha,\beta)}\mathcal{P}_k^{(\alpha,\beta)}=\lambda_k^{\alpha,\beta}\mathcal{P}_k^{(\alpha,\beta)}.$$

Moreover, the system $\{\mathcal{P}_k^{(\alpha,\beta)}\}_{k\geq 0}$ forms a complete orthonormal basis of $L^2((0,\pi),d\mu_{\alpha,\beta}(\theta))$. For further references about Jacobi polynomials, see Chapter IV in [**11**].

We have the decomposition (see [**1**])

$$\mathcal{J}_{(\alpha,\beta)}=\delta^*\delta+\left(\frac{\alpha+\beta+1}{2}\right)^2,$$

where $\delta=\frac{d}{d\theta}$, and δ^* is its formal adjoint in $L^2((0,\pi),d\mu_{\alpha,\beta}(\theta))$, that is,

$$\delta^*=-\frac{d}{d\theta}-\left(\alpha+\frac{1}{2}\right)\cot\frac{\theta}{2}+\left(\beta+\frac{1}{2}\right)\tan\frac{\theta}{2}.$$

The Jacobi–Riesz transform is formally defined as $\mathcal{R}_{(\alpha,\beta)}=\delta(\mathcal{J}_{(\alpha,\beta)})^{-1/2}$. For a function $f\in L^2((0,\pi),d\mu_{\alpha,\beta}(\theta))$ we can write

$$f=\sum_{k=0}^{\infty}c_k^{\alpha,\beta}(f)\mathcal{P}_k^{(\alpha,\beta)}$$

in $L^2((0,\pi), d\mu_{\alpha,\beta}(\theta))$, where $c_k^{\alpha,\beta}(f) = (f, \mathcal{P}_k^{(\alpha,\beta)})_{L^2((0,\pi),d\mu_{\alpha,\beta}(\theta))}$. And then the Riesz transform can be precisely written as

$$\begin{aligned}\mathcal{R}_{(\alpha,\beta)} f(\theta) &= \sum_{k=0}^{\infty} \frac{1}{(k+\frac{\alpha+\beta+1}{2})} c_k^{\alpha,\beta}(f) \delta \mathcal{P}_k^{(\alpha,\beta)}(\theta) \\ &= -\frac{1}{2} \sum_{k=1}^{\infty} \frac{(k(k+\alpha+\beta+1))^{1/2}}{(k+\frac{\alpha+\beta+1}{2})} c_k^{\alpha,\beta}(f)(\sin\theta)\mathcal{P}_{k-1}^{(\alpha+1,\beta+1)}(\theta).\end{aligned} \tag{2.4}$$

It can be checked that $\mathcal{R}_{(\alpha,\beta)}$ is a bounded operator on $L^2((0,\pi), d\mu_{\alpha,\beta}(\theta))$. It is known that (see [**4**]) $\mathcal{R}_{(\alpha,\beta)}$ is a Calderón–Zygmund operator on the space $((0,\pi), |\cdot|, d\mu_{\alpha,\beta})$ of homogeneous type.

2.3. Spherical harmonics and eigenfunctions of $\boldsymbol{\Delta}$. Let $\mathcal{H}_m(\mathbf{S}^3)$ stand for the space of spherical harmonics of degree m on $\mathbf{S}^3$. Then it is well known that $L^2(\mathbf{S}^3)$ is the orthogonal direct sum of $\mathcal{H}_m(\mathbf{S}^3)$ as m ranges over $\mathbb{N}$, the set of all natural numbers (including 0). Moreover, every element $Y \in \mathcal{H}_m(\mathbf{S}^3)$ is an eigenfunction of $\Delta_{\mathbf{S}^3}$ with eigenvalue $m(m+2)$. The space $\mathcal{H}_m(\mathbf{S}^3)$ can be further decomposed in terms of spherical harmonics on $\mathbf{S}^2$.

For each $\lambda > -\frac{1}{2}$, $x \in (-1,1)$ let $C_k^\lambda(x)$ stand for the ultraspherical polynomials defined by

$$C_k^\lambda(x) = \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(k+2\lambda)}{\Gamma(2\lambda)\Gamma(k+\lambda+\frac{1}{2})} P_k^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x),$$

see (4.7.1) in [**11**]. Let $\psi_{m,n}(\theta) = a_{m,n}(\sin\theta)^n C_{m-n}^{n+1}(\cos\theta)$ where $a_{m,n}$ are normalising constants chosen in such a way that

$$\int_0^\pi |\psi_{m,n}(\theta)|^2 \sin^2\theta \, d\theta = 1.$$

Let $Y_{n,j}(\omega)$, $j = 1, 2, \ldots, (2n+1)$ be an orthonormal basis for $\mathcal{H}_n(\mathbf{S}^2)$, the space of spherical harmonics of degree n on $\mathbf{S}^2$. Then it can be shown that the functions $\psi_{m,n}(\theta)Y_{n,j}(\omega)$, $m-n \geq 0$ are eigenfunctions of $\Delta_{\mathbf{S}^3}$ with eigenvalues $m(m+2)$. In fact, these functions are the spherical harmonics of degree m on $\mathbf{S}^3$ in the coordinate system (θ,ω), see (1.5.6) in [**2**] and also [**8**]. Hence if $g = \Phi(\theta,\omega)$ then the functions

$$\varphi_{m,n,j}(g) = \psi_{m,n}(\theta) Y_{n,j}(\omega), \quad 0 \leq n \leq m, \quad 0 \leq j \leq 2n+1$$

are eigenfunctions of the operator Δ on $SU(2)$.

The spectral decomposition of the operator $(\Delta + 1) = \Delta_{\mathbf{S}^3} + 1$ is therefore given by

$$(\Delta + 1)f(g) = \sum_{m=0}^{\infty} (m+1)^2 P_m f(g),$$

where the projections P_m are defined by

$$P_m f(g) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} (f, \varphi_{m,n,j})_{L^2(SU(2))} \varphi_{m,n,j}(g).$$

By the spectral theorem the operator $(\Delta + 1)^{-\frac{1}{2}}$ is given by

$$(\Delta+1)^{-\frac{1}{2}} f(g) = \sum_{m=0}^{\infty} (m+1)^{-1} \sum_{n=0}^{m} \sum_{j=1}^{2n+1} (f, \varphi_{m,n,j})_{L^2(SU(2))} \varphi_{m,n,j}(g). \tag{2.5}$$

Since we are interested in mixed norm estimates it is useful to rewrite $(\Delta + 1)^{-\frac{1}{2}} f$ in a more convenient form.

By letting $F(\theta, \omega) = f \circ \Phi(\theta, \omega)$ and defining

$$F_{n,j}(\theta) = \int_{\mathbf{S}^2} F(\theta, \omega) Y_{n,j}(\omega)\, d\sigma(\omega)$$

we have the expansion

$$F(\theta, \omega) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} F_{n,j}(\theta) Y_{n,j}(\omega). \tag{2.6}$$

From this it is clear that

$$\int_{\mathbf{S}^2} |F(\theta, \omega)|^2\, d\sigma(\omega) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} |F_{n,j}(\theta)|^2.$$

For each n and j the function $F_{n,j}(\theta)$ can be expanded in terms of $\psi_{m+n,n}(\theta)$:

$$F_{n,j}(\theta) = \sum_{m=0}^{\infty} \left(\int_0^{\pi} F_{n,j}(\eta) \psi_{m+n,n}(\eta) \sin^2 \eta\, d\eta \right) \psi_{m+n,n}(\theta).$$

Recalling the definition of $\varphi_{m,n,j}$ we see that

$$\int_0^{\pi} F_{n,j}(\theta) \psi_{m+n,n}(\theta) \sin^2 \theta\, d\theta = \int_{SU(2)} f(g) \varphi_{m+n,n,j}(g)\, dg. \tag{2.7}$$

Thus we have

$$F_{n,j}(\theta) = \sum_{m=0}^{\infty} (f, \varphi_{m+n,n,j})_{L^2(SU(2))} \psi_{m+n,n}(\theta). \tag{2.8}$$

Since $\varphi_{m,n,j}(g)$ are eigenfunctions of $(\Delta+1)$ with eigenvalues $(m+1)^2$ we see that

$$(\Delta+1)^{-\frac{1}{2}} f(g) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \widetilde{F}_{n,j}(\theta) Y_{n,j}(\omega),$$

where

$$\widetilde{F}_{n,j}(\theta) = \sum_{m=0}^{\infty} (m+n+1)^{-1} (F_{n,j}, \psi_{m+n,n})_{L^2((0,\pi),\sin^2\theta\, d\theta)} \psi_{m+n,n}(\theta). \tag{2.9}$$

In view of the definition of $\psi_{m,n}(\theta)$ it is not difficult to see that

$$\begin{aligned}&(F_{n,j}, \psi_{m+n,n})_{L^2((0,\pi),\sin^2\theta\, d\theta)} \psi_{m+n,n}(\theta)\\ &= (\sin\theta)^n \mathcal{P}_m^{(n+\frac{1}{2},n+\frac{1}{2})}(\theta) \int_0^\pi F_{n,j}(\eta)(\sin\eta)^{-n} \mathcal{P}_m^{(n+\frac{1}{2},n+\frac{1}{2})}(\eta)(\sin\eta)^{2n+2} d\eta,\end{aligned}$$

where $\mathcal{P}_m^{(\alpha,\beta)}(\theta)$ stand for the Jacobi polynomials normalised in the space $L^2((0,\pi), d\mu_{\alpha,\beta})$. By letting J_n stand for the Jacobi operator $\mathcal{J}_{(n+\frac{1}{2},n+\frac{1}{2})}$ and recalling that $\mathcal{P}_m^{(n+\frac{1}{2},n+\frac{1}{2})}(\theta)$ are eigenfunctions of J_n with eigenvalues $(m+n+1)^2$ we infer that

$$\widetilde{F}_{n,j}(\theta) = u_n(\theta) J_n^{-\frac{1}{2}} (u_n^{-1} F_{n,j})(\theta), \tag{2.10}$$

where $u_n(\theta) = (\sin\theta)^n$. Thus we have the expression

$$(\Delta+1)^{-\frac{1}{2}} f(g) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} u_n(\theta) J_n^{-\frac{1}{2}} (u_n^{-1} F_{n,j})(\theta) Y_{n,j}(\omega). \tag{2.11}$$

This formula plays a crucial role in the study of mixed norm estimates for Riesz transforms on $SU(2)$.

3. Mixed norm estimates for the Riesz transforms

In this section we present a proof of Theorem 1.2. As we have mentioned in the introduction we will prove the result by making use of a lemma of Herz and Rivière and following an idea of Rubio de Francia [**6**].

Consider the vector of Riesz transforms $\mathcal{R}f = (Z, X, Y)(\Delta+1)^{-\frac{1}{2}} f$. Denote the components of $\mathcal{R}$ by $\mathcal{R}_j$, $j = 1, 2, 3$. Thus $\mathcal{R}_1 f = Z(\Delta + 1)^{-\frac{1}{2}} f$, etc. Given a unit vector $\zeta \in \mathbf{S}^2$ let us consider the operator

$$T_\zeta f = \sum_{j=1}^{3} \zeta_j \mathcal{R}_j f = \zeta \cdot \mathcal{R} f.$$

Since the Riesz transforms are bounded on $L^p(SU(2))$, $1 < p < \infty$, it follows that T_ζ are bounded on $L^p(SU(2))$ for the same range of p. Now we make use of the following lemma of Herz and Rivière [**3**].

Lemma 3.1. *Let (G,μ) and (H,ν) be arbitrary measure spaces and $T\colon L^p(G) \to L^p(G)$ a bounded linear operator. Then if $p \le q \le 2$ or $p \ge q \ge 2$, there exists a bounded linear operator $\tilde{T}\colon L^p(G;L^q(H)) \to L^p(G;L^q(H))$ with $\|\tilde{T}\| \le \|T\|$ such that for $F \in L^p(G;L^q(H))$ of the form $F(x,\xi) = f(\xi)u(x)$ where $f \in L^p(G)$ and $u \in L^q(H)$ we have*

$$(\tilde{T}F)(\xi,x) = (Tf)(\xi)u(x).$$

Let $K = SO(3)$. By taking $H = K$ in the lemma we see that T_ζ has an extension denoted by $\widetilde{T_\zeta}$ to $L^p(SU(2), L^2(K))$ as a bounded linear operator. Given $f \in L^p(SU(2))$ and $k \in K$ let us define

$$\rho(k)f(g) = (f \circ \Phi)(\theta, k\,\omega), \quad g = \Phi(\theta,\omega).$$

Then $\widetilde{f}(g,k) = \rho(k)f(g)$ belongs to $L^p(SU(2), L^2(K))$ and hence by the lemma above

$$\int_{SU(2)} \left(\int_K |\widetilde{T_\zeta}\widetilde{f}(g,k)|^2\,dk \right)^{\frac{p}{2}} dg \le C \int_{SU(2)} \left(\int_K |\widetilde{f}(g,k)|^2\,dk \right)^{\frac{p}{2}} dg.$$

Note that C can be taken independent of $\zeta \in \mathbf{S}^2$ since the norm of T_ζ on $L^p(SU(2))$ is a bounded function of ζ and the norm of the extended operator $\widetilde{T_\zeta}$ is bounded by that of T_ζ. We can now easily prove the following result.

Theorem 3.2. *We have for $j = 1,2,3$ and $1 < p < \infty$ the mixed norm estimates*

$$\|\mathcal{R}_j f\|_{L^{p,2}(SU(2))} \le C\|f\|_{L^{p,2}(SU(2))}.$$

Proof: A simple calculation, using the fact that $\mathbf{S}^2 = SO(3)/SO(2)$ shows that

$$\int_{SU(2)} \left(\int_K |\widetilde{f}(g,k)|^2\,dk \right)^{\frac{p}{2}} dg = \int_0^\pi \left(\int_{\mathbf{S}^2} |f \circ \Phi(\theta,\omega)|^2\,d\sigma(\omega) \right)^{\frac{p}{2}} \sin^2\theta\,d\theta.$$

In view of this it is enough to prove the inequality

$$\int_{SU(2)} \left(\int_K |(\rho(k)T_\zeta f)(g)|^2\,dk \right)^{\frac{p}{2}} dg \le C \int_{SU(2)} \left(\int_K |\rho(k)f(g)|^2\,dk \right)^{\frac{p}{2}} dg.$$

Now we claim that

$$(3.1)\quad (\rho(k)T_\zeta f)(g) = (k^{-1}\zeta)\cdot \mathcal{R}(\rho(k)f)(g) = \sum_{j=1}^{3}(k^{-1}\zeta)_j\mathcal{R}_j(\rho(k)f)(g).$$

Assuming the claim for a moment, we first complete the proof.

Since $\mathcal{R}_j$ are bounded on $L^p(SU(2))$, by Lemma 3.1, they have extensions $\widetilde{\mathcal{R}}_j$ to $L^p(SU(2), L^2(K))$. Moreover, the extensions satisfy $\widetilde{\mathcal{R}}_j\widetilde{f}(g,k) = \mathcal{R}_j(\rho(k)f)(g)$ on a dense class of functions. To see this, note that in view of Peter–Weyl Theorem for K the functions $\widetilde{f}(g,k)$ can be expanded in terms of matrix coefficients of irreducible unitary representations π of K. Thus

$$\widetilde{f}(g,k) = \sum_{\pi\in\hat{K}}\sum_{i,l=1}^{d_\pi} f_{i,l}(g)\pi_{i,l}(k),$$

where $\hat{K}$ is the unitary dual of K. From the definition of the extension, it follows that for each summand $\widetilde{\mathcal{R}}_j(f_{i,l}\pi_{i,l})(g,k) = \mathcal{R}_j(f_{i,l})(g)\pi_{i,l}(k)$. Hence $\widetilde{\mathcal{R}}_j\widetilde{f}(g,k) = \mathcal{R}_j(\rho(k)f)(g)$ for all functions f for which $\widetilde{f}(g,k)$ has a finite Peter–Weyl expansion. Consequently, in view of the assumed claim, we have

$$\rho(k)T_\zeta f(g) = \sum_{j=1}^{3}(k^{-1}\zeta)_j\widetilde{\mathcal{R}}_j\widetilde{f}(g,k)$$

which gives the estimate

$$\begin{aligned}
&\left(\int_{SU(2)}\left(\int_K |\rho(k)T_\zeta f(g)|^2\,dk\right)^{\frac{p}{2}}dg\right)^{\frac{1}{p}}\\
&\qquad\le \sum_{j=1}^{3}\left(\int_{SU(2)}\left(\int_K |\widetilde{\mathcal{R}}_j\widetilde{f}(g,k)|^2\,dk\right)^{\frac{p}{2}}dg\right)^{\frac{1}{p}}\\
&\qquad\le C\left(\int_{SU(2)}\left(\int_K |\widetilde{f}(g,k)|^2\,dk\right)^{\frac{p}{2}}dg\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus we are left with proving claim (3.1).

In order to prove claim (3.1) we make use of the following expression for $\mathcal{R}f$. By writing the vector fields X, Y, and Z in geodesic polar coordinates we obtain

$$(\mathcal{R}f)\circ\Phi(\theta,\omega)=\left(\omega\frac{\partial}{\partial\theta}+\cot\theta\,\nabla_{\mathbf{S}^2}+N\right)(\Delta+1)^{-\frac{1}{2}}(f\circ\Phi)(\theta,\omega),$$

where $Nf=\nabla_{\mathbf{S}^2}f\times\omega$ is the cross product of the vectors $\nabla_{\mathbf{S}^2}f$ and ω in $\mathbb{R}^3$. In view of this

$$T_\zeta f=\left(\zeta\cdot\omega\frac{\partial}{\partial\theta}+\cot\theta\,\zeta\cdot\nabla_{\mathbf{S}^2}+\zeta\cdot N\right)(\Delta+1)^{-\frac{1}{2}}f. \tag{3.2}$$

In order to compute $T_\zeta(\rho(k)f)$ we make use of the following facts: (i) since $\rho(k)$ acts on the ω-variable, it commutes with $\Delta=\Delta_{\mathbf{S}^3}$; (ii) the spherical gradient $\nabla_{\mathbf{S}^2}$ is related to the gradient ∇ on $\mathbb{R}^3$ via the equation $\nabla_{\mathbf{S}^2}h(\omega)=\nabla h(\omega)-\omega(\omega\cdot\nabla_{\mathbf{S}^2}h(\omega))$ (see equation 1.8.12 in [**2**]) and (iii) for any $x,u\in\mathbb{R}^3$ and $k\in K$, $k\,u\cdot\nabla h(kx)=u\cdot\nabla(\rho(k)h)(x)$ which follows by direct calculation. In order to deal with the term Nf we also need the relation $\nabla=\frac{1}{r}\nabla_{\mathbf{S}^2}+\omega\frac{\partial}{\partial r}$ and the fact that for $k\in K$, $k(u\times v)=k\,u\times k\,v$ for any two vectors. Using these facts in (3.2) it is easy to check that

$$\Big(T_\zeta\rho(k)f\Big)(g)=\Big(\rho(k)T_{k\zeta}f\Big)(g)$$

or

$$\rho(k)(T_\zeta f)(g)=(k^{-1}\zeta)\cdot\mathcal{R}(\rho(k)f)(g)=\sum_{j=1}^{3}(k^{-1}\zeta)_j\mathcal{R}_j(\rho(k)f)(g).$$

Hence the claim is proved. $\square$

4. Vector valued inequalities for Jacobi–Riesz transforms

With the aim of proving Theorem 1.3 let us return to the expression

$$(\Delta+1)^{-\frac{1}{2}}f(g)=\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\widetilde{F}_{n,j}(\theta)Y_{n,j}(\omega)$$

obtained in Section 2. Recall (2.9) that

$$\widetilde{F}_{n,j}(\theta)=\sum_{m=0}^{\infty}(m+n+1)^{-1}(F_{n,j},\psi_{m+n,n})_{L^2((0,\pi),\sin^2\theta\,d\theta)}\psi_{m+n,n}(\theta).$$

In view of the expression

$$\mathcal{R}f(g)=\left(\omega\frac{\partial}{\partial\theta}+\cot\theta\,\nabla_{\mathbf{S}^2}+N\right)(\Delta+1)^{-\frac{1}{2}}f(g)$$

we infer that $\mathcal{R}f(g) = \sum_{j=0}^{3} \mathcal{R}^{(j)} f(g)$ with

$$\mathcal{R}^{(1)} f(g) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{\partial}{\partial \theta} \widetilde{F}_{n,j}(\theta) Y_{n,j}(\omega)\omega,$$

$$\mathcal{R}^{(2)} f(g) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \cot\theta \widetilde{F}_{n,j}(\theta) \nabla_{\mathbf{S}^2} Y_{n,j}(\omega),$$

$$\mathcal{R}^{(3)} f(g) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \widetilde{F}_{n,j}(\theta) \nabla_{\mathbf{S}^2} Y_{n,j} \times \omega.$$

Using these expressions along with some orthogonality properties of the spherical harmonics $Y_{n,j}$ and $\nabla_{\mathbf{S}^2} Y_{n,j}$ we can prove the following result.

Theorem 4.1. *With notations as above we have*

$$\int_{\mathbf{S}^2} \langle \mathcal{R}f(\theta,\omega), \mathcal{R}f(\theta,\omega)\rangle \, d\sigma(\omega)$$
$$= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\left| \frac{\partial}{\partial \theta} \widetilde{F}_{n,j}(\theta) \right|^2 + \frac{n(n+1)}{\sin^2\theta} |\widetilde{F}_{n,j}(\theta)|^2 \right).$$

In order to prove this theorem we need several properties of the spherical harmonics. It is known that $\omega \cdot \nabla_{\mathbf{S}^2} Y_{n,j}(\omega) = 0$ for any spherical harmonic and also

$$\int_{\mathbf{S}^2} \nabla_{\mathbf{S}^2} Y_{n,j}(\omega) \cdot \nabla_{\mathbf{S}^2} Y_{m,l}(\omega) \, d\sigma(\omega) = n(n+1)\delta_{n,m}\delta_{j,l}$$

see (1.4.9) and (1.8.14) in [**2**]. We also require the following results.

Proposition 4.2. *Let Y_n and Y_m be spherical harmonics of degree n and m respectively on $\mathbf{S}^2$. Then*

(a) $\nabla_{\mathbf{S}^2} \times \nabla_{\mathbf{S}^2} Y_n(\omega) = \omega \times \nabla_{\mathbf{S}^2} Y_n(\omega)$,
(b) $(\nabla_{\mathbf{S}^2} Y_n(\omega) \times \omega) \cdot (\nabla_{\mathbf{S}^2} Y_m(\omega) \times \omega) = \nabla_{\mathbf{S}^2} Y_n(\omega) \cdot \nabla_{\mathbf{S}^2} Y_m(\omega)$,
(c) $\int_{\mathbf{S}^2} \nabla_{\mathbf{S}^2} Y_n(\omega) \cdot (\nabla_{\mathbf{S}^2} Y_m(\omega) \times \omega) \, d\sigma(\omega) = 0$.

We postpone a proof of this proposition to the next section. Using the result of this proposition along with the expressions for the components $\mathcal{R}^{(j)} f$ of $\mathcal{R}f$, it is immediately seen that Theorem 4.1 is true.

The mixed norm estimates proved in Theorem 3.2 leads to the following result.

Theorem 4.3. *With notations as above we have*

(a) $$\int_0^\pi \left(\sum_{n=0}^\infty \sum_{j=1}^{2n+1} \left|\frac{\partial}{\partial\theta}\widetilde{F}_{n,j}(\theta)\right|^2\right)^{\frac{p}{2}} \sin^2\theta\, d\theta$$
$$\leq C\int_0^\pi \left(\sum_{n=0}^\infty \sum_{j=1}^{2n+1} |F_{n,j}(\theta)|^2\right)^{\frac{p}{2}} \sin^2\theta\, d\theta,$$

(b) $$\int_0^\pi \left(\sum_{n=0}^\infty \sum_{j=1}^{2n+1} \frac{n^2}{\sin^2\theta}|\widetilde{F}_{n,j}(\theta)|^2\right)^{\frac{p}{2}} \sin^2\theta\, d\theta$$
$$\leq C\int_0^\pi \left(\sum_{n=0}^\infty \sum_{j=1}^{2n+1} |F_{n,j}(\theta)|^2\right)^{\frac{p}{2}} \sin^2\theta\, d\theta.$$

In order to prove Theorem 1.3 we only need to reinterpret the inequality (a) of the theorem above as a vector valued inequality for Jacobi–Riesz transforms. We have already made the observation that

$$\widetilde{F}_{n,j}(\theta) = u_n(\theta)J_n^{-\frac{1}{2}}(u_n^{-1}F_{n,j})(\theta).$$

Recalling that $u_n(\theta) = (\sin\theta)^n$, this gives us

$$\frac{\partial}{\partial\theta}\widetilde{F}_{n,j}(\theta) = \frac{n}{\sin\theta}\widetilde{F}_{n,j}(\theta) + u_n(\theta)\frac{\partial}{\partial\theta}J_n^{-\frac{1}{2}}(u_n^{-1}F_{n,j})(\theta).$$

Since $\frac{\partial}{\partial\theta}J_n^{-\frac{1}{2}}(u_n^{-1}F_{n,j})(\theta) = \mathcal{R}_n(u_n^{-1}F_{n,j})$, the inequalities (a) and (b) together give us

$$\int_0^\pi \left(\sum_{n=0}^\infty \sum_{j=1}^{2n+1} u_n(\theta)^2|\mathcal{R}_n(u_n^{-1}F_{n,j})|^2\right)^{\frac{p}{2}} \sin^2\theta\, d\theta$$
$$\leq C\int_0^\pi \left(\sum_{n=0}^\infty \sum_{j=1}^{2n+1} |F_{n,j}(\theta)|^2\right)^{\frac{p}{2}} \sin^2\theta\, d\theta.$$

Thus Theorem 1.3 is proved.

5. Some calculations related to spherical harmonics

In this section we prove some properties of spherical harmonics. First we introduce some notations which we use further before going to state the properties. Let ∇ stand for the gradient on $\mathbb{R}^d$, which can be written in spherical coordinates as

$$\nabla = \frac{1}{r}\nabla_{\mathbf{S}^{d-1}} + \omega\frac{\partial}{\partial r},$$

where $\nabla_{\mathbf{S}^{d-1}}$ stands for the spherical (angular) part of the gradient ∇ on $\mathbb{R}^d$, and ω is the unit vector along the direction of x, i.e, $x = r\,\omega$, $r = |x|$. Let ∇^j, $\nabla^j_{\mathbf{S}^{d-1}}$, and ω_j stand for the j^{th} component of ∇, $\nabla_{\mathbf{S}^{d-1}}$, and ω respectively. In the same way, the Laplacian $\Delta_{\mathbb{R}^d} = -\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ on $\mathbb{R}^d$ can be written in spherical coordinates as

$$\Delta_{\mathbb{R}^d} = -\left(\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathbf{S}^{d-1}}\right),$$

where $\Delta_{\mathbf{S}^{d-1}}$ is the spherical part of $\Delta_{\mathbb{R}^d}$ (also called the Laplace–Beltrami operator on $\mathbf{S}^{d-1}$). Note that $\nabla_{\mathbf{S}^{d-1}} \cdot \nabla_{\mathbf{S}^{d-1}} = -\Delta_{\mathbf{S}^{d-1}}$. Let $\mathcal{H}_m(\mathbf{S}^{d-1})$ stand for the space of spherical harmonics of degree m on $\mathbf{S}^{d-1}$. Then it is well known that $L^2(\mathbf{S}^{d-1})$ is the orthogonal direct sum of $\mathcal{H}_m(\mathbf{S}^{d-1})$ as m ranges over $\mathbb{N}$, the set of all natural numbers (including 0). Moreover, every element $Y \in \mathcal{H}_m(\mathbf{S}^{d-1})$ is an eigenfunction of $\Delta_{\mathbf{S}^{d-1}}$ with eigenvalue $m(m+d-2)$. Now we state and prove some properties of the spherical gradient and spherical harmonics.

Proposition 5.1.

(1) (a) *For $f, g \in C^1(\mathbf{S}^{d-1})$, the space of all continuously differentiable functions on $\mathbf{S}^{d-1}$,*

$$\int_{\mathbf{S}^{d-1}} f(\omega)\nabla_{\mathbf{S}^{d-1}} g(\omega)\, d\sigma(\omega) = -\int_{\mathbf{S}^{d-1}} g(\omega)[\nabla_{\mathbf{S}^{d-1}} f(\omega) - (d-1)f(\omega)\omega]\, d\sigma(\omega),$$

where $d\sigma(\omega)$ is the surface measure on $\mathbf{S}^{d-1}$.

(b) *Furthermore, for $f \in C^2(\mathbf{S}^{d-1})$, the space of all twice continuously differentiable functions defined on $\mathbf{S}^{d-1}$, and $g \in C^1(\mathbf{S}^{d-1})$*

$$\int_{\mathbf{S}^{d-1}} \nabla_{\mathbf{S}^{d-1}} f(\omega) \cdot \nabla_{\mathbf{S}^{d-1}} g(\omega)\, d\sigma(\omega) = \int_{\mathbf{S}^{d-1}} g(\omega)\Delta_{\mathbf{S}^{d-1}} f(\omega)\, d\sigma(\omega).$$

(c) *For any smooth vector-valued function F and any smooth scalar valued function g defined on $\mathbf{S}^{d-1}$,*

$$\int_{\mathbf{S}^{d-1}} \nabla_{\mathbf{S}^{d-1}} g(\omega)\cdot F(\omega)\, d\sigma(\omega) = \int_{\mathbf{S}^{d-1}} g(\omega)[(d-1)\omega - \nabla_{\mathbf{S}^{d-1}}]\cdot F(\omega)\, d\sigma(\omega).$$

(2) *Let $Y \in \mathcal{H}_m(\mathbf{S}^{d-1})$. Then we have the following.*

(a) *For $i, j = 1, 2, \dots, d$,* $\nabla^i_{\mathbf{S}^{d-1}}\omega_j = \begin{cases} -\omega_i\,\omega_j, & \text{if } i \neq j; \\ 1-\omega_j^2, & \text{if } i = j. \end{cases}$

(b) $\nabla_{\mathbf{S}^{d-1}} \cdot \omega = d-1$.

(c) $\nabla^i_{\mathbf{S}^{d-1}}\omega_j - \nabla^j_{\mathbf{S}^{d-1}}\omega_i = 0$; *in particular for* $d=3$, *we have* $\nabla_{\mathbf{S}^2} \times \omega = \bar{0}$, *(here the symbol "×" cross denotes the cross product of vectors in* $\mathbb{R}^3$*).*

(d) $(\nabla^i_{\mathbf{S}^{d-1}}\nabla^j_{\mathbf{S}^{d-1}} - \nabla^j_{\mathbf{S}^{d-1}}\nabla^i_{\mathbf{S}^{d-1}})Y(\omega) = (\omega_i\nabla^j_{\mathbf{S}^{d-1}} - \omega_j\nabla^i_{\mathbf{S}^{d-1}})Y(\omega)$.

Proof: For the proof of (1)(a) and (1)(b), see Proposition 1.8.7 in [**2**]. And (1)(c) follows from (1)(a) by applying componentwise.

For (2)(a) consider the identity $\nabla^i x_j = \frac{\partial}{\partial x_i}(x_j) = \delta_{i\,j}$. If we write this identity in spherical coordinates, then we have

$$\left(\frac{1}{r}\nabla^i_{\mathbf{S}^{d-1}} + \omega_i\frac{\partial}{\partial r}\right)(r\,\omega_j) = \delta_{i\,j}$$

which implies (2)(a). For (2)(b), consider $\nabla_{\mathbf{S}^{d-1}} \cdot \omega = \sum_{j=1}^d \nabla^j_{\mathbf{S}^{d-1}}\omega_j$. By making use of (2)(a), we have $\nabla_{\mathbf{S}^{d-1}} \cdot \omega = \sum_{j=1}^d (1-\omega_j^2) = (d-1)$, which proves (2)(b). Moreover, (2)(c) is obvious by (2)(a) and (2)(b).

For (2)(d), consider the identity

$$(\nabla^i\nabla^j - \nabla^j\nabla^i)Y(r\,\omega) = 0,$$

where $Y(r\,\omega) = r^m Y(\omega)$. We will write the above in spherical coordinates to get (2)(d). First consider

$$\begin{aligned}\nabla^i\nabla^j Y(r\,\omega) &= \left(\frac{1}{r}\nabla^i_{\mathbf{S}^{d-1}} + \omega_i\frac{\partial}{\partial r}\right)\left(\frac{1}{r}\nabla^j_{\mathbf{S}^{d-1}} + \omega_j\frac{\partial}{\partial r}\right)Y(r\omega)\\ &= r^{m-2}\Big[\nabla^i_{\mathbf{S}^{d-1}}\nabla^j_{\mathbf{S}^{d-1}}Y(\omega) m\,Y(\omega)\nabla^i_{\mathbf{S}^{d-1}}\omega_j\\ &\quad + (m-1)\omega_i\nabla^j_{\mathbf{S}^{d-1}}Y(\omega)\\ &\quad + m\,\omega_j\nabla^i_{\mathbf{S}^{d-1}}Y(\omega) + m(m-1)\omega_i\,\omega_j Y(\omega)\Big].\end{aligned} \tag{5.1}$$

Similarly we have that

$$\begin{aligned}\nabla^j\nabla^i Y(r\,\omega) = r^{m-2}\Big[&\nabla^j_{\mathbf{S}^{d-1}}\nabla^i_{\mathbf{S}^{d-1}}Y(\omega) + m\,Y(\omega)\nabla^j_{\mathbf{S}^{d-1}}\omega_i\\ &+ (m-1)\omega_j\nabla^i_{\mathbf{S}^{d-1}}Y(\omega)\\ &+ m\,\omega_i\nabla^j_{\mathbf{S}^{d-1}}Y(\omega) + m(m-1)\omega_i\,\omega_j Y(\omega)\Big].\end{aligned} \tag{5.2}$$

If we subtract (5.2) from (5.1), then we have that

$$0 = r^{m-2}[(\nabla^i_{\mathbf{S}^{d-1}}\nabla^j_{\mathbf{S}^{d-1}} - \nabla^j_{\mathbf{S}^{d-1}}\nabla^i_{\mathbf{S}^{d-1}})Y(\omega) + (\omega_j\nabla^i_{\mathbf{S}^{d-1}} - \omega_i\nabla^j_{\mathbf{S}^{d-1}})Y(\omega)]$$

which implies (2)(d). Note that we have used (2)(c) and the fact that $Y(r\,\omega) = r^m Y(\omega)$ to get the above. □

Proof of Proposition 4.2: By making use of the definition of cross product of vectors in $\mathbb{R}^3$ and (2)(d) in Proposition 5.1, (a) can be proved.

For (b), if we use the formula, for vectors a, b, c, and $d \in \mathbb{R}^3$,

$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

and the fact $\omega \cdot \nabla_{\mathbf{S}^{d-1}} Y(\omega) = 0$ for any spherical harmonic $Y \in \mathcal{H}_m(\mathbf{S}^{d-1})$, then we can see that

$$\begin{aligned}(\nabla_{\mathbf{S}^{d-1}} Y_n(\omega) \times \omega) \cdot (\nabla_{\mathbf{S}^{d-1}} Y_m(\omega) \times \omega) &= (\nabla_{\mathbf{S}^{d-1}} Y_n \cdot \nabla_{\mathbf{S}^{d-1}} Y_m)(\omega \cdot \omega) \\ &= \nabla_{\mathbf{S}^{d-1}} Y_n(\omega) \cdot \nabla_{\mathbf{S}^{d-1}} Y_m(\omega).\end{aligned}$$

For (c), first we note that, for $\mathbb{R}^3$-valued smooth functions F and G

$$\nabla_{\mathbf{S}^2} \cdot (F \times G) = G \cdot (\nabla_{\mathbf{S}^2} \times F) - F \cdot (\nabla_{\mathbf{S}^2} \times G). \tag{5.3}$$

By appealing to (1)(c) of Proposition 5.1, the integral in (c) is equal to

$$\int_{\mathbf{S}^2} Y_n(\omega)(2\omega - \nabla_{\mathbf{S}^2}) \cdot (\nabla_{\mathbf{S}^2} Y_m(\omega) \times \omega)\, d\sigma(\omega)$$

which is equal to

$$-\int_{\mathbf{S}^2} Y_n(\omega) \nabla_{\mathbf{S}^2} \cdot (\nabla_{\mathbf{S}^2} Y_m(\omega) \times \omega)\, d\sigma(\omega).$$

By using (5.3), (a) of Proposition 4.2, (2)(c) of Proposition 5.1 and the fact that $a \cdot (a \times b) = 0$, we have

$$= -\int_{\mathbf{S}^2} Y_n(\omega)[\omega \cdot (\nabla_{\mathbf{S}^2} \times \nabla_{\mathbf{S}^2} Y_m(\omega)) - \nabla_{\mathbf{S}^2} Y_m(\omega) \cdot (\nabla_{\mathbf{S}^2} \times \omega)]\, d\sigma(\omega)$$

$$= -\int_{\mathbf{S}^2} Y_n(\omega)[\omega \cdot (\omega \times \nabla_{\mathbf{S}^2} Y_m(\omega)) - 0]\, d\sigma(\omega) = 0.$$

This completes the proof of Proposition 4.2. □

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