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CONVERGENCE OF LAGRANGE INTERPOLATION SERIES IN THE FOCK SPACES

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Abstract: We study the uniqueness sets, the weak interpolation sets, and convergence of the Lagrange interpolation series in radial weighted Fock spaces.

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1. Introduction and main results

In this paper we study the weighted Fock spaces $\mathcal{F}^2_\varphi(\mathbb{C})$

$$\mathcal{F}^2_\varphi(\mathbb{C}) = \left\{ f \in \mathrm{Hol}(\mathbb{C}) : \|f\|^2_\varphi = \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(|z|)}\, dm(z) < \infty \right\};$$

here dm is area measure and φ is an increasing function defined on $[0, +\infty)$, $\lim_{r\to\infty} \varphi(r) = \infty$. We assume that the radial weight $\varphi(z) = \varphi(|z|)$ is C^2 smooth and strictly subharmonic on $\mathbb{C}$, and set

$$\rho(z) = (\Delta\varphi(z))^{-1/2},$$

so that $\Delta\varphi(r) = \varphi''(r) + \varphi'(r)/r$ $(r > 0)$. One more condition on φ is that for every fixed C we have

$$\rho(r + C\rho(r)) \asymp \rho(r), \quad 0 < r < \infty.$$

(The notation $A \asymp B$ means that there is a constant C independent of the relevant variables such that $C^{-1}B \leq A \leq CB$.) In particular, this holds if $\rho'(r) = o(1)$, $r \to \infty$. The function ρ plays the role of a scaling parameter, see the definition of d_ρ below.

Typical φ are power functions,

$$\varphi(r) = r^a, \quad a > 0.$$

For such functions φ we have

$$\rho(r) \asymp r^{1-a/2}, \quad r > 1.$$

Furthermore, if

$$\varphi(r) = (\log r)^2,$$

then

$$\rho(r) \asymp r, \quad r > 1.$$

Given $z, w \in \mathbb{C}$, we define a scaled distance function

$$d_\rho(z,w) = \frac{|z-w|}{\min(\rho(z), \rho(w))}.$$

We say that a subset Λ of $\mathbb{C}$ of is d_ρ-separated if

$$\inf_{\lambda \neq \lambda^*} \{d_\rho(\lambda, \lambda^*),\ \lambda, \lambda^* \in \Lambda\} > 0.$$

Next, we introduce a family of sufficiently regular subsets Λ in $\mathbb{C}$ defined as the zero sets for the functions in a special class.

Definition 1. Given $\gamma \in \mathbb{R}$, we say that an entire function S belongs to the class $\mathcal{S}_\gamma$ if

(1) the zero set Λ of S is d_ρ-separated, and
(2)

$$|S(z)| \asymp e^{\varphi(z)} \frac{d(z,\Lambda)}{\rho(z)} \frac{1}{(1+|z|)^\gamma}, \quad z \in \mathbb{C}.$$

For constructions of such functions in radial weighted Fock spaces see, for example, [**1, 2, 5, 7**].

In the standard Fock spaces ($\varphi(r) = r^2$) the classes $\mathcal{S}_\gamma$ were introduced by Lyubarskii in [**4**]. They are analogs of the sine type functions for the Paley-Wiener space, and their zero sets include rectangular lattices and their perturbations.

Definition 2. A set $\Lambda \subset \mathbb{C}$ is called *a weak interpolation set* for $\mathcal{F}^2_\varphi(\mathbb{C})$ if for every $\lambda \in \Lambda$ there exists $f_\lambda \in \mathcal{F}^2_\varphi(\mathbb{C})$ such that $f_\lambda(\lambda) = 1$ and $f_\lambda|\Lambda\backslash\{\lambda\} = 0$.

A set $\Lambda \subset \mathbb{C}$ is called *a uniqueness set* for $\mathcal{F}^2_\varphi(\mathbb{C})$ if for every $f \in \mathcal{F}^2_\varphi(\mathbb{C})$, the relation $f|\Lambda = 0$ implies that $f = 0$.

Theorem 3. *Let ϕ and ρ be as above. Given $S \in \mathcal{S}_\gamma$, consider its zero set Λ. Then*

(a) *Λ is a uniqueness set for $\mathcal{F}^2_\varphi(\mathbb{C})$ if and only if $\gamma \le 1$,*
(b) *Λ is a weak interpolation set for $\mathcal{F}^2_\varphi(\mathbb{C})$ if and only if $\gamma > 0$.*

Denote by $\mathbf{k}_z$ the reproducing kernel of $\mathcal{F}^2_\varphi(\mathbb{C})$:

$$\langle f, \mathbf{k}_z \rangle_{\mathcal{F}^2_\varphi(\mathbb{C})} = f(z), \quad f \in \mathcal{F}^2_\varphi(\mathbb{C}), \quad z \in \mathbb{C}.$$

The sequence $\Lambda \subset \mathbb{C}$ is called *sampling* for $\mathcal{F}^2_\varphi(\mathbb{C})$ if

$$\|f\|^2_\varphi \asymp \|f\|^2_{\varphi,\Lambda} := \sum_{\lambda\in\Lambda} \frac{|f(\lambda)|^2}{\mathbf{k}_\lambda(\lambda)}, \quad f \in \mathcal{F}^2_\varphi(\mathbb{C}),$$

and *interpolating* for $\mathcal{F}^2_\varphi(\mathbb{C})$ if for every $v = (v_\lambda)_{\lambda\in\Lambda}$ with $\|v\|_{\varphi,\Lambda} < \infty$ there exists $f \in \mathcal{F}^2_\varphi(\mathbb{C})$ such that

$$v = f|\Lambda.$$

It is obvious that each sampling sequence for $\mathcal{F}^2_\varphi(\mathbb{C})$ is a set of uniqueness for $\mathcal{F}^2_\varphi(\mathbb{C})$ and each interpolation sequence for $\mathcal{F}^2_\varphi(\mathbb{C})$ is a weak interpolation set for $\mathcal{F}^2_\varphi(\mathbb{C})$.

The sequence $\Lambda \subset \mathbb{C}$ is called *complete interpolating* sequence for $\mathcal{F}^2_\varphi(\mathbb{C})$ if it is simultaneously interpolating and sampling for $\mathcal{F}^2_\varphi(\mathbb{C})$.

Let $\Bbbk_\lambda = \mathbf{k}_\lambda / \|\mathbf{k}_\lambda\|_{\varphi,2}$ be the normalized reproducing kernel at λ. Let $\Lambda \subset \mathbb{C}$. We say that $\{\Bbbk_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}^2_\varphi(\mathbb{C})$ if it is complete and for some $C > 0$ and each finite sequence $\{a_\lambda\}$ we have

$$\frac{1}{C} \sum_{\lambda\in\Lambda} |a_\lambda|^2 \le \left\| \sum_{\lambda\in\Lambda} a_\lambda \Bbbk_\lambda \right\|^2_\varphi \le C \sum_{\lambda\in\Lambda} |a_\lambda|^2.$$

Note that in $\mathcal{F}^2_\varphi(\mathbb{C})$, interpolation and sampling can be expressed in terms of geometric properties of reproducing kernels: interpolation means that the sequence of the associated reproducing kernels is a Riesz basis in its closed linear span; sampling means that this sequence is a frame (see [**9**, Chapter 3]). Standard duality arguments show that the system $\{\Bbbk_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}^2_\varphi(\mathbb{C})$ if and only if Λ is a complete interpolating sequence for $\mathcal{F}^2_\varphi(\mathbb{C})$.

In 1992 Seip and Wallstén [**8, 10**] characterized interpolating and sampling sequences in these spaces when $\varphi(r) = r^2$. Their results show that there are no sequences which are simultaneously interpolating and sampling, and hence there are no unconditional or Riesz bases in this situation. The situation changes in small Fock spaces when the weight increases slowly. Borichev and Lyubarskii [**2**] have shown that for $\varphi(r) = (\log r)^2$ there exist Riesz bases in $\mathcal{F}^2_\varphi(\mathbb{C})$. Furthermore [**9, 3, 2**], the space $\mathcal{F}^2_\varphi(\mathbb{C})$ does not admit Riesz bases of the (normalized) reproducing kernels for regular φ, $(\log r)^2 = o(\varphi(r))$, $r \to \infty$.

By Theorem 3, for $0 < \gamma \le 1$ the family $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a complete minimal family in $\mathcal{F}^2_\varphi(\mathbb{C})$. Thus the family

$$\big\{S/[S'(\lambda)(\cdot-\lambda)]\big\}_{\lambda\in\Lambda}$$

is the biorthogonal system and we associate to any $f \in \mathcal{F}^2_\varphi(\mathbb{C})$ the formal (Lagrange interpolation) series

$$f \sim \sum_{\lambda\in\Lambda} f(\lambda)\frac{S}{S'(\lambda)(\cdot-\lambda)}.$$

This series converges unconditionally in $\mathcal{F}^2_\varphi(\mathbb{C})$ if $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}^2_\varphi(\mathbb{C})$. Otherwise, it does not necessarily unconditionally converge in $\mathcal{F}^2_\varphi(\mathbb{C})$, and it is natural to ask whether this series admits a summation method if we modify (slightly) the norm of the space.

Denote by $\Lambda = \{\lambda_k\}$ the zero sequence of S ordered in such a way that $|\lambda_k| \le |\lambda_{k+1}|$, $k \ge 1$. Similarly to Lyubarskii [**4**] and Lyubarskii-Seip [**6**], we obtain the following result:

Theorem 4. *Let $0 \le \beta \le 1$, $\gamma + \beta \in (1/2, 1)$, and let $S \in \mathcal{S}_\gamma$. Suppose that*

$$r^{1-2\beta} = O(\rho(r)), \quad r \to +\infty. \tag{1}$$

Then for every $f \in \mathcal{F}^2_\varphi(\mathbb{C})$ we have

$$\lim_{N\to\infty}\left\| f - S\sum_{k=1}^{N}\frac{f(\lambda_k)}{S'(\lambda_k)(\cdot-\lambda_k)}\right\|_{\varphi_\beta} = 0, \tag{2}$$

where $\varphi_\beta(r) = \varphi(r) + \beta\log(1+r)$.

The result corresponding to $\beta = 1/2$, $\varphi(r) = r^2$, $\rho(r) \asymp 1$ is contained in [**6**, Theorem 10]. On the other hand, in the case $\varphi(r) = (\log r)^a$, $1 < a \le 2$, $\rho(r) \asymp r(\log r)^{1-\frac{a}{2}}$, $r > 2$, the space $\mathcal{F}^2_\varphi(\mathbb{C})$ contains Riesz bases of (normalized) reproducing kernels [**2**]. Furthermore, our theorem shows that in the case $r \lesssim \rho(r)$, $r > 1$, when $S \in \mathcal{S}_\gamma$, $\gamma \in (1/2, 1)$, the interpolation series converges already in $\mathcal{F}^2_\varphi(\mathbb{C})$. (The notation $A \lesssim B$ means that there is a constant C independent of the relevant variables such that $A \le CB$.)

In the case $0 < a \le 2$, $\varphi(r) = r^a$, $\rho(r) \asymp r^{1-a/2}$, $r > 1$, we can choose $a/4 \le \beta \le 1$ with appropriate γ as in Theorem 4. Thus, the closer we are to $\phi(r) = (\log r)^2$, $\rho(r) \asymp r$, $r > 2$, the less we should modify the norm (by the smaller β) to get convergence in (2). Now, it is interesting to find out how sharp is condition (1) in Theorem 4.

Theorem 5. *Let* $0 < a \le 2$, $\varphi(r) = r^a$, $r > 1$. *If* $0 \le \beta < a/4$, $\gamma \in \mathbb{R}$, *and* $S \in \mathcal{S}_\gamma$, *then there exists* $f \in \mathcal{F}^2_\varphi(\mathbb{C})$ *such that*

$$\left\| f - S\sum_{k=1}^{N} \frac{f(\lambda_k)}{S'(\lambda_k)(\cdot - \lambda_k)} \right\|_{\varphi_\beta} \not\to 0, \quad N \to \infty.$$

Thus, for the power weights $\varphi(r) = r^a$, $0 < a \le 2$, we really need to modify the norm to get the convergence, and the critical value of β is $a/4$.

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2. Proofs

2.1. Proof of Theorem 3. (a) If $\gamma > 1$ then $\mathcal{S}_\gamma \subset \mathcal{F}^2_\varphi(\mathbb{C})$ and $S|\Lambda = 0$. Hence Λ is not a uniqueness set.

If $\gamma \le 1$, then $\mathcal{S}_\gamma \cap \mathcal{F}^2_\varphi(\mathbb{C}) = \emptyset$. Suppose that there exists $g \in \mathcal{F}^2_\varphi(\mathbb{C})$ such that $g|\Lambda = 0$. Then $g = FS$ for an entire function F, and we have

$$\int_{\mathbb{C}} |F(w)|^2|S(w)|^2 e^{-2\varphi(w)}\, dm(w) < \infty. \tag{3}$$

Given $\Omega \subset \mathbb{C}$, denote

$$\mathcal{I}[\Omega] = \int_\Omega |F(w)|^2 \frac{d^2(w,\Lambda)}{(1+|w|)^{2\gamma}\rho^2(w)}\, dm(w).$$

By (3), we have

$$\mathcal{I}[\mathbb{C}] < \infty.$$

Denote by $D(z,r)$ the disc of radius r centered at z. Let

$$\Omega_\varepsilon = \bigcup_{\lambda\in\Lambda} D(\lambda, \varepsilon\rho(\lambda)),$$

where ε is such that the discs $D(\lambda, 2\varepsilon\rho(\lambda))$ are pairwise disjoint. We have

$$\mathcal{I}[\mathbb{C}] = \mathcal{I}[\mathbb{C}\setminus\Omega_{2\varepsilon}] + \sum_{\lambda\in\Lambda} \mathcal{I}[D(\lambda,2\varepsilon\rho(\lambda))\setminus D(\lambda,\varepsilon\rho(\lambda))] + \sum_{\lambda\in\Lambda} \mathcal{I}[D(\lambda,\varepsilon\rho(\lambda))].$$

It is clear that

$$\mathcal{I}[\mathbb{C}\setminus\Omega_{2\varepsilon}] \ge c_1 \int_{\mathbb{C}\setminus\Omega_{2\varepsilon}} \frac{|F(w)|^2}{(1+|w|)^{2\gamma}}\, dm(w).$$

On the other hand,

$$\int_{D(\lambda,\varepsilon\rho(\lambda))} |F(w)|^2\, dm(w) \le c_2 \int_{D(\lambda,2\varepsilon\rho(\lambda))\setminus D(\lambda,\varepsilon\rho(\lambda))} |F(w)|^2\, dm(w),$$

and, hence,

$$\mathcal{I}[D(\lambda, 2\varepsilon\rho(\lambda))\setminus D(\lambda, \varepsilon\rho(\lambda))] \ge c_3 \mathcal{I}[D(\lambda, \varepsilon\rho(\lambda))].$$

Therefore

$$\int_{\mathbb{C}} \frac{|F(w)|^2}{(1+|w|)^{2\gamma}}\, dm(w) < \infty,$$

the function F is constant, and $g = cS$. Since $\mathcal{S}_\gamma \cap \mathcal{F}^2_\varphi(\mathbb{C}) = \emptyset$, we get a contradiction. Statement (a) is proved.

(b) Let $\gamma > 0$. Set

$$f_\lambda(z) = \frac{S(z)}{S'(\lambda)(z-\lambda)}, \quad \lambda \in \Lambda.$$

It is obvious that $f_\lambda \in \mathcal{F}^2_\varphi(\mathbb{C})$, $f_\lambda|(\Lambda\setminus\{\lambda\}) = 0$ and $f_\lambda(\lambda) = 1$. Hence Λ is a weak interpolation set. If $\gamma \le 0$, $\lambda \in \Lambda$, then, by (a), $\Lambda\setminus\{\lambda\}$ is a uniqueness set for $\mathcal{F}^2_\varphi(\mathbb{C})$. Therefore, Λ is not a weak interpolation set for $\mathcal{F}^2_\varphi(\mathbb{C})$. □

2.2. Proof of Theorem 4. We follow the scheme of proof proposed in [**4, 6**] and concentrate mainly on the places where the proofs differ. We need some auxiliary notions and lemmas. The proof of the first lemma is the same as in [**1**, Lemma 4.1].

Lemma 6. *For every $\delta > 0$, there exists $C > 0$ such that for functions f holomorphic in $D(z, \delta\rho(z))$ we have*

$$|f(z)|^2 e^{-2\varphi(z)} \le \frac{C}{\rho(z)^2} \int_{D(z,\delta\rho(z))} |f(w)|^2 e^{-2\varphi(w)}\, dm(w).$$

Definition 7. A simple closed curve $\gamma = \{r(\theta)e^{-i\theta},\ \theta \in [0, 2\pi]\}$ is called K-bounded if r is C^1-smooth and 2π-periodic on the real line and $|r'(\theta)| \le K$, $\theta \in \mathbb{R}$.

Let $\gamma \in \mathbb{R}$, $S \in \mathcal{S}_\gamma$, and let $\Lambda = \{\lambda_k\}$ be the zero set of S ordered in such a way that $|\lambda_k| \le |\lambda_{k+1}|$, $k \ge 1$. We can construct a sequence of numbers $R_N \to \infty$ and a sequence of contours Γ_N such that:

(1) $\Gamma_N = R_N \gamma_N$, where γ_N are K-bounded with $K > 0$ independent of N.

(2) $d_\rho(\Lambda, \Gamma_N) \ge \varepsilon$ for some $\varepsilon > 0$ independent of N.

(3) $\{\lambda_k\}_1^N$ lie inside Γ_N and $\{\lambda_k\}_{N+1}^\infty$ lie outside Γ_N.

(4) $\Gamma_N \subset \{z : R_N - \rho(R_N) < |z| < R_N + \rho(R_N)\}$.

Indeed, for some $0 < \varepsilon < 1$ the discs $D_k = D(\lambda_k, \varepsilon\rho(\lambda_k))$ are disjoint. For some $\delta = \delta(\varepsilon) > 0$ we have

$$\varepsilon\rho(\lambda_k) > 4\delta\rho(\lambda_N), \quad \big||\lambda_k| - |\lambda_N|\big| < \delta\rho(\lambda_N).$$

Fix $\psi \in C_0^\infty[-1,1]$, $0 < \psi < 1$, such that $\psi > 1/2$ on $[-1/2, 1/2]$.

Put

$$\Xi = \left\{k : \big||\lambda_k| - |\lambda_N|\big| < \frac{1}{4}\delta\rho(\lambda_N)\right\},$$

denote $\lambda_k = r_k e^{i\theta_k}$, $k \in \Xi$, and set

$$r(\theta) = 1 + \sum_{k\in\Xi} s_k \frac{\delta\rho(\lambda_N)}{|\lambda_N|}\psi\left(\frac{|\lambda_N|}{\delta\rho(\lambda_N)}(\theta - \theta_k)\right),$$

where $s_k = 1$, $k \le N$, $s_k = -1$, $k > N$.

Finally, set

$$\gamma_N = \big\{r(\theta)e^{i\theta},\, \theta \in [0, 2\pi]\big\}.$$

Lemma 8.

$$R_N\rho(R_N)\int_{\gamma_N} |f(R_N\zeta)|^2 e^{-2\varphi(R_N\zeta)}|d\zeta| \to 0, \quad N \to \infty.$$

Proof: Set

$$C_N = \bigcup_{\zeta\in\gamma_N} D(R_N\zeta, \rho(R_N\zeta)).$$

Since

$$\rho(R_N\zeta) \asymp \rho(R_N), \quad \zeta \in \gamma_N,$$

by Lemma 6 we have

$$\begin{aligned}
&R_N\rho(R_N)\int_{\gamma_N} |f(R_N\zeta)|^2 e^{-2\varphi(R_N\zeta)}|d\zeta| \\
&\quad\lesssim R_N\rho(R_N)\int_{\gamma_N}\left[\frac{1}{\rho(R_N\zeta)^2}\int_{D(R_N\zeta,\rho(R_N\zeta))}|f(w)|^2e^{-2\varphi(w)}\,dm(w)\right]|d\zeta| \\
&\quad\asymp \frac{R_N}{\rho(R_N)}\int_{C_N}|f(w)|^2e^{-2\varphi(w)}\left(\int_{\gamma_N}\chi_{D(R_N\zeta,\rho(R_N\zeta))}(w)|d\zeta|\right)dm(w) \\
&\quad\lesssim \int_{C_N}|f(w)|^2e^{-2\varphi(w)}\,dm(w) \to 0, \quad N \to \infty.
\end{aligned}$$

□

Proof of Theorem 4. Let $\chi_N(z) = 1$ if z lies inside Γ_N and 0 otherwise. Put

$$\Sigma_N(z,f) = S(z)\sum_{k=1}^{N} \frac{f(\lambda_k)}{S'(\lambda_k)(z-\lambda_k)},$$

and set

$$I_N(z,f) = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{f(\zeta)}{S(\zeta)(z-\zeta)}\,d\zeta.$$

The Cauchy formula gives us that

$$I_N(z,f) = \sum_{k=1}^{N} \frac{f(\lambda_k)}{S'(\lambda_k)(z-\lambda_k)} - \chi_N(z)\frac{f(z)}{S(z)}, \quad z \notin \Gamma_N.$$

Hence,

$$\Sigma_N(z,f) - f(z) = S(z)I_N(z,f) + (\chi_N(z)-1)f(z),$$

and to complete the proof of the theorem, it remains only to verify that

$$\|SI_N(\cdot,f)\|_{\varphi_\beta} \to 0, \quad N \to \infty.$$

Let ω be a Lebesgue measurable function such that

$$\int_0^\infty \int_0^{2\pi} |\omega(re^{it})|^2 e^{-2\varphi(r)}(1+r)^{-2\beta} r\,dr\,dt \le 1, \tag{4}$$

and let

$$J_N(f,\omega) = \int_0^\infty \int_0^{2\pi} \omega(re^{it})S(re^{it})I_N(re^{it},f)e^{-2\varphi(r)}(1+r)^{-2\beta} r\,dr\,dt.$$

By duality, it remains to show that

$$\sup |J_N(f,\omega)| \to 0, \quad N \to \infty,$$

where the supremum is taken over all ω satisfying (4).

We have

$$\begin{aligned} 2\pi i J_N(f,\omega) &= \int_{\Gamma_N} \frac{f(\zeta)}{S(\zeta)} \int_{\mathbb{C}} \frac{\omega(z)S(z)}{z-\zeta} e^{-2\varphi(z)}(1+|z|)^{-2\beta}\,dm(z)\,d\zeta \\ &= \int_{\Gamma_N} \frac{f(\zeta)}{S(\zeta)} \int_{\mathbb{C}} \frac{\phi(z)}{z-\zeta}(1+|z|)^{-\beta-\gamma}\,dm(z)\,d\zeta, \end{aligned}$$

where

$$\phi(z) = [\omega(z)e^{-\varphi(z)}(1+|z|)^{-\beta}][S(z)e^{-\varphi(z)}(1+|z|)^{\gamma}].$$

Note that

$$\int_{\mathbb{C}} |\phi(z)|^2\,dm(z) \le C.$$

Set

$$\psi(z) = R_N \phi(R_N z)$$

(here ψ depends on N). We have

$$\int_{\mathbb{C}} |\psi(z)|^2 \, dm(z) \le C.$$

Changing the variables $z = R_N w$ and $\zeta = R_N \eta$, we get

$$2\pi i J_N(f,\omega) = R_N \int_{\gamma_N} \frac{f(R_N\eta)}{S(R_N\eta)} \int_{\mathbb{C}} \frac{\psi(w)}{w-\eta} (1 + R_N|w|)^{-\beta-\gamma} \, dm(w) \, d\eta.$$

Consider the operators

$$T_N(\psi)(\eta) = \int_{\mathbb{C}} \frac{\psi(w)}{w-\eta} |w|^{-\beta-\gamma} \, dm(w), \quad \psi \in L^2(\mathbb{C}, dm(w)).$$

Since $\gamma + \beta \in (1/2, 1)$, by [**6**, Lemma 13], the operators T_N are bounded from $L^2(\mathbb{C}, dm(w))$ into $L^2(\gamma_N)$ and

$$\sup_N \|T_N\| < \infty.$$

Hence, by Lemma 8 and by the property $r^{1-2\beta} = O(\rho(r))$, $r \to \infty$, we get

$$\begin{aligned}
J_N(f,\omega) &\lesssim R_N^{1-\beta-\gamma} \left| \int_{\gamma_N} \frac{f(R_N\eta)}{S(R_N\eta)} T_N(\psi)(\eta) \, d\eta \right| \\
&\lesssim R_N^{1-\beta} \int_{\gamma_N} |f(R_N\eta)| e^{-\varphi(R_N\eta)} |T_N(\psi)(\eta)| \, |d\eta| \\
&\lesssim \left(R_N \rho(R_N) \int_{\gamma_N} |f(R_N\eta)|^2 e^{-2\varphi(R_N\eta)} |d\eta| \right)^{1/2} \\
&\qquad \times \|T_N(\psi)\|_{L^2(\gamma_N)} \to 0, \qquad N \to \infty.
\end{aligned}$$

This completes the proof. □

2.3. Proof of Theorem 5. It suffices to find $f \in \mathcal{F}^2_\varphi(\mathbb{C})$ and a sequence N_k such that (in the notations of the proof of Theorem 4)

$$A_k = \left\| S\chi_{N_k} \int_{\Gamma_{N_k}} \frac{f(\zeta)}{S(\zeta)(\cdot - \zeta)} \, d\zeta \right\|_{\varphi_\beta} \not\to 0, \quad k \to \infty. \tag{5}$$

We follow the method of the proof of [**6**, Theorem 11]. Let us write down the Taylor series of S:

$$S(z) = \sum_{n \ge 0} s_n z^n.$$

Since $S \in \mathcal{S}_\gamma$, by Cauchy's inequality, we have

$$\begin{aligned}|s_n| &\lesssim \inf_{r>0} e^{r^a} \frac{r}{r^{1-\frac{a}{2}}} \frac{1}{(1+r)^\gamma} r^{-n} \\ &\lesssim \inf_{r>0} e^{r^a} r^{-n-\gamma+\frac{a}{2}} \\ &\lesssim \exp\left(-\frac{n}{a}\ln\frac{n}{ae} - \frac{\gamma}{a}\ln n\right), \quad n > 0.\end{aligned}$$

Choose $0 < \varepsilon < \frac{a}{2} - 2\beta$. Given $R > 0$ consider

$$S_R = \sum_{|n-aR^a|<R^{\frac{a}{2}+\varepsilon}} s_n z^n.$$

Then for every n we have

$$|S(z) - S_R(z)|e^{-|z|^a} = O(|z|^{-n}), \quad \big||z| - R\big| < \rho(R), \quad R \to \infty.$$

Next we use that for some $c > 0$ independent of n, we have

$$\int_0^\infty r^{2n+1}e^{-2r^a}\,dr \le c \int_{|r-(\frac{n}{a})^{1/a}|<n^{(1/a)-(1/2)}} r^{2n+1}e^{-2r^a}\,dr.$$

Therefore,

$$\begin{aligned}\|S_R\|_\varphi^2 &= \sum_{|n-aR^a|<R^{\frac{a}{2}+\varepsilon}} \pi|s_n|^2 \int_0^\infty r^{2n+1}e^{-2r^a}\,dr \\ &\le \sum_{|n-aR^a|<R^{\frac{a}{2}+\varepsilon}} c|s_n|^2 \int_{|r-(\frac{n}{a})^{1/a}|<n^{(1/a)-(1/2)}} r^{2n+1}e^{-2r^a}\,dr \\ &\le \sum_{|n-aR^a|<R^{\frac{a}{2}+\varepsilon}} c|s_n|^2 \int_{|r-R|<c_1R^{1-\frac{a}{2}+\varepsilon}} r^{2n+1}e^{-2r^a}\,dr \\ &\le \sum_{n\ge 0} c|s_n|^2 \int_{|r-R|<c_1R^{1-\frac{a}{2}+\varepsilon}} r^{2n+1}e^{-2r^a}\,dr \\ &= c\int_{|r-R|<c_1R^{1-\frac{a}{2}+\varepsilon}} \sum_{n\ge 0} |s_n|^2 r^{2n+1}e^{-2r^a}\,dr \\ &= c\int_{\left||z|-R\right|<c_1R^{1-\frac{a}{2}+\varepsilon}} |S(z)|^2 e^{-2|z|^a}\,dm(z) \le c_2R^{2-\frac{a}{2}} + \varepsilon - 2\gamma.\end{aligned}$$

Fix $\varkappa$ such that
$$1-\frac{a}{4}+\frac{\varepsilon}{2}-\gamma<\varkappa<1-\beta-\gamma.$$

Choose a sequence N_k, $k\ge 1$, such that for $R_k=|\lambda_{N_k}|$ we have $R_{k+1}>2R_k$, $k\ge 1$, and
$$\Bigl|e^{-|z|^a}\sum_{m\ne k}S_{R_m}(z)R_m^{-\varkappa}\Bigr|\le\frac{1}{|z|^{\gamma+1}},\quad \bigl||z|-R_k\bigr|<\rho(R_k),\quad k\ge 1.$$

Set
$$f=\sum_{k\ge 1}S_{R_k}R_k^{-\varkappa}.$$

Then $f\in\mathcal{F}^2_\varphi(\mathbb{C})$, and
$$\frac{f}{S}=R_k^{-\varkappa}+O(R_k^{-1-\varkappa})\quad\text{on }\Gamma_{N_k},\quad k\to\infty.$$

Hence,
$$\Bigl|S(z)\int_{\Gamma_{N_k}}\frac{f(\zeta)}{S(\zeta)(z-\zeta)}\,d\zeta\Bigr|\ge cR_k^{-\varkappa}\frac{e^{|z|^a}}{(1+|z|)^\gamma},\quad |z|<\frac{R_k}{2},$$

and finally
$$A_k\ge cR_k^{-\varkappa}\left(\int_0^{R_k/2}\frac{r^{1-2\beta}\,dr}{(1+r)^{2\gamma}}\right)^{1/2}\to\infty,\quad k\to\infty.$$

This proves (5) and thus completes the proof of the theorem. □

References

[1] A. Borichev, R. Dhuez, and K. Kellay, Sampling and interpolation in large Bergman and Fock spaces, *J. Funct. Anal.* **242(2)** (2007), 563–606. `DOI: 10.1016/j.jfa.2006.09.002`.

[2] A. Borichev and Y. Lyubarskii, Riesz bases of reproducing kernels in Fock-type spaces, *J. Inst. Math. Jussieu* **9(3)** (2010), 449–461. `DOI: 10.1017/S147474800900019X`.

[3] K. P. Isaev and R. S. Yulmukhametov, The absence of unconditional bases of exponentials in Bergman spaces on non-polygonal domains, (Russian), *Izv. Ross. Akad. Nauk Ser. Mat.* **71(6)** (2007), 69–90; translation in: *Izv. Math.* **71(6)** (2007), 1145–1166 `DOI: 10.1070/IM2007v071n06ABEH002385`.

[4] Yu. I. Lyubarskiĭ, Frames in the Bargmann space of entire functions, in: *“Entire and subharmonic functions”*, Adv. Soviet Math. **11**, Amer. Math. Soc., Providence, RI, 1992., pp. 167–180.

[5] Y. Lyubarskii and E. Malinnikova, On approximation of subharmonic functions, *J. Anal. Math.* **83** (2001), 121–149. DOI: 10.1007/BF02790259.

[6] Yu. I. Lyubarskii and K. Seip, Convergence and summability of Gabor expansions at the Nyquist density, *J. Fourier Anal. Appl.* **5(2–3)** (1999), 127–157. DOI: 10.1007/BF01261606.

[7] N. Marco, X. Massaneda, and J. Ortega-Cerdà, Interpolating and sampling sequences for entire functions, *Geom. Funct. Anal.* **13(4)** (2003), 862–914. DOI: 10.1007/s00039-003-0434-7.

[8] K. Seip, Density theorems for sampling and interpolation in the Bargmann-Fock space. I, *J. Reine Angew. Math.* **429** (1992), 91–106. DOI: 10.1515/crll.1992.429.91.

[9] K. Seip, *"Interpolation and sampling in spaces of analytic functions"*, University Lecture Series **33**, American Mathematical Society, Providence, RI, 2004.

[10] K. Seip and R. Wallstén, Density theorems for sampling and interpolation in the Bargmann-Fock space. II, *J. Reine Angew. Math.* **429** (1992), 107–113. DOI: 10.1515/crll.1992.429.107.

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