Publ. Mat. **58** (2014), 353–377
DOI: 10.5565/PUBLMAT_58214_18

ATOMIC DECOMPOSITION OF REAL-VARIABLE TYPE FOR BERGMAN SPACES IN THE UNIT BALL OF $\mathbb{C}^n$

Zeqian Chen and Wei Ouyang

Abstract: In this paper we show that, for any $0 < p \leq 1$ and $\alpha > -1$, every (weighted) Bergman space $\mathcal{A}^p_\alpha(\mathbb{B}_n)$ admits an atomic decomposition of real-variable type. More precisely, for each $f \in \mathcal{A}^p_\alpha(\mathbb{B}_n)$ there exist a sequence of $(p,\infty)_\alpha$-atoms a_k with compact support and a scalar sequence $\{\lambda_k\}$ such that $f = \sum_k \lambda_k a_k$ in the sense of distribution and $\sum_k |\lambda_k|^p \lesssim \|f\|^p_{p,\alpha}$; and moreover, $f = \sum_k \lambda_k P_\alpha(a_k)$ in $\mathcal{A}^p_\alpha(\mathbb{B}_n)$, where P_α is the orthogonal projection from $L^2_\alpha(\mathbb{B}_n)$ onto $\mathcal{A}^2_\alpha(\mathbb{B}_n)$. The proof is constructive and our construction is based on analysis inside the unit ball $\mathbb{B}_n$ associated with a quasimetric.

2010 Mathematics Subject Classification: 32A36, 32A50.

Key words: Bergman space, atomic decomposition, Bergman kernel, homogeneous space, maximal function.

1. Introduction

Let $\mathbb{B}_n$ be the unit ball of $\mathbb{C}^n$. Given $\alpha > -1$ and $p > 0$, the (weighted) Lebesgue space $L^p_\alpha(\mathbb{B}_n)$ consists of measurable functions f on $\mathbb{B}_n$ such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) \right)^{\frac{1}{p}} < \infty,$$

where the weighted Lebesgue measure dv_α on $\mathbb{B}_n$ is defined by

$$dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha \, dv(z)$$

and $c_\alpha = \Gamma(n + \alpha + 1)/[n!\, \Gamma(\alpha + 1)]$ is a normalizing constant so that dv_α is a probability measure on $\mathbb{B}_n$. The (weighted) Bergman space $\mathcal{A}^p_\alpha$ on $\mathbb{B}_n$ is then defined by

$$\mathcal{A}^p_\alpha = \mathcal{H}(\mathbb{B}_n) \cap L^p_\alpha(\mathbb{B}_n),$$

where $\mathcal{H}(\mathbb{B}_n)$ is the space of all holomorphic functions in $\mathbb{B}_n$. When $\alpha = 0$ we simply write $\mathcal{A}^p$ for $\mathcal{A}^p_0$. These are the usual Bergman spaces. Note that for $1 \leq p < \infty$, $\mathcal{A}^p_\alpha$ is a Banach space under the norm $\| \ \|_{p,\alpha}$. If $0 < p < 1$, the space $\mathcal{A}^p_\alpha$ is a quasi-Banach space with p-norm $\|f\|^p_{p,\alpha}$.

There is a well known atomic decomposition for $\mathcal{A}^p_\alpha$ with non-compactly supported atoms established by Coifman and Rochberg [**3**]. However, to the best of our knowledge, an atomic decomposition of these spaces with compactly supported atoms is presented in the literature only for the case $p = 1$ (see e.g. [**4**]). Recently, the present authors [**2**] proved an atomic decomposition of these spaces in terms of compactly supported atoms with respect to Carleson tubes again for $p = 1$. But the approaches in [**2, 4**] are both based on duality and therefore, are not constructive and cannot be applied to the case $0 < p < 1$.

The aim of this paper is to prove an atomic decomposition of $\mathcal{A}^p_\alpha$ with compactly supported atoms for all $0 < p < 1$, through using a constructive method. This is the analogous of the one of Hardy spaces in the complex ball presented in [**7**] (see [**8, 9**] for the more general and complicated setting). However, our proof is based on analysis inside the complex ball instead of the sphere for Hardy spaces (cf. [**7**]). This particularly includes the pointwise estimates for the Bergman kernel and its derivatives in place of the ones for the Szegö kernel associated with Hardy spaces, and some geometrical properties of the complex ball associated with a quasimetric which are different from the ones of the sphere with the nonisotropic metric involved for Hardy spaces. We remark that the framework of this proof is suited for the more general domains such as strictly pseudoconvex domains (see [**8, 9**] for the case of Hardy spaces), but for simplicity, we focus on the case of the complex ball.

As a straightforward application of this atomic decomposition, we can characterize the dual space of $\mathcal{A}^p_\alpha$ as a Lipschitz type space of integral form for all $0 < p < 1$, as done in [**7**, Theorem 6] for Hardy spaces. Further applications of this result to the regularity of small Hankel operators and a factorization theorem on $\mathcal{A}^p_\alpha$ are in order, just to name a few (see e.g. [**9**] in the case of Hardy spaces).

The paper is organized as follows. In Section 2, we present preliminaries and, in particular we introduce local coordinates which reflect the complex structure and define real-variable Bergman spaces. In Section 3, we introduce several maximal functions associated with Bergman spaces, the analogous of the ones for Hardy spaces. In Section 4, we state the corresponding atomic decomposition and prove some auxiliary lemmas. Finally, Section 5 is devoted to the proof of the associated atomic decomposition.

For two nonnegative (possibly infinite) quantities X and Y, by $X \lesssim Y$ we mean that there exists a positive constant C such that $X \leq CY$, and by $X \approx Y$ that $X \lesssim Y$ and $Y \lesssim X$. Here, the constant C does not depend on the important parameters on which X and Y depend. Any notation and terminology not otherwise explained, are as used in [**12**] for spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$.

2. Preliminaries and notation

2.1. Homogeneous spaces. Define

$$\varrho(z,w) = \begin{cases} \big||z|-|w|\big| + \left|1 - \dfrac{1}{|z||w|}\langle z,w\rangle\right|, & \text{if } z,w \in \mathbb{B}_n\backslash\{0\}, \\ |z|+|w|, & \text{otherwise.} \end{cases}$$

Then ϱ is a quasimetric on $\mathbb{B}_n$, i.e.,

(1) $\varrho(z,w) = 0$ if and only if $z = w$;
(2) $\varrho(z,w) = \varrho(w,z)$;
(3) there exists a positive constant $K \geq 1$ such that

$$\varrho(z,w) \leq K[\varrho(z,u) + \varrho(u,w)], \quad \forall\, z,w,u \in \mathbb{B}_n, \tag{2.1}$$

(the quasi-triangular inequality with $K = 2$ in the present case).

For any $z \in \mathbb{B}_n$ and $r > 0$, the set $B^\varrho(z,r) = \{w \in \mathbb{B}_n : \varrho(z,w) < r\}$ is called a ϱ-ball of center z and radius r. Moreover, $(\mathbb{B}_n, \varrho, dv_\alpha)$ is a homogeneous space for $\alpha > -1$, that is,

- for each $z \in \mathbb{B}_n$, the balls $B^\varrho(z,r)$ form a basis of open neighborhoods of z and, also, $v_\alpha(B^\varrho(z,r)) > 0$ whenever $r > 0$;
- (doubling property) there exists a constant $A > 0$ such that for each $z \in \mathbb{B}_n$ and $r > 0$, one has

$$v_\alpha(B^\varrho(z,2r)) \leq A v_\alpha(B^\varrho(z,r)). \tag{2.2}$$

We refer to [**1, 11**] for the details.

The following are some basic properties of the quasimetric ϱ which will be used later.

Lemma 2.1 (cf. [**11**, Lemma 2.10])**.** *Let $\alpha > -1$. For $z \in \mathbb{B}_n \setminus \{0\}$ and $0 < r < 3$,*

$$v_\alpha\big(B^\varrho(z,r)\big) \approx r^{n+1}[\max(r, 1-|z|)]^\alpha,$$

where "$\approx$" depends only on α and n.

Lemma 2.2 (cf. [**11**, Lemma 2.12])**.** *For $z \in \mathbb{B}_n$ and $0 < r_0 < 1$, if $z_0 = (r_0, 0, \ldots, 0)$ one has*

- $|1 - r_0 z_1| \geq \frac{1}{3}\varrho(z,z_0)$;
- $|z_1 - r_0| \leq \varrho(z,z_0)$;
- $\sum_{j=2}^n |z_j|^2 \leq 2\varrho(z,z_0)$;
- $|1 - \langle z,z_0\rangle| \leq 1 - r_0^2 + \varrho(z,z_0)$.

Lemma 2.3. *Let $\alpha > -1$ and $0 < p < \infty$. For every $\gamma > 0$ there exists a constant $C > 0$ such that for any $f \in \mathcal{H}(\mathbb{B}_n)$,*

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{B^\varrho(z,\gamma(1-|z|))} |f(w)|^p \, dv_\alpha(w), \quad \forall\, z \in \mathbb{B}_n.$$

This mean-value inequality with the quasi-metric ϱ should be well-known, although we have no concrete reference. The proof can be done as in [**12**, Lemma 2.24].

2.2. Local coordinates. For $z \in \mathbb{B}_n$ and $\xi \in \mathbb{C}^n$ a unit vector, we denote by $\tau(z, \xi)$ the distance from z to the boundary $\mathbb{S}_n$ along the complex line determined by ξ. For each $z_0 \in \mathbb{B}_n$ there exists a special set of real coordinate basis $\{v_1(z_0), \tau_1(z_0), \ldots, v_n(z_0), \tau_n(z_0)\}$ in $\mathbb{R}^{2n} = \mathbb{C}^n$ defined as follows, which we call *τ-extremal*. The first vector

$$v_1(z_0) = \begin{cases} \dfrac{z_0}{|z_0|}, & \text{if } z_0 \neq 0, \\ \mathbf{1}, & \text{if } z_0 = 0, \end{cases}$$

where $\mathbf{1} = (1, 0, \ldots, 0)$. $v_1(z_0)$ is clearly the direction transversal to the boundary $\mathbb{S}_n$, in the sense that the shortest distance from z_0 to $\mathbb{S}_n$ is attained in the complex line determined by $v_1(z_0)$. The vector $v_2(z_0)$ is chosen among the vectors orthogonal to $v_1(z_0)$ in such a way that $\tau(z_0, v_2(z_0))$ is maximal. The vector $v_3(z_0)$ is chosen among the vectors orthogonal to both $v_1(z_0)$ and $v_2(z_0)$ such that $\tau(z_0, v_3(z_0))$ is maximal. We repeat this process until we obtain an orthonormal basis $\{v_1(z_0), \ldots, v_n(z_0)\}$ in $\mathbb{C}^n$. Put

$$\tau_j(z_0) = \mathrm{i} v_j(z_0), \quad j = 1, \ldots, n.$$

Then $\{v_1(z_0), \tau_1(z_0), \ldots, v_n(z_0), \tau_n(z_0)\}$ is an orthonormal basis in $\mathbb{R}^{2n}$.

For $w \in \mathbb{B}_n$, if

$$w - z_0 = \alpha_1 v_1(z_0) + \beta_1 \tau_1(z_0) + \cdots + \alpha_n v_n(z_0) + \beta_n \tau_n(z_0),$$

we denote by $(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$ the real coordinates of w with respect to this basis. Precisely, we define a mapping $\Theta \colon \mathbb{B}_n \times \mathbb{B}_n \to \mathbb{R}^{2n}$ such that if

$$w - z = \alpha_1 v_1(z) + \beta_1 \tau_1(z) + \cdots + \alpha_n v_n(z) + \beta_n \tau_n(z)$$

then $\Theta(z, w) = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$. One can easily verify that this coordinate mapping Θ is a C^∞ diffeomorphism.

For a multi-index $J = (j_1, j_2, j_3, \ldots, j_{2n})$ of non negative integers, let

$$d(J) = j_1 + j_2 + \frac{j_3}{2} + \cdots + \frac{j_{2n}}{2}$$

and $|J| = j_1 + \cdots + j_{2n}$. For any $f \in C^\infty(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$, we define a differential operator

$$D_z^J f(w) = \frac{\partial^{j_1+\cdots+j_{2n}}}{\partial\alpha_1^{j_1}\partial\beta_1^{j_2}\cdots\partial\alpha_n^{j_{2n-1}}\partial\beta_n^{j_{2n}}} f\big(z + \alpha_1 v_1(z) + \beta_1\tau_1(z) + \cdots \\ \cdots + \alpha_n v_n(z) + \beta_n\tau_n(z)\big),$$

whenever $\Theta(z,w) = (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$. Let Ω be a domain in $\mathbb{B}_n$, we say $f \in \mathcal{C}^N(\Omega)$ if for each $z \in \Omega$ and J with $|J| \le N$, $D_z^J f(w)$ exists and is continuous in a neighborhood of z.

2.3. Real-variable Bergman spaces. Let $0 < p \le 1$ and $\alpha > -1$. Set

$$N_{p,\alpha} = \max\left\{\left[2(n+1)\left(\frac{1}{p}-1\right)\right], \left[2(n+1+\alpha)\left(\frac{1}{p}-1\right)\right]\right\} + 1,$$

where $[x]$ denotes the greatest integer less than x. Let $z_0 \in \mathbb{B}_n$ and $r_0 > 0$. For any $\phi \in \mathcal{C}^\infty(B^\varrho(z_0, r_0))$, we define the quantity for a nonnegative integer N,

$$\|\phi\|_{\mathcal{S}_N(B^\varrho(z_0,r_0))} := \sum_{|J|=N} r_0^{d(J)} \big\|D_{z_0}^J \phi\big\|_{L^\infty(B^\varrho(z_0,r_0))}.$$

Definition 2.1. Let $0 < p \le 1$ and $\alpha > -1$. Let $N \ge N_{p,\alpha}$ be an integer. A measurable function a on $\mathbb{B}_n$ is a $(p,\infty,N)_\alpha$-atom if there exist $z_0 \in \mathbb{B}_n$ and $r_0 > 0$ such that

(1) a is supported in $B^\varrho(z_0, r_0)$;
(2) $|a(z)| \le v_\alpha(B^\varrho(z_0,r_0))^{-\frac{1}{p}}$ for all $z \in \mathbb{B}_n$;
(3) $\int_{\mathbb{B}_n} a(z)\, dv_\alpha(z) = 0$;
(4) for all $\phi \in \mathcal{C}^\infty(B^\varrho(z_0, r_0))$,

$$\left|\int_{\mathbb{B}_n} a(z)\phi(z)\, dv_\alpha(z)\right| \le \|\phi\|_{\mathcal{S}_N(B^\varrho(z_0,r_0))} v_\alpha(B^\varrho(z_0,r_0))^{1-\frac{1}{p}}.$$

Any bounded function a with $\|a\|_{L^\infty} \le 1$ is also considered to be a $(p,\infty,N)_\alpha$-atom.

We regard a $(p,\infty,N_{p,\alpha})_\alpha$-atom as a $(p,\infty)_\alpha$-atom.

Remark 2.1. (i) As in the case of Hardy spaces (cf. [**7**, Section 4]), it seems necessary to consider all bounded functions a with $\|a\|_{L^\infty} \le 1$ as atoms in the complex ball.

(ii) We remark that condition (4) replaces the classical higher moment condition and is similar to the one found in [**8, 9**].

(iii) Let $1 \leq q < \infty$ and $p < q$. Replacing condition (2) by $\|a\|_{q,\alpha} \leq v_\alpha(B^\varrho(z_0, r_0))^{\frac{1}{q}-\frac{1}{p}}$, we get the concept of $(p,q)_\alpha$-atoms. A $(p,\infty)_\alpha$-atom is necessarily a $(p,q)_\alpha$-atom. Hence, atomic decomposition in terms of $(p,\infty)_\alpha$-atoms implies the one involving $(p,q)_\alpha$-atoms.

For $0 < p \leq 1$ and $\alpha > -1$, the real-variable (atomic) Bergman space $\mathrm{A}^p_\alpha(\mathbb{B}_n)$ is defined to be the space of distributions f on $\mathbb{B}_n$ which can be written as $f = \sum_j \lambda_j a_j$, where $\sum_j |\lambda_j|^p < \infty$, the a_j's are $(p,\infty)_\alpha$-atoms, and the series is assumed to converge in the sense of distributions. As usual, we put

$$\|f\|_{\mathrm{A}^p_\alpha} = \inf\left\{ \left(\sum_j |\lambda_j|^p \right)^{\frac{1}{p}} : f = \sum_j \lambda_j a_j \right\},$$

where the infimum is taken over all decompositions of f described above.

We note that $\mathrm{A}^1_\alpha(\mathbb{B}_n)$ is a Banach space and for any $0 < p < 1$, $\mathrm{A}^p_\alpha(\mathbb{B}_n)$ is a complete metric space under the metric $d(f,g) = \|f-g\|^p_{\mathrm{A}^p_\alpha}$.

Let P_α be the orthogonal projection from $L^2_\alpha(\mathbb{B}_n)$ onto $\mathcal{A}^2_\alpha$, given by

$$P_\alpha f(z) = \int_{\mathbb{B}_n} K_\alpha(z,w) f(w)\, dv_\alpha(w), \quad \forall\, f \in L^1(\mathbb{B}_n, dv_\alpha),$$

where the Bergman kernel function $K_\alpha(z,w)$ is expressed as

$$K_\alpha(z,w) = \frac{1}{\big(1 - \langle z, w\rangle\big)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}_n.$$

Proposition 2.1. *Let $0 < p \leq 1$ and $\alpha > -1$. Then $P_\alpha \mathrm{A}^p_\alpha(\mathbb{B}_n) \subset \mathcal{A}^p_\alpha(\mathbb{B}_n)$. More precisely, there exists a constant C such that $\|P_\alpha(f)\|_{p,\alpha} \leq C\|f\|_{\mathrm{A}^p_\alpha}$ for all $f \in \mathrm{A}^p_\alpha(\mathbb{B}_n)$.*

The proof of this proposition is based on the following lemma.

Lemma 2.4. *Let N be a nonnegative integer. Let $z_0 \in \mathbb{B}_n$ and $J = (j_1, j_2, j_3, \ldots, j_{2n})$ be a multi-index such that $|J| = N$. If $\varrho(w, z_0) < \frac{1}{4}\varrho(z, z_0)$, then there exists a constant $C_{N,n,\alpha}$ depending only on N, n, α such that*

$$\big|D^J_{z_0}\big(K_\alpha(z,\cdot)\big)(w)\big| \leq \frac{C_{N,n,\alpha}}{\varrho(z,w)^{d(J)} v_\alpha(B^\varrho(w, \varrho(z,w)))}.$$

This lemma is a special case of a classical result of C. Fefferman [**5**] about the pointwise estimates of the Bergman kernel and its derivatives in smooth strictly pseudoconvex domains in $\mathbb{C}^n$. Thanks to the explicit formula of the Bergman kernel in the complex ball, it can be proved by a straightforward computation. We omit the details.

We now can proceed with the proof of Proposition 2.1 as done in the one of [**8**, Theorem 2.2] but with the help of Lemma 2.4 in place of the estimates for the Szegö kernel and its derivatives. We do not repeat the details here. The goal of this paper is to prove the converse inclusion $\mathcal{A}^p_\alpha(\mathbb{B}_n) \subset P_\alpha \mathrm{A}^p_\alpha(\mathbb{B}_n)$.

3. Maximal functions

In order to prove the atomic decomposition of Bergman spaces $\mathcal{A}^p_\alpha$ for all $0 < p \leq 1$, we need to introduce some maximal functions. These maximal functions are variants of the ones used for Hardy spaces (see e.g. [**8, 9, 10**]).

Let $\delta > 0$ and $z \in \mathbb{B}_n$. The 'approach region' $A_\delta(z)$ is defined by

$$A_\delta(z) = \bigl\{w \in \mathbb{B}_n : \varrho(z,w) < \delta(1-|w|)\bigr\}.$$

For any $f \in \mathcal{H}(\mathbb{B}_n)$, we define respectively the non-tangential maximal function

$$f^\star_\delta(z) := \sup_{w\in A_\delta(z)} |f(w)|$$

and the tangential function

$$f^{\star\star}_M(z) := \sup_{w\in\mathbb{B}_n} \left(\frac{1-|w|}{1-|w|+\varrho(z,w)}\right)^M |f(w)|,$$

where M is a positive constant.

We need also to introduce the so-called grand maximal functions. Given $z \in \mathbb{B}_n$, we denote by $\mathcal{G}^L_\delta(z)$ the space of smooth bump functions at z for δ and L, that consists of all functions $g \in C^\infty(\mathbb{B}_n)$ for which there exist $z_0 \in \mathbb{B}_n$ and $r_0 > 0$ such that

$$\operatorname{supp} g \subset B^\varrho(z_0, r_0), \quad \varrho(z,z_0) < \delta r_0, \quad \text{and} \quad \|g\|_{L,z_0,r_0} \leq 1,$$

where

$$\|g\|_{L,z_0,r_0} = v_\alpha(B^\varrho(z_0,r_0)) \sup_{|J|\leq L} r_0^{d(J)} \|D^J_{z_0} g\|_{L^\infty(B^\varrho(z_0,r_0))}.$$

Recall that $d(J) = j_1 + j_2 + \frac{1}{2}(j_3 + \cdots + j_{2n})$ with $J = (j_1, j_2, \ldots, j_{2n-1}, j_{2n})$. The grand maximal function on $\mathbb{B}_n$ is then defined as

$$\mathcal{K}_{\delta,L}(f)(z) = \sup_{g\in\mathcal{G}^L_\delta(z)} \left| \int_{\mathbb{B}_n} f(w) g(w)\, dv_\alpha(w) \right|.$$

The following lemma is a straightforward consequence of Lemma 2.3 and of the boundedness of the Hardy–Littlewood maximal operator on L^2 (see e.g. [**6**]).

Lemma 3.1. *Let* $0 < p < \infty$ *and* $\alpha > -1$. *Then* $\|f_\delta^\star\|_{p,\alpha} \lesssim \|f\|_{p,\alpha}$ *for all* $f \in \mathcal{A}_\alpha^p$.

By adapting the classical argument of [**6**, Lemma VI.1], we immediately obtain the following lemma.

Lemma 3.2. *Let* $0 < p < \infty$ *and* $\alpha > -1$. *If* M *is a constant such that* $Mp > n+1+\alpha$, *then* $\|f_M^{\star\star}\|_{p,\alpha} \lesssim \|f\|_{p,\alpha}$ *for all* $f \in \mathcal{A}_\alpha^p$.

In the sequel, we mainly prove the following lemma.

Lemma 3.3. *Let* $0 < p < \infty$ *and* $\delta > 0$. *Let* $L > M$ *be an integer, where* M *is a constant such that* $Mp > n+1+\alpha$. *Then*

$$\mathcal{K}_{\delta,L}(f)(z) \lesssim f_{3+2\delta}^\star(z) + f_M^{\star\star}(z), \quad \forall\, f \in \mathcal{H}(\mathbb{B}_n),$$

for all $z \in \mathbb{B}_n$. *Consequently,* $\|\mathcal{K}_{\delta,L}(f)\|_{L^p(\mathbb{B}_n, dv_\alpha)} \lesssim \|f\|_{p,\alpha}$ *for all* $f \in \mathcal{A}_\alpha^p$.

Proof: This lemma can be proved by using the argument of [**9**, Theorem 3.2], but geometrical properties of the complex ball instead of the sphere are involved. For the sake of convenience, we present the details. We want to estimate $\left|\int_{\mathbb{B}_n} f(w)g(w)\,dv_\alpha(w)\right|$ for $g \in \mathcal{G}_\delta^L(z)$. Given such a function g, there exist $z_0 \in \mathbb{B}_n$ and $r_0 > 0$ such that $\operatorname{supp} g \subset B^\varrho(z_0, r_0)$, $\varrho(z, z_0) < \delta r_0$, and $\|g\|_{L,z_0,r_0} \le 1$. Note that

$$\begin{aligned}
&\int_{\mathbb{B}_n} f(w)g(w)\,dv_\alpha(w)\\
&\quad = 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha \left(\int_{\mathbb{S}_n} f(r\xi)g(r\xi)\,d\sigma(\xi)\right) dr\\
&\quad = 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha \left(\int_{\mathbb{S}_n} [f(r\xi) - f(r\xi - r_0 r\xi)]g(r\xi)\,d\sigma(\xi)\right) dr\\
&\qquad + 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha \left(\int_{\mathbb{S}_n} f(r\xi - r_0 r\xi)g(r\xi)\,d\sigma(\xi)\right) dr\\
&\quad \triangleq I_1 + I_2.
\end{aligned}$$

We need to estimate I_1 and I_2.

We first estimate I_2. Since $r\xi \in B^\varrho(z_0, r_0)$, it follows that

$$\varrho(r\xi, z) \le 2[\varrho(r\xi, z_0) + \varrho(z_0, z)] \le (2+2\delta)r_0,$$

and

$$\varrho(r\xi - r_0 r\xi, z) \le \big||r\xi - r_0 r\xi| - |r\xi|\big| + \varrho(r\xi, z) \le (3+2\delta)r_0.$$

Also, $1-|r\xi - r_0 r\xi| > r_0$. Hence, $r\xi - r_0 r\xi \in A_{(3+2\delta)}(z)$. Therefore,

$$\begin{aligned}|I_2| &= \left|2n\int_0^1 r^{2n-1}(1-r^2)^\alpha \left(\int_{\mathbb{S}_n} f(r\xi - r_0 r\xi)g(r\xi)\,d\sigma(\xi)\right)\,dr\right| \\ &\le \frac{f^\star_{3+2\delta}(z)}{v_\alpha(B^\varrho(z_0,r_0))}2n\int_0^1 r^{2n-1}(1-r^2)^\alpha \left(\int_{\{\xi\in\mathbb{S}_n:\varrho(r\xi,z_0)<r_0\}} 1\,d\sigma(\xi)\right)\,dr \\ &\le f^\star_{3+2\delta}(z).\end{aligned}$$

In what follows, we estimate the term I_1. First of all, for a differential operator Y_ℓ of order ℓ defined as

$$Y_\ell = \sum_{\substack{k_1+\cdots+k_n \\ +m_1+\cdots+m_n=\ell}} C_{k_1,\dots,k_n,m_1,\dots,m_n}(w)\frac{\partial^{k_1+\cdots+k_n+m_1+\cdots+m_n}}{\partial x_1^{k_1}\partial y_1^{m_1}\cdots\partial x_n^{k_n}\partial y_n^{m_n}}$$

with smooth coefficients $C_{k_1,\dots,k_n,m_1,\dots,m_n}$, where $w=(x_1+\mathrm{i}y_1,\dots,x_n+\mathrm{i}y_n)$, one has

$$\|Y_\ell g\|_{L^\infty(B^\varrho(z_0,r_0))} \lesssim \sum_{|J|=\ell}\frac{1}{r_0^{d(J)}v_\alpha(B^\varrho(z_0,r_0))},\quad \forall\,\ell\le L, \tag{3.1}$$

for all $g\in\mathcal{G}^L_\delta(z)$. The proof can be done by the chain rule and change of coordinates.

For $0<r<1$ and $\xi\in\mathbb{S}_n$, we put

$$G_r(w)=\begin{cases} g(rw)g_1(1-|w|), & \text{if } 1-2r_0\le|w|\le 1,\\ 0, & \text{if } |w|<1-2r_0,\end{cases}$$

where $g_1\in C_0^\infty([0,2r_0])$, $g_1=1$ if $0\le t\le r_0$ and $|g_1^{(j)}|\le c_j r_0^{-j}$ (c_j depending only on j). Then, by (3.1) we have

$$|Y_\ell G_r(w)| \lesssim \sum_{k=0}^{\ell}\frac{1}{r_0^k}|(Y_{\ell-k}g)(rw)| \le \frac{c_\ell}{r_0^\ell v_\alpha(B^\varrho(z_0,r_0))}. \tag{3.2}$$

Now, denote by v_ξ the unit outward vector at $\xi\in\mathbb{S}_n$, we have

$$\begin{aligned}I_1 &= 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha\left(\int_{\mathbb{S}_n}[f(r\xi)-f(r\xi-r_0r\xi)]g(r\xi)\,d\sigma(\xi)\right)\,dr\\ &= 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha\left(\int_{\mathbb{S}_n}\left[\int_0^{r_0}-\frac{df}{dr_1}(r\xi-r_1r\xi)\,dr_1\right]g(r\xi)\,d\sigma(\xi)\right)\,dr\\ &= 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha\int_0^{r_0} r\left(\int_{\mathbb{S}_n}\frac{df}{dv_\xi}(r\xi-r_1r\xi)G_r(\xi)\,d\sigma(\xi)\right)\,dr_1\,dr.\end{aligned}$$

By applying the Green's formula, we have

$$
\begin{aligned}
I_1 &= 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha \int_0^{r_0} r\left(\int_{\mathbb{S}_n} f(r\xi - r_1 r\xi)\frac{dG_r(\xi)}{dv_\xi}\,d\sigma(\xi)\right) dr_1\,dr \\
&\quad + 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha \int_0^{r_0} r\left(\int_{\mathbb{B}_n} \triangle f(rw - r_1 rw)G_r(w)\,dv(w)\right) dr_1\,dr \\
&\quad - 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha \int_0^{r_0} r\left(\int_{\mathbb{B}_n} f(rw - r_1 rw)\triangle G_r(w)\,dv(w)\right) dr_1\,dr \\
&= I_{11} + 0 - I_{12},
\end{aligned}
$$

since $f(rw - r_1 rw)$ is holomorphic in $\mathbb{B}_n$. Let $Y_2 G_r(w) \triangleq \triangle G_r(w)$. We rewrite $I_{12} = I_{121} + I_{122}$, where

$$
I_{121} = 2n\int_0^1\int_0^{r_0} r^{2n-1}(1-r^2)^\alpha r \int_{\mathbb{B}_n\setminus\mathbb{D}_{1-r_0}} f(rw - r_1 rw)Y_2 G_r(w)\,dv(w)\,dr_1\,dr
$$

and

$$
I_{122} = 2n\int_0^1\int_0^{r_0} r^{2n-1}(1-r^2)^\alpha r \int_{\mathbb{D}_{1-r_0}} f(rw - r_1 rw)Y_2 G_r(w)\,dv(w)\,dr_1\,dr.
$$

Hereafter, $\mathbb{D}_r = \{z \in \mathbb{C}^n : |z| < r\}$ for $0 < r < 1$. For the term I_{122}, since $w \in \mathbb{D}_{1-r_0}$ we have

$$
\varrho(rw - r_1 rw, z) < (3+2\delta)r_0 < (3+2\delta)(1 - |rw - r_1 rw|).
$$

Then

$$
\begin{aligned}
|I_{122}| &= 2n\left|\int_0^1 r^{2n-1}(1-r^2)^\alpha \int_0^{r_0} r\int_{1-2r_0}^{1-r_0} 2ns^{2n-1}\right. \\
&\qquad\qquad \left.\times\left(\int_{\mathbb{S}_n} f(rs\xi - r_1 rs\xi)Y_2 G_r(s\xi)\,d\sigma(\xi)\right) ds\,dr_1\,dr\right| \\
&\lesssim \frac{f^\star_{3+2\delta}(z)r_0}{r_0^2 v_\alpha(B^\varrho(z_0,r_0))} 2n\int_0^1 r^{2n-1}(1-r^2)^\alpha \\
&\qquad\qquad \times\left(\int_{1-2r_0}^{1-r_0} 2ns^{2n-1}\int_{\{\xi\in\mathbb{S}_n : \varrho(rs\xi, z_0) < r_0\}} 1\,d\sigma(\xi)\,ds\right) dr \\
&\lesssim f^\star_{3+2\delta}(z).
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
\left|\int_{\mathbb{B}_n} f(w)g(w)\,dv_\alpha(w)\right| &\lesssim f^\star_{3+2\delta}(z)\\
&+\left|\int_0^1 r^{2n-1}(1-r^2)^\alpha\int_0^{r_0} r\left(\int_{\mathbb{S}_n} f(r\xi-r_1r\xi)\frac{dG_r(\xi)}{dv_\xi}\,d\sigma(\xi)\right)\,dr_1\,dr\right|\\
&+\left|\int_0^1 r^{2n-1}(1-r^2)^\alpha\int_0^{r_0}\left(\int_{\mathbb{B}_n\setminus\mathbb{D}_{1-r_0}} f(rw-r_1rw)Y_2G_r(w)\,dv(w)\right)\,dr_1\,dr\right|\\
&= f^\star_{3+2\delta}(z)\\
&+\left|\int_0^1 r^{2n-1}(1-r^2)^\alpha\int_0^{r_0}\int_{\mathbb{S}_n} f\big((1-r_1)r\xi\big)Y_1G_r(\xi)\,d\sigma(\xi)\,dr_1\,dr\right|\\
&+\left|\int_0^1 r^{2n-1}(1-r^2)^\alpha\int_0^{r_0}\int_{1-r_0}^1\int_{\mathbb{S}_n} f\big((1-r_1)rs_1\xi\big)Y_2G_r(s_1\xi)\,d\sigma(\xi)\,ds_1\,dr_1\,dr\right|,
\end{aligned}$$

where $Y_1G_r(\xi)\triangleq r\frac{dG_r(\xi)}{dv_\xi}$ and we write $Y_2G_r(s_1\xi)$ by abuse of notation for $2ns_1^{2n-1}Y_2G_r(s_1\xi)$ up to a change of the constant in Y_2.

By using (3.2) and repeating the method used in the estimations of I_1 and I_2, we obtain the following estimate

$$\begin{aligned}
\left|\int_{\mathbb{B}_n} f(w)g(w)\,dv_\alpha(w)\right| &\lesssim f^\star_{3+2\delta}(z)\\
&+\sum_{0\le k\le \ell}\int_0^1 r^{2n-1}(1-r^2)^\alpha\int_0^{r_0}\int_{r_1}^{r_0}\cdots\int_{r_{\ell-1}}^{r_0}\int_{1-r_0}^1\cdots\int_{1-r_0}^1\\
&\times\left|\int_{\mathbb{S}_n} f\big((1-r_\ell)rs_1\cdots s_k\xi\big)Y_{\ell+k}G_r(s_1\cdots s_k\xi)\,d\sigma(\xi)\right|\\
&\times ds_k\cdots ds_1\,dr_\ell\cdots dr_1\,dr.
\end{aligned}$$

Notice that

$$\begin{aligned}
\left|f\big((1-r_\ell)rs_1\cdots s_k\xi\big)\right| &\le f^{\star\star}_M(z)\left(1+\frac{\varrho\big((1-r_\ell)rs_1\cdots s_k\xi,z\big)}{1-(1-r_\ell)rs_1\cdots s_k}\right)^M\\
&\le f^{\star\star}_M(z)\left(1+\frac{(3+2\delta)r_0}{r_\ell}\right)^M\\
&\le f^{\star\star}_M(z)(4+2\delta)^M\left(\frac{r_0}{r_\ell}\right)^M
\end{aligned}$$

and

$$\begin{aligned}\varrho(r\xi, z_0) &\leq |r - rs_1 \cdots s_k| + \varrho(rs_1 \cdots s_k\xi, z_0) \\ &\leq |r - rs_1| + \sum_{i=1}^{k-1} |rs_1 \cdots s_i - rs_1 \cdots s_i s_{i+1}| + \varrho(rs_1 \cdots s_k\xi, z_0) \\ &\leq kr_0 + r_0 \leq (k+1)r_0.\end{aligned}$$

Thus, fix $\ell = L > M$, we have

$$\begin{aligned}&\left|\int_{\mathbb{B}_n} f(w)g(w)\, dv_\alpha(w)\right| \\ &\lesssim f^\star_{3+2\delta}(z) + f^{\star\star}_M(z) \sum_{0\leq k\leq \ell} \int_0^1 r^{2n-1}(1-r^2)^\alpha \int_0^{r_0}\int_{r_1}^{r_0}\cdots\int_{r_{\ell-1}}^{r_0}\int_{1-r_0}^1\cdots\int_{1-r_0}^1 \\ &\qquad \times \int_{\{\xi\in\mathbb{S}_n:\varrho(rs_1\cdots s_k\xi, z_0)<r_0\}} \left(\frac{r_0}{r_\ell}\right)^M \frac{1}{r_0^{\ell+k} v_\alpha(B^\varrho(z_0, r_0))}\, d\sigma(\xi) \\ &\qquad\qquad \times ds_k \cdots ds_1\, dr_\ell \cdots dr_1\, dr \\ &\lesssim f^\star_{3+2\delta}(z) + \sum_{0\leq k\leq\ell} \frac{f^{\star\star}_M(z) r_0^{k+M}}{r_0^{\ell+k} v_\alpha(B^\varrho(z_0, r_0))} \int_0^1 r^{2n-1}(1-r^2)^\alpha \int_0^{r_0}\int_{r_1}^{r_0}\cdots\int_{r_{\ell-1}}^{r_0} \\ &\qquad \times \int_{\{\xi\in\mathbb{S}_n:\varrho(r\xi, z_0)<(k+1)r_0\}} \left(\frac{1}{r_\ell}\right)^M d\sigma(\xi)\, dr_\ell\cdots dr_1\, dr \\ &\lesssim f^\star_{3+2\delta}(z) + f^{\star\star}_M(z).\end{aligned}$$

This completes the proof. □

4. The main result and auxiliary lemmas

The following is the main result of this paper.

Theorem 4.1. *Let $0 < p \leq 1$ and $\alpha > -1$. Let $N \geq N_{p,\alpha}$ be a integer. If $f \in \mathcal{A}^p_\alpha$, then there exist a scalar sequence $\{\lambda_j\}$ in $\mathbb{C}$ with $\sum_j |\lambda_j|^p < \infty$, and a sequence of $(p, \infty, N)_\alpha$-atoms $\{a_j\}$ such that $f = \sum_j \lambda_j a_j$ in the sense of distributions and $\left(\sum_j |\lambda_j|^p\right)^{\frac{1}{p}} \lesssim \|f\|_{p,\alpha}$.*

Moreover, $f = \sum_j \lambda_j P_\alpha(a_j)$, *i.e., the series* $\sum_j \lambda_j P_\alpha(a_j)$ *converges* f *in* $\mathcal{A}^p_\alpha$. *Consequently, for any* $f \in \mathcal{A}^p_\alpha$ *one has*

$$\|f\|_{p,\alpha} \approx \inf\biggl\{ \Bigl(\sum_j |\lambda_j|^p\Bigr)^{\frac{1}{p}} : f = \sum_j \lambda_j P_\alpha(a_j) \biggr\},$$

where the infimum is taken over all decompositions of f *described above.*

To prove Theorem 4.1, we need some auxiliary lemmas.

Lemma 4.1. *Let* $\mathcal{O} \subsetneq \mathbb{B}_n$ *be an open set. Then there are a sequence of balls* $\{B^\varrho(z_i, r_i)\}$ *in* $\mathbb{B}_n$, *positive constants* $\mu > 1 > \nu > \lambda > 0$ *and* N_0 *depending only on* n *such that*

(1) *for any* i,

$$r_i = \frac{1}{2K}\varrho(z_i, \mathcal{O}^c),$$

where K *is the constant occurring in the quasi-triangular inequality* (2.1) *satisfied by the quasi-metric* ϱ;

(2) $\mathcal{O} = \bigcup_i B^\varrho(z_i, \nu r_i)$;

(3) *for each* i, $B^\varrho(z_i, \mu r_i) \cap \mathcal{O}^c \neq \emptyset$;

(4) *the balls* $B^\varrho(z_i, \lambda r_i)$ *are pairwise disjoint;*

(5) *no point in* $\mathcal{O}$ *lies in more than* N_0 *of the ball* $B^\varrho(z_i, r_i)$.

Proof: See [**11**, Lemma 2.4] for the details. □

Lemma 4.2. *Let* $\mathcal{O} \subsetneq \mathbb{B}_n$ *be an open subset. Then there exist a collection of balls* $B^\varrho(z_i, r_i)$, *a sequence of functions* $\varphi_i \in C^\infty(\mathbb{B}_n)$ $(i = 1, 2, \dots)$, *and a constant* $\mu > 1$ *depending only on* n, *such that*

(1) $0 \le \varphi_i \le 1$;

(2) $\operatorname{supp} \varphi_i \subset B^\varrho(z_i, r_i)$;

(3) $\sum_{i=1}^\infty \varphi_i = \chi_{\mathcal{O}}$;

(4) *for any nonnegative integer* L *there is a constant* $c_L > 0$ *depending only on* L *and* n *such that for each* i *and any* $w_i \in B^\varrho(z_i, \mu r_i) \cap \mathcal{O}^c$,

$$\frac{c_L}{\|\varphi_i\|_{1,\alpha}}\varphi_i \in \mathcal{G}^L_\mu(w_i).$$

Proof: See Section 3 for the definition of $\mathcal{G}^L_\mu(w)$. Then, the proof proceeds as the one of [**9**, Lemma 4.3] with the help of Lemma 4.1. □

Let $f \in \mathcal{A}^p_\alpha$ and $\mu > 1$ be the constant appearing in Lemma 4.1. Given an integer $N \ge N_{p,\alpha}$, let $L \ge \max\bigl\{N, \bigl[\frac{1}{p}(n+1+\alpha)\bigr] + 1\bigr\}$ be an integer. By Lemmas 3.1 and 3.3, we have

$$\mathcal{K}_{\mu,L}(f) + f^\star_\delta \in L^p(\mathbb{B}_n, dv_\alpha).$$

Let k_0 be the least integer such that

$$\left\|\mathcal{K}_{\mu,L}(f)+f_\delta^\star\right\|_{L^p_\alpha(\mathbb{B}_n)}\le 2^{k_0}.$$

For any nonnegative integer k, we define

$$\mathcal{O}_k=\left\{z\in\mathbb{B}_n:\mathcal{K}_{\mu,L}(f)(z)+f_\delta^\star(z)>2^{k_0+k}\right\}.$$

Then $\mathcal{O}_k\subsetneq\mathbb{B}_n$ for any $k=0,1,\dots$. For each k we fix the Whitney type covering $\{B^\varrho(z_i^k,r_i^k)\}_{i=1}^\infty$ and the partition of unity $\{\varphi_i^k\}$ with respect to $\mathcal{O}_k$, as constructed in Lemma 4.2.

For each i and k, we denote by $L^2_{\alpha,\varphi_i^k}(\mathbb{B}_n)$ the L^2-space with respect to the probability measure $dv_{\alpha,\varphi_i^k}:=\frac{\varphi_i^k}{\|\varphi_i^k\|_{1,\alpha}}\,dv_\alpha$. The norm on this space will be denoted by $\|\cdot\|_{\alpha,\varphi_i^k}$. Then we define a subspace $V^L_{\varphi_i^k}(z_i^k)$ of $L^2_{\alpha,\varphi_i^k}(\mathbb{B}_n)$ consisting of 'polynomials' of the form

$$P(z)=\sum_{|J|\le L}c_J\Theta(z_i^k,z)^J,$$

where

$$\Theta(z_i^k,z)^J=\alpha_1^{j_1}\beta_1^{j_2}\cdots\alpha_i^{j_{2i-1}}\beta_i^{j_{2i}}\cdots\alpha_n^{j_{2n-1}}\beta_n^{j_{2n}}$$

when $\Theta(z_i^k,z)=(\alpha_1,\beta_1,\dots,\alpha_n,\beta_n)$ and $J=(j_1,j_2,\dots,j_{2n-1},j_{2n})$. It is clear that $V^L_{\varphi_i^k}(z_i^k)$ is a finite-dimensional Hilbert space.

Let $\pi_J(z)$ ($|J|\le L$) be an orthonormal basis for $V^L_{\varphi_i^k}(z_i^k)$.

Lemma 4.3. *Let L be a nonnegative integer. Then there is a constant $c_L>0$ depending only on L and n such that*

$$\frac{c_L}{\|\varphi_i^k\|_{1,\alpha}}\pi_J\varphi_i^k\in\mathcal{G}^L_\mu(w_i^k),\tag{4.1}$$

for all $w_i^k\in B^\varrho(z_i^k,\mu r_i^k)\cap\mathcal{O}_k^c$.

Proof: The proof of this lemma follows the argument of Claim 1 in the proof of [**9**, Lemma 4.6], but some additional properties of the quasimetric ϱ in $\mathbb{B}_n$ are involved. For simplicity, by replacing $\Theta(z_i^k,z)^J$ with $\Theta(z)^J$, we let

$$\pi_J(z)=\sum_{|I|\le|J|\le L}a_{J,I}\Theta(z)^I.$$

Then we will prove that

$$|a_{J,I}|\le c_J(r_i^k)^{-d(I)},\quad |I|\le|J|\le L,\tag{4.2}$$

by mathematical induction, where the positive constant c_J depending only on J and n.

To this end, we introduce a linear order $\prec$ on the multi-indices set $\{J : |J| \leq L\}$ such that $|I| < |J|$ implies $I \prec J$. Note that the orthonormal basis π_J can be constructed by the Gram–Schmidt process beginning with $\pi_0 = 1$, and then

$$\pi_J(z) = \frac{\Theta(z)^J - \sum_{I\prec J} \pi_I(z) \int \Theta(w)^J \pi_I(w)\, dv_{\alpha,\varphi_i^k}(w)}{\left\| \Theta(z)^J - \sum_{I\prec J} \pi_I(z) \int \Theta(w)^J \pi_I(w)\, dv_{\alpha,\varphi_i^k}(w) \right\|_{\alpha,\varphi_i^k}}. \tag{4.3}$$

Using mathematical induction, we assume that if $O \preceq I \prec J$ then

$$|a_{I,O}| \leq c_I (r_i^k)^{-d(O)}.$$

Because, for any $O \preceq I$ we have $\left|\Theta(z)^O \varphi_i^k(z)\right| \leq (r_i^k)^{d(O)}$, it follows that $\pi_I(z)\varphi_i^k(z) \leq c_I$. Therefore, in the numerator of (4.3), the coefficient of $\Theta(z)^I$ $(I \preceq J)$ is dominated by $c_J(r_i^k)^{d(J)-d(I)}$.

In the following, we shall estimate the denominator of (4.3). Recalling the constant ν in Lemma 4.1, we claim that there exists a constant $C = C(\nu, n)$ such that

$$\frac{(1-|z|)^\alpha}{v_\alpha(B^\varrho(z_i^k, r_i^k))} \gtrsim \frac{C}{(r_i^k)^{n+1}}, \quad \forall\, z \in \left\{ w \in \mathbb{B}_n : \varrho(z_i^k, w) < \frac{1}{4}\nu r_i^k \right\}. \tag{4.4}$$

Indeed, since $1 - |z_i^k| = \varrho\big(\frac{z_i^k}{|z_i^k|}, z_i^k\big) > \nu r_i^k$, the proof of (4.4) can be divided into two cases: $0 \leq \alpha < \infty$ and $-1 < \alpha < 0$.

• *Case* $0 \leq \alpha < \infty$. Suppose $z \in \{w \in \mathbb{B}_n : \varrho(z_i^k, w) < \frac{1}{4}\nu r_i^k\}$. If $r_i^k \geq 1 - |z_i^k|$, we have

$$1 - |z| \geq \nu r_i^k - \varrho(z_i^k, z) \geq \frac{1}{2}\nu r_i^k,$$

consequently,

$$\frac{(1-|z|)^\alpha}{v_\alpha(B^\varrho(z_i^k, r_i^k))} \gtrsim \frac{C(r_i^k)^\alpha}{(r_i^k)^{n+1+\alpha}} \geq \frac{C}{(r_i^k)^{n+1}};$$

if $r_i^k < 1 - |z_i^k|$, we also have

$$1 - |z| \geq 1 - |z_i^k| - \varrho(z_i^k, z) \geq 1 - |z_i^k| - \frac{1}{4}\nu r_i^k \geq \left(1 - \frac{1}{4}\nu\right)(1 - |z_i^k|),$$

hence,

$$\frac{(1-|z|)^\alpha}{v_\alpha(B^\varrho(z_i^k, r_i^k))} \gtrsim \frac{C(1-|z_i^k|)^\alpha}{(r_i^k)^{n+1}(1-|z_i^k|)^\alpha} \geq \frac{C}{(r_i^k)^{n+1}}.$$

• *Case* $-1<\alpha<0$. Let $z\in\{w\in\mathbb{B}_n:\varrho(z_i^k,w)<\frac{1}{4}\nu r_i^k\}$. If $r_i^k\geq 1-|z_i^k|$, we have

$$1-|z|\leq 1-|z_i^k|+\varrho(z_i^k,z)\lesssim r_i^k,$$

therefore,

$$\frac{(1-|z|)^\alpha}{v_\alpha(B^\varrho(z_i^k,r_i^k))}\gtrsim\frac{(r_i^k)^\alpha}{(r_i^k)^{n+1+\alpha}}\geq\frac{C}{(r_i^k)^{n+1}};$$

if $r_i^k<1-|z_i^k|$, we also have

$$1-|z|\leq 1-|z_i^k|+\varrho(z_i^k,z)\lesssim 1-|z_i^k|,$$

hence,

$$\frac{(1-|z|)^\alpha}{v_\alpha(B^\varrho(z_i^k,r_i^k))}\gtrsim\frac{C(1-|z_i^k|)^\alpha}{(r_i^k)^{n+1}(1-|z_i^k|)^\alpha}\geq\frac{C}{(r_i^k)^{n+1}}.$$

In summary, (4.4) is proved.

We now come back to estimate the denominator of (4.3). Let

$$F_t=\left\{(\alpha,\beta)\triangleq(\alpha_1,\ldots,\beta_n):|\alpha_1|,|\beta_1|<t;\ \alpha_2^2+\cdots+\beta_n^2<t\right\}.$$

Then, by (4.4) we have

$$\begin{aligned}
&\left\|\Theta(z)^J-\sum_{I\prec J}\pi_I(z)\int\Theta(w)^J\pi_I(w)\,dv_{\alpha,\varphi_i^k}(w)\right\|^2_{\alpha,\varphi_i^k}\\
&\gtrsim\frac{1}{v_\alpha(B^\varrho(z_i^k,r_i^k))}\int_{B^\varrho(z_i^k,\nu r_i^k)}\left|\Theta(z)^J-\sum_{I\prec J}\pi_I(z)\int\Theta^J\pi_I\,dv_{\alpha,\varphi_i^k}\right|^2\varphi_i^k\,dv_\alpha(z)\\
&\gtrsim\frac{1}{(r_i^k)^{n+1}}\int_{\left\{z\in\mathbb{B}_n:\varrho(z_i^k,z)<\frac{1}{4}\nu r_i^k\right\}}\left|\Theta(z)^J-\sum_{I\prec J}\pi_I(z)\int\Theta^J\pi_I\,dv_{\alpha,\varphi_i^k}\right|^2dv(z)\\
&=\int_{F_{\frac{1}{4}\nu}}\Bigg|(r_i^k)^{d(J)}(\alpha,\beta)^J-\sum_{I\prec J}\pi_I\big(r_i^k\alpha_1,r_i^k\beta_1,(r_i^k)^{\frac{1}{2}}\alpha_2,\ldots,(r_i^k)^{\frac{1}{2}}\beta_n\big)\\
&\qquad\qquad\times\int\Theta^J\pi_I\,dv_{\alpha,\varphi_i^k}\Bigg|^2dv(\alpha,\beta),
\end{aligned}$$

in the last equality we have made the change of variables

$$(\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots,\alpha_n,\beta_n)\to\left(\frac{\alpha_1}{r_i^k},\frac{\beta_1}{r_i^k},\frac{\alpha_2}{(r_i^k)^{\frac{1}{2}}},\frac{\beta_2}{(r_i^k)^{\frac{1}{2}}},\ldots,\frac{\alpha_n}{(r_i^k)^{\frac{1}{2}}},\frac{\beta_n}{(r_i^k)^{\frac{1}{2}}}\right).$$

We continue to estimate the last integral, which is equal to

$$(r_i^k)^{2d(J)}\int_{F_{\frac14\nu}}\left|(\alpha,\beta)^J-\sum_{I\prec J}\frac{1}{(r_i^k)^{d(J)}}\pi_I\big(r_i^k\alpha_1,r_i^k\beta_1,(r_i^k)^{\frac12}\alpha_2,\dots,(r_i^k)^{\frac12}\beta_n\big)\right.$$

$$\left.\times\int\Theta^J\pi_I\,dv_{\alpha,\varphi_i^k}\right|^2 dv(\alpha,\beta)$$

$$\gtrsim (r_i^k)^{2d(J)}\int_{F_{\frac14\nu}}|(\alpha,\beta)^J-P_J(\alpha,\beta)|^2\,dv(\alpha,\beta)\geq c_J(r_i^k)^{2d(J)},$$

where $P_J(\alpha,\beta)$ is the projection of $(\alpha,\beta)^J$ into the Hilbert space of polynomials spanned by $\{(\alpha,\beta)^I : I\prec J\}$ with the norm $\|P\| = \big(\int_{F_{\frac14\nu}}|P(\alpha,\beta)|^2\,dv(\alpha,\beta)\big)^{\frac12}$. Combining this estimation on the denominator of (4.3) with the previous estimation for its numerator yields that the coefficient of $\Theta(z)^I$ $(I\preceq J)$ is dominated by $c_J(r_i^k)^{-d(I)}$, i.e.,

$$|a_{J,I}|\leq c_J(r_i^k)^{d(J)-d(I)}(r_i^k)^{-d(J)}=c_J(r_i^k)^{-d(I)},\quad I\preceq J.$$

Therefore, the claim (4.2) is proved.

Now we return to the proof of (4.1). Indeed, by the Leibniz rule and the fact $\frac{c_L}{\|\varphi_i^k\|_{1,\alpha}}\varphi_i^k\in\mathcal{G}_\mu^L(w_i^k)$, we have

$$\frac{c_L(r_i^k)^{-d(I)}}{\|\varphi_i^k\|_{1,\alpha}}\Theta(z_i^k,z)^I\varphi_i^k\in\mathcal{G}_\mu^L(w_i^k),\quad |I|\leq L.$$

Thus, by (4.2) we also have

$$\frac{c_L}{\|\varphi_i^k\|_{1,\alpha}}\pi_J\varphi_i^k\in\mathcal{G}_\mu^L(w_i^k),\quad |J|\leq L.$$

This completes the proof. □

Lemma 4.4. *Let $\mathcal{P}_{\varphi_i^k}$ be the orthogonal projection of $L^2_{\varphi_i^k}(\mathbb{B}_n)$ onto $V^L_{\varphi_i^k}(z_i^k)$. With the notation introduced above, there exists a constant $C>0$ such that for $f\in\mathcal{A}^p_\alpha$,*

$$\left|\mathcal{P}_{\varphi_i^k}(f)(z)\varphi_i^k(z)\right|\leq C2^{k_0+k}, \tag{4.5}$$

and

$$\left|\mathcal{P}_{\varphi_j^{k+1}}\Big([f-\mathcal{P}_{\varphi_j^{k+1}}(f)]\varphi_i^k\Big)(z)\varphi_j^{k+1}(z)\right|\leq C2^{k_0+k+1}, \tag{4.6}$$

for all i, j, k.

Proof: With the help of Lemma 4.3, the proof proceeds as the one of [**8**, Lemma 5.2]. □

5. Proof of the main result

This section is devoted to the proof of Theorem 4.1 by using a constructive method. The proof proceeds essentially as the ones in [**8, 9**], but some analysis inside the complex ball is involved. For the sake of completeness, we include the details.

Proof of Theorem 4.1: Let $f \in \mathcal{A}^p_\alpha$. Given an integer $N \geq N_{p,\alpha}$, let $L \geq \max\big\{N, \big[\frac{1}{p}(n+1+\alpha)\big]+1\big\}$ be an integer. Recall that

$$\mathcal{O}_k = \big\{z \in \mathbb{B}_n : \mathcal{K}_{\mu,L}(f)(z) + f^\star_\delta(z) > 2^{k_0+k}\big\}, \quad k = 0, 1, \dots .$$

For each k we fix the Whitney type covering $\{B^\varrho(z^k_i, r^k_i)\}_{i=1}^\infty$ and the partition of unity $\{\varphi^k_i\}$ with respect to $\mathcal{O}_k$, as constructed in Lemma 4.2. Then, we can write

$$f = \left(f - \sum_{i=1}^\infty f\varphi^k_i\right) + \sum_{i=1}^\infty f\varphi^k_i = h_k + \sum_{i=1}^\infty \Big(f - \mathcal{P}_{\varphi^k_i}(f)\Big)\,\varphi^k_i,$$

where

$$h_k = \left(f - \sum_{i=1}^\infty f\varphi^k_i\right) + \sum_{i=1}^\infty \mathcal{P}_{\varphi^k_i}(f)\varphi^k_i. \tag{5.1}$$

Also, by (4.5) one has

$$\left|\sum_{i=1}^\infty \mathcal{P}_{\varphi^k_i}(f)(z)\varphi^k_i(z)\right| \leq c2^{k_0+k}, \tag{5.2}$$

because no point in $\mathcal{O}_k$ lies in more than N_0 of the balls $B^\varrho(z^k_i, r^k_i)$. Note that

$$\operatorname{supp}\left(\sum_{i=1}^\infty \Big[f - \mathcal{P}_{\varphi^k_i}(f)\Big]\,\varphi^k_i\right) \subset \mathcal{O}_k.$$

This implies that $\sum_{i=1}^\infty \big[f - \mathcal{P}_{\varphi^k_i}(f)\big]\varphi^k_i \to 0$ as $k \to \infty$ almost everywhere on $\mathbb{B}_n$. Hence, by (5.1) one concludes that $f - h_k \to 0$ as $k \to \infty$ for almost all $z \in \mathbb{B}_n$. This implies that

$$f = h_0 + \sum_{k=0}^\infty \big(h_{k+1} - h_k\big), \quad \text{a.e.} \quad z \in \mathbb{B}_n. \tag{5.3}$$

Now, since

$$\sum_{i=1}^{\infty} \mathcal{P}_{\varphi_j^{k+1}}\Big(\big[f-\mathcal{P}_{\varphi_j^{k+1}}(f)\big]\varphi_i^k\Big)=\mathcal{P}_{\varphi_j^{k+1}}\Big(\big[f-\mathcal{P}_{\varphi_j^{k+1}}(f)\big]\chi_{\mathcal{O}_k}\Big)$$
$$=\mathcal{P}_{\varphi_j^{k+1}}\big[f\chi_{\mathcal{O}_k}\big]-\mathcal{P}_{\varphi_j^{k+1}}\big[\mathcal{P}_{\varphi_j^{k+1}}(f)\chi_{\mathcal{O}_k}\big]=0,$$

we can write

$$\begin{aligned}
h_{k+1}-h_k &= \big(f-h_k\big)-\big(f-h_{k+1}\big)\\
&= \sum_{i=1}^{\infty}\big[f-\mathcal{P}_{\varphi_i^k}(f)\big]\varphi_i^k-\sum_{j=1}^{\infty}\big[f-\mathcal{P}_{\varphi_j^{k+1}}(f)\big]\varphi_j^{k+1}\\
&= \sum_{i=1}^{\infty}\big[f-\mathcal{P}_{\varphi_i^k}(f)\big]\varphi_i^k\\
&\quad -\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\Big\{\big[f-\mathcal{P}_{\varphi_j^{k+1}}(f)\big]\varphi_i^k-\mathcal{P}_{\varphi_j^{k+1}}\Big(\big[f-\mathcal{P}_{\varphi_j^{k+1}}(f)\big]\varphi_i^k\Big)\Big\}\varphi_j^{k+1}\\
&=: \sum_{i=1}^{\infty} b_i^k,
\end{aligned}$$

where

$$\begin{aligned}
b_i^k &= \big[f-\mathcal{P}_{\varphi_i^k}(f)\big]\varphi_i^k\\
&\quad -\sum_{j=1}^{\infty}\Big\{\big[f-\mathcal{P}_{\varphi_j^{k+1}}(f)\big]\varphi_i^k-\mathcal{P}_{\varphi_j^{k+1}}\Big(\big[f-\mathcal{P}_{\varphi_j^{k+1}}(f)\big]\varphi_i^k\Big)\Big\}\varphi_j^{k+1}.
\end{aligned}\tag{5.4}$$

Put

$$a_0=\frac{1}{\lambda_0}h_0 \quad \text{with} \quad \lambda_0=\|h_0\|_{L^\infty(\mathbb{B}_n)},$$

and

$$a_i^k=\frac{1}{\lambda_i^k}b_i^k \quad \text{with} \quad \lambda_i^k=2^{k_0+k+1}v_\alpha(B^\varrho(z_i^k,Cr_i^k))^{\frac{1}{p}},$$

where C is a constant which will be fixed later. Therefore, we can write formally

$$f=\lambda_0a_0+\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\lambda_i^k a_i^k. \tag{5.5}$$

This is the desired atomic decomposition. For clarity, we check it in several steps as follows.

I. Support of the b_i^k's. We note that the first term in (5.4) is clearly supported in $B^\varrho(z_i^k, r_i^k)$. Note that if the terms in the series (5.4) are not identically 0, then the condition

$$B^\varrho(z_i^k, r_i^k) \cap B^\varrho(z_j^{k+1}, r_j^{k+1}) \neq \emptyset$$

must be satisfied for some j. We claim that there is a constant $C > 0$ depending only on ϱ such that $r_j^{k+1} \leq C r_i^k$. Assuming this claim, we conclude that $B^\varrho(z_j^{k+1}, r_j^{k+1}) \subset B^\varrho(z_i^k, C r_i^k)$. Thus b_i^k is supported in $B^\varrho(z_i^k, C r_i^k)$ and so does a_i^k.

To prove the previous claim, denote by K be the constant occurring in the quasi-triangle inequality satisfied by ϱ. By Lemma 4.1, we know that $r_i^k = \frac{1}{2K}\varrho(z_i^k, \mathcal{O}_k^c)$ for all i and k. Let $w \in B^\varrho(z_i^k, r_i^k) \cap B^\varrho(z_j^{k+1}, r_j^{k+1})$. Since $\mathcal{O}_{k+1} \subset \mathcal{O}_k$, we have

$$r_j^{k+1} \leq \frac{1}{2} r_j^{k+1} + \frac{1}{2}\varrho(w, \mathcal{O}_k^c) \leq \frac{1}{2} r_j^{k+1} + \frac{1}{2} K\big[\varrho(w, z_i^k) + \varrho(z_i^k, \mathcal{O}_k^c)\big].$$

Then,

$$r_j^{k+1} \leq K r_i^k + K\varrho(z_i^k, \mathcal{O}_k^c) \leq K r_i^k + 2K^2 r_i^k \leq K(1+2K) r_i^k,$$

and so the claim is proved.

II. Size estimates for h_0 and b_i^k's. Firstly, by (5.1) and (5.2) we have

$$|h_0| = \left| f\chi_{\mathcal{O}_0^c} + \sum_{i=1}^{\infty} \mathcal{P}_{\varphi_i^0}(f)\varphi_i^0 \right| r \leq \|f_\delta^\star\|_{L^\infty(\mathcal{O}_0^c)} + \left| \sum_{i=1}^{\infty} \mathcal{P}_{\varphi_i^0}(f)\varphi_i^0 \right| \leq c2^{k_0}.$$

Thus $\|h_0\|_{L^\infty} \leq c2^{k_0}$, so a_0 is a $(p, \infty, N)_\alpha$-atom.

On the other hand, by (5.4) we have

$$\begin{aligned} |b_i^k| \leq& \left| \big[f - \mathcal{P}_{\varphi_i^k}(f)\big]\varphi_i^k - \sum_{j=1}^{\infty} \big[f - \mathcal{P}_{\varphi_j^{k+1}}(f)\big]\varphi_i^k \varphi_j^{k+1} \right| \\ &+ \left| \sum_{j=1}^{\infty} P_{\varphi_j^{k+1}}\Big(\big[f - \mathcal{P}_{\varphi_j^{k+1}}(f)\big]\varphi_i^k\Big)\varphi_j^{k+1} \right|. \end{aligned}$$

The second term on the right hand side is bounded by $c2^{k_0+k+1}$ by (5.2), while the first term is equal to

$$\left|[f - \mathcal{P}_{\varphi_i^k}(f)]\varphi_i^k \chi_{\mathcal{O}_{k+1}^c} + \sum_{j=1}^{\infty} \Big([f - \mathcal{P}_{\varphi_i^k}(f)] - [f - \mathcal{P}_{\varphi_j^{k+1}}(f)]\Big)\varphi_i^k \varphi_j^{k+1}\right|$$

$$\le |f\chi_{\mathcal{O}_k \setminus \mathcal{O}_{k+1}}| + |\mathcal{P}_{\varphi_i^k}(f)\varphi_i^k| + \left|\sum_{j=1}^{\infty} \mathcal{P}_{\varphi_j^{k+1}}(f)\varphi_j^{k+1}\right| + |\mathcal{P}_{\varphi_i^k}(f)\varphi_i^k|$$

$$\le f_\delta^\star \chi_{\mathcal{O}_k \setminus \mathcal{O}_{k+1}} + c2^{k_0+k+1} \lesssim 2^{k_0+k+1},$$

where we have used Lemma 4.4. Thus, $|b_i^k| \lesssim 2^{k_0+k+1}$.

III. Vanishing condition. Notice that $1 \in V^L_{\varphi_i^k}(z_i^k) \cap V^L_{\varphi_j^{k+1}}(z_i^{k+1})$. Then,

$$\int_{\mathbb{B}_n} [f - \mathcal{P}_{\varphi_i^k}(f)]\varphi_i^k \, dv_\alpha = 0$$

and

$$\int_{\mathbb{B}_n} \Big([f - \mathcal{P}_{\varphi_j^{k+1}}(f)]\varphi_i^k - P_{\varphi_j^{k+1}}\Big([f - \mathcal{P}_{\varphi_j^{k+1}}(f)]\varphi_i^k\Big)\Big)\varphi_j^{k+1} \, dv_\alpha = 0.$$

Therefore, $\int_{\mathbb{B}_n} b_i^k \, dv_\alpha = 0$ and so $\int_{\mathbb{B}_n} a_i^k \, dv_\alpha = 0$.

IV. Moment condition. We shall prove that

$$\left|\int_{\mathbb{B}_n} b_i^k(z)\Phi(z)\, dv_\alpha(z)\right| \lesssim 2^{k_0+k+1} v_\alpha(B^\varrho(z_i^k, Cr_i^k))\, \|\Phi\|_{\mathcal{S}_N(B^\varrho(z_i^k, Cr_i^k))}$$

for any $\Phi \in \mathcal{C}^\infty(B^\varrho(z_i^k, Cr_i^k))$. To this end, we first note that there exists a unitary operator $U_{z_i^k}$ such that $U_{z_i^k} z_i^k = (|z_i^k|, 0, \ldots, 0)$. For any $z \in B^\varrho(z_i^k, Cr_i^k)$ we assume $U_{z_i^k} z = (x_1 + \mathrm{i}y_1, \ldots, x_n + \mathrm{i}y_n)$. Then, by Lemma 2.2 we have

$$\big|x_1 + \mathrm{i}y_1 - |z_i^k|\big| \le \varrho(z_i^k, z) \quad \text{and} \quad \sum_{j=2}^{n} |x_j + \mathrm{i}y_j|^2 \le 2\varrho(z_i^k, z).$$

Hence, $|x_1 - |z_i^k||, |y_1| \lesssim r_i^k$ and $|x_j|, |y_j| \lesssim \sqrt{r_i^k}$ for $j \geq 2$. Thus, if $\Theta(z_i^k, z) = (\alpha_1, \beta_1, \dots, \alpha_n, \beta_n)$,

$$|\alpha_1|, |\beta_1| \lesssim r_i^k \quad \text{and} \quad |\alpha_j|, |\beta_j| \lesssim \sqrt{r_i^k} \quad \text{for} \quad j \geq 2.$$

Using local coordinates $(\alpha_1, \beta_1, \dots, \alpha_n, \beta_n)$, we denote by $P^{\Phi}_{z_k^i, N}$ the Taylor expansion of order $N-1$ of Φ around z_i^k on $B^{\varrho}(z_i^k, Cr_i^k)$, i.e.,

$$P^{\Phi}_{z_k^i, N}(z) = \sum_{|J| \leq N-1} c_J \frac{\partial^{k_1 + \dots + k_n + m_1 + \dots + m_n} \Phi}{\partial \alpha_1^{k_1} \partial \beta_1^{m_1} \cdots \partial \alpha_n^{k_n} \partial \beta_n^{m_n}} (z_i^k) \Theta(z_i^k, z)^J,$$

where $J = (k_1, m_1, \dots, k_n, m_n)$. Note that $P^{\Phi}_{z_k^i, N}$ is in $V^L_{\varphi_i^k}(z_i^k)$. Then, we have

$$\left\| \Phi - P^{\Phi}_{z_k^i, N} \right\|_{L^\infty(B^{\varrho}(z_i^k, Cr_i^k))} \lesssim \|\Phi\|_{\mathcal{S}_N(B^{\varrho}(z_i^k, Cr_i^k))}.$$

In addition, if $B^{\varrho}(z_i^k, r_i^k) \cap B^{\varrho}(z_j^{k+1}, r_j^{k+1}) \neq \emptyset$, there exists a constant $C > 0$ such that $r^{k+1} \leq Cr_i^k$ and so $B^{\varrho}(z_j^{k+1}, r_j^{k+1}) \subset B^{\varrho}(z_i^k, Cr_i^k)$ (see Step I). In this case, for all $|J| \leq N-1$ we make the change of variable such that $\Theta(z_i^k, z)^J$ becomes the element in $V^L_{\varphi_j^{k+1}}(z_j^{k+1})$ and its order is still less than $N-1$. Therefore,

$$\begin{aligned}
\left| \int_{\mathbb{B}_n} b_i^k(z) \Phi(z) \, dv_\alpha(z) \right| &= \left| \int_{\mathbb{B}_n} b_i^k(z) \left(\Phi(z) - P^{\Phi}_{z_k^i, N} \right) \, dv_\alpha(z) \right| \\
&\leq 2^{k_0 + k + 1} v_\alpha(B^{\varrho}(z_i^k, Cr_i^k)) \left\| \Phi - P^{\Phi}_{z_i^k, N} \right\|_{L^\infty(B^{\varrho}(z_i^k, Cr_i^k))} \\
&\lesssim 2^{k_0 + k + 1} v_\alpha(B^{\varrho}(z_i^k, Cr_i^k)) \|\Phi\|_{\mathcal{S}_N(B^{\varrho}(z_i^k, Cr_i^k))},
\end{aligned}$$

by the size estimate for b_i^k's as above. Thus,

$$\left| \int_{\mathbb{B}_n} a_i^k(z) \Phi(z) \, dv_\alpha(z) \right| \lesssim \|\Phi\|_{\mathcal{S}_N(B^{\varrho}(z_i^k, Cr_i^k))} v_\alpha(B^{\varrho}(z_i^k, Cr_i^k))^{1 - \frac{1}{p}},$$

and so a_i^k is a $(p, \infty, N)_\alpha$-atom.

V. Convergence in the sense of distributions. In order to show that (5.5) holds in the sense of distributions, it suffices to verify that $\sum_{k=0}^m \sum_{i=1}^\infty b_i^k$ convergence in the sense of distributions. Let $\Psi \in C^\infty(\mathbb{B}_n)$. By the estimate of b_i^k in Step IV, we have for any $m>n$,

$$\begin{aligned}\left|\int_{\mathbb{B}_n}\sum_{k=n}^{m}\sum_{i=1}^{\infty}b_i^k\Psi\,dv_\alpha\right| &\le \sum_{k=n}^{m}\sum_{i=1}^{\infty}\left|\int_{\mathbb{B}_n}b_i^k\Psi\,dv_\alpha\right|\\ &\lesssim \sum_{k=n}^{m}\sum_{i=1}^{\infty}2^{k_0+k+1}v_\alpha\big(B^\varrho(z_i^k,Cr_i^k)\big)\big(r_i^k\big)^{\frac{N}{2}}\|\Psi\|_{C^N(\mathbb{B}_n)}\\ &\lesssim \sum_{k=n}^{m}\sum_{i=1}^{\infty}2^{k_0+k+1}v_\alpha\big(B^\varrho(z_i^k,r_i^k)\big)^{\frac{1}{p}}\,\|\Psi\|_{C^N(\mathbb{B}_n)}\,,\end{aligned}$$

where we use the fact $N\ge N_{p,\alpha}$ and Lemma 2.1. Hence, since $\frac{1}{p}\ge 1$ we have

$$\begin{aligned}&\left|\int_{\mathbb{B}_n}\sum_{k=n}^{m}\sum_{i=1}^{\infty}b_i^k(z)\Psi(z)\,dv_\alpha(z)\right|\\ &\quad\lesssim \sum_{k=n}^{m}2^{k_0+k+1}\left(\sum_{i=1}^{\infty}v_\alpha\big(B^\varrho(z_i^k,r_i^k)\big)\right)^{\frac{1}{p}}\|\Psi\|_{C^N(\mathbb{B}_n)}\\ &\quad\le \left(\sum_{k=n}^{m}2^{(k_0+k+1)p}v_\alpha(\mathcal{O}_k)\right)^{\frac{1}{p}}\|\Psi\|_{C^N(\mathbb{B}_n)}\\ &\quad\lesssim \|\Psi\|_{C^N(\mathbb{B}_n)}\left(\sum_{k=n}^{m}\int_{2^{k_0+k-1}}^{2^{k_0+k}}t^{p-1}v_\alpha\{z\in\mathbb{B}_n:\mathcal{K}_{\mu,L}(f)(z)+f_\delta^\star(z)>t\}\;dt\right)^{\frac{1}{p}}\\ &\quad= \left(\int_{\mathcal{O}_{n-1}}|\mathcal{K}_{\mu,L}(f)+f_\delta^\star|^p\,dv_\alpha\right)^{\frac{1}{p}}\|\Psi\|_{C^N(\mathbb{B}_n)}\,,\end{aligned}$$

which tends to 0 as $n\to\infty$. Thus, the equality (5.5) holds in the sense of distributions.

VI. Coefficients in ℓ^p. Indeed,

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}|\lambda_i^k|^p &= \sum_{k=0}^{\infty}\sum_{i=1}^{\infty}2^{(k_0+k+1)p}v_\alpha(B^\varrho(z_i^k,Cr_i^k))\\ &\lesssim \sum_{k=0}^{\infty}\sum_{i=1}^{\infty}2^{(k_0+k+1)p}v_\alpha(B^\varrho(z_i^k,r_i^k))\lesssim\sum_{k=0}^{\infty}2^{(k_0+k+1)p}v_\alpha(\mathcal{O}_k)\\ &\lesssim \int_{2^{k_0}}^{\infty}t^{p-1}v_\alpha\{z\in\mathbb{B}_n:\ \mathcal{K}_{\mu,L}(f)(z)+f_\delta^\star(z)>t\}\;dt\\ &\lesssim \|\mathcal{K}_{\mu,L}(f)+f_\delta^\star\|_{p,\alpha}^p\lesssim\|f\|_{p,\alpha}^p.\end{aligned}$$

In conclusion, we have shown that the representation (5.5) is an atomic decomposition for f.

VII. Completion of the proof of Theorem 4.1. It remains to prove that $f = \sum_j \lambda_j P_\alpha(a_j)$ in $\mathcal{A}^p_\alpha$. Indeed, assuming that $f \in \mathcal{A}^p_\alpha \cap \mathcal{A}^2_\alpha$ with $f = \sum_j \lambda_j a_j$ in the sense of distributions, we have

$$f(z) = P_\alpha(f)(z) = \left\langle \sum_j \lambda_j a_j, K_\alpha(\cdot, z) \right\rangle = \sum_j \lambda_j P_\alpha(a_j)(z).$$

Therefore, $f = \sum_j \lambda_j P_\alpha(a_j)$ for $f \in \mathcal{A}^p_\alpha \cap \mathcal{A}^2_\alpha$. Since $\mathcal{A}^p_\alpha \cap \mathcal{A}^2_\alpha$ is dense in $\mathcal{A}^p_\alpha$, by a standard argument we conclude the assertion for all $f \in \mathcal{A}^p_\alpha$. This completes the proof of Theorem 4.1. $\square$

Acknowledgements. We are grateful to the anonymous referee for making helpful suggestions and comments, which have been incorporated into this version of the paper. This research was supported in part by the NSFC under grant no. 11171338.

References

[1] D. Bekollé, Inégalité à poids pour le projecteur de Bergman dans la boule unité de $\mathbf{C}^n$, *Studia Math.* **71(3)** (1981/82), 305–323.

[2] Z. Chen and W. Ouyang, Maximal and area integral characterizations of Bergman spaces in the unit ball of $\mathbb{C}^n$, *J. Funct. Spaces Appl.* Art. ID 167514 (2013), 13 pp.

[3] R. R. Coifman and R. Rochberg, Representation theorems for holomorphic and harmonic functions in L^p, in: *"Representation theorems for Hardy spaces"*, Astérisque **77**, Soc. Math. France, Paris, 1980, pp. 11–66.

[4] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83(4)** (1977), 569–645. DOI: 10.1090/S0002-9904-1977-14325-5.

[5] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, *Invent. Math.* **26** (1974), 1–65. DOI: 10.1007/BF01406845.

[6] C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.* **129(3–4)** (1972), 137–193. DOI: 10.1007/BF02392215.

[7] J. B. Garnett and R. H. Latter, The atomic decomposition for Hardy spaces in several complex variables, *Duke Math. J.* **45(4)** (1978), 815–845. DOI: 10.1215/S0012-7094-78-04539-8.

[8] S. Grellier and M. M. Peloso, Decomposition theorems for Hardy spaces on convex domains of finite type, *Illinois J. Math.* **46(1)** (2002), 207–232.

[9] S. G. Krantz and S.-Y. Li, On decomposition theorems for Hardy spaces on domains in $\mathbf{C}^n$ and applications, *J. Fourier Anal. Appl.* **2(1)** (1995), 65–107. DOI: 10.1007/s00041-001-4023-6.

[10] E. M. Stein, *"Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals"*, With the assistance of Timothy S. Murphy, Princeton Mathematical Series **43**, Monographs in Harmonic Analysis III, Princeton University Press, Princeton, NJ, 1993.

[11] E. Tchoundja, Carleson measures for the generalized Bergman spaces via a T(1)-type theorem, *Ark. Mat.* **46(2)** (2008), 377–406. DOI: 10.1007/s11512-008-0070-4.

[12] K. Zhu, *"Spaces of holomorphic functions in the unit ball"*, Graduate Texts in Mathematics **226**, Springer-Verlag, New York, 2005.

Zeqian Chen:
Wuhan Institute of Physics and Mathematics
Chinese Academy of Sciences
West District 30
Xiao-Hong-Shan, Wuhan 430071
China
E-mail address: zqchen@wipm.ac.cn

Wei Ouyang:
Institute of Geodesy and Geophysics
Chinese Academy of Sciences
340 Xu-Dong Road
Wuhan 430077
China

and

University of Chinese Academy of Sciences
Beijing 100049
China
E-mail address: oyw1@163.com

Primera versió rebuda el 18 de març de 2013,
darrera versió rebuda el 14 d'octubre de 2013.