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SOME NON-AMENABLE GROUPS

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Abstract: We generalise a result of R. Thomas to establish the non-vanishing of the first ℓ^2 Betti number for a class of finitely generated groups.

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In this note we give the following generalisation of a result of Richard Thomas [**8**].

Theorem 1. *Let G be a finitely generated group given by the presentation*

$$\langle x_1,\dots,x_d : u_1^{m_1},\dots,u_r^{m_r}\rangle$$

such that each relator u_i has order m_i in G.

(1) *If G is finite then $1-d+\sum_{i=1}^{r}\frac{1}{m_i}>0$ and $|G|\geq\frac{1}{1-d+\sum_{i=1}^{r}\frac{1}{m_i}}$.*

(2) *If the first ℓ^2 Betti number $\beta_1^2(G)$ of G is zero, then*

$$1-d+\sum_{i=1}^{r}\frac{1}{m_i}\geq 0.$$

In particular, the case when all the exponents m_i in the presentation are equal to 1 yields the well known observation that when the first ℓ^2 Betti number is zero the deficiency of the presentation $d-r$ must be at most 1. The vanishing of the first ℓ^2 Betti number of a group G holds for example if G is finite, if it satisfies Kazhdan's property (T) or if it admits an infinite normal amenable subgroup (in particular if it is infinite amenable). We refer to [**4**] for other interesting examples. We obtain as a corollary:

Corollary 2. *Let G be a finitely generated group given by the presentation*

$$\langle x_1,\dots,x_d : u_1^{m_1},\dots,u_r^{m_r}\rangle$$

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such that each relator u_i has order m_i in G. If $d > 1 + \sum_{i=1}^{r} \frac{1}{m_i}$, then G is infinite, does not satisfy Kazhdan's property (T) and has no amenable infinite normal subgroups.

Thomas established the inequality in (1) above by providing a simple but elegant computation of the dimension of the $\mathbb{F}_2$-vector space of 1-cycles of the cellular chain complex of the Cayley graph of G (Thomas refers to this space as the cycle space of Γ.) If Γ has d edges and v vertices then the dimension of this vector space is $d - v + 1$. An alternative approach, yielding information about the classical first Betti number of G and its finite index subgroups is explored by Allcock in [**1**].

We generalise this idea to give the additional inequality in (2) above by using elementary observations about the ℓ^2 Betti numbers β_i^2 of the orbihedral presentation 2-complex of G. For an introduction to ℓ^2 Betti numbers, we refer the reader to [**3**]. The first ℓ^2 Betti number vanishes for all finite groups. Cheeger and Gromov have shown that if a group G is amenable then $\beta_1^2(G) = 0$ [**2**, Theorem 0.2]. More generally, $\beta_1^2(G)$ is zero for any group G which contains an infinite normal amenable subgroup.

Remark 3. Theorem 1 can be derived from deeper results of Peterson and Thom; in particular, Equation (3) yields the inequality $\beta_1^2(G) \geq \frac{1}{|G|} + d - 1 - \sum_i \frac{1}{m_i}$ from [**7**]. Here, $|G|$ denotes the size of G and $\frac{1}{|G|}$ is understood to be zero when G is infinite.

Finitely generated but not finitely presented groups. Lück has defined ℓ^2 Betti numbers for any countable discrete group. The notion agrees with the cellular ℓ^2 Betti numbers for finitely presented groups and the basic properties including a generalised Euler-Poincaré formula for G-CW complexes may be found in Chapter 6 of [**6**]. Working in this context and arguing as in the proof of Theorem 1, we obtain the following generalisation.

Theorem 4. *Suppose a group G is given by the presentation*

$$G = \langle x_1, \ldots, x_d : u_i^{m_i},\ i \in I \rangle$$

where I is a countable set and each relator u_i has order m_i in G. If $\sum_{i \in I} \frac{1}{m_i}$ converges then $\beta_1^2(G) \geq \frac{1}{|G|} + d - 1 - \sum_{i \in I} \frac{1}{m_i}$. In particular if $\beta_1^2(G) = 0$ then $\sum_{i \in I} \frac{1}{m_i} - d + 1 \geq 0$.

Before we embark on the proof of Theorem 1, we need a short lemma which says that the orbihedral Euler characteristic of a G-CW complex Y may be computed from its ℓ^2 Betti numbers. The lemma is well known and may be found in [**6**].

Lemma 5 ([**6**, Theorem 6.80])**.** *If G acts on a connected CW complex $\tilde{Y}$ with finite quotient Y such that stabilisers of cells are finite, then the ℓ^2-Euler characteristic of Y is equal to the orbihedral Euler characteristic of Y. More precisely, if for each i, Σ_i is a choice of representatives for the orbits of i-cells in $\tilde{Y}$ and the stabiliser of a cell σ in G is written G_σ, then*

$$\sum_i (-1)^i \beta_i^2(Y) = \sum_i (-1)^i \sum_{\sigma \in \Sigma_i} \frac{1}{|G_\sigma|}. \tag{1}$$

We now proceed with the proof of Theorem 1.

Proof of Theorem 1: Let G be a group given by the presentation $\langle x_1, \ldots, x_d : u_1^{m_1}, \ldots, u_r^{m_r} \rangle$ where each relator u_i has order m_i in G. The orbihedral presentation 2-complex of G, which we will denote by $\mathcal{P}$, has one vertex and d edges forming a bouquet of d circles. Identifying each of the circles with one of the generators x_i we identify the fundamental group of this bouquet with the free group on $\{x_1, \ldots, x_d\}$. Attached to this are r discs, $\mathcal{D}_1, \ldots, \mathcal{D}_r$. For each $i = 1, \ldots, r$, the disc $\mathcal{D}_i$ is endowed with a cone point of cone angle $\frac{2\pi}{m_i}$ and its boundary is attached by a degree 1 map along the loop in the bouquet of circles corresponding to the element u_i.

Attaching the corresponding stabilisers to cells we obtain, in the language of Haefliger [**5**], a developable complex of groups, meaning that the orbihedral universal cover X of $\mathcal{P}$ exists. In fact, X has a simple description in terms of the Cayley graph $\mathcal{C}$ of G. The 1-skeleton of the orbihedral universal cover is the Cayley graph of G with respect to the generating set $\{x_1, \ldots, x_d\}$, while the 2-skeleton is obtained from the 2-skeleton of the topological universal cover of the presentation 2-complex by collapsing stacks of relator discs having common boundaries. Specifically, the relator $u_i^{m_i}$ corresponds to a loop γ_i in $\mathcal{P}$ bounding a disc and there is a unique lift $\tilde{\gamma}_i$ of γ_i based at the identity vertex in $\mathcal{C}$. In the topological universal cover of the presentation 2-complex there are additional copies of this disc (glued along the same loop) based at the elements $u_i, \ldots, u_i^{m_i - 1}$ and the action of the subgroup $\langle u_i \rangle$ permutes these discs so that each has trivial stabiliser. In contrast, these copies are identified in the orbihedral cover to give a single disc and it is preserved by the element u_i. The hypothesis that u_i has order m_i controls the order of the cell stabiliser.

We now apply the identity in (1) to our complex X. The action of G on the vertices and the edges of X is both free and transitive. On the other hand, by hypothesis, the stabiliser of a lift of a 2-cell $\mathcal{D}_i$ has order m_i. Hence, $\beta_0^2(\mathcal{P})-\beta_1^2(\mathcal{P})+\beta_2^2(\mathcal{P}) = 1-d+\sum_i \frac{1}{m_i}$. We also know that $\beta_0^2(\mathcal{P}) = \frac{1}{|G|}$ where $\frac{1}{|G|}$ is understood to be zero when G is infinite. Therefore,

$$\frac{1}{|G|} - \beta_1^2(\mathcal{P}) + \beta_2^2(\mathcal{P}) = 1 - d + \sum_i \frac{1}{m_i}. \tag{2}$$

Finally we remark that the first ℓ^2 Betti number of the group G may be computed as the first ℓ^2 Betti number of the orbihedral presentation complex used above. By definition, $\beta_1^2(G)$ is the von Neumann dimension of the first ℓ^2 homology group of Y with coefficients in the von-Neumann algebra of G, where Y is the universal cover of the (topological) presentation 2 complex for G. Since both X and Y are simply connected we deduce from Theorem 6.54(3) of [**6**] that $\beta_1^2(G) = \beta_1^2(\mathcal{P})$. Therefore, Equation (2) becomes

$$\frac{1}{|G|} - \beta_1^2(G) + \beta_2^2(\mathcal{P}) = 1 - d + \sum_i \frac{1}{m_i}. \tag{3}$$

Now assume that $\beta_1^2(G) = 0$. Since $\beta_2^2(\mathcal{P}) \geq 0$, we get the identity we are looking for, namely

$$1 - d + \sum_{i=1}^{r} \frac{1}{m_i} \geq \frac{1}{|G|}.$$

In particular, if G is finite, then the ℓ^2 cohomology of G is just the group cohomology with real coefficients, and this vanishes so we obtain Thomas's result that $1 - d + \sum_{i=1}^r \frac{1}{m_i} > 0$ and $|G| \geq \frac{1}{1-d+\sum_{i=1}^r \frac{1}{m_i}}$. On the other hand, if G is infinite and its first ℓ^2 Betti number is zero, in particular if G is an infinite amenable group, then we obtain the inequality $1 - d + \sum_{i=1}^r \frac{1}{m_i} \geq 0$, as required. □

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