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A DUALITY APPROACH TO THE FRACTIONAL LAPLACIAN WITH MEASURE DATA

Kenneth H. Karlsen, Francesco Petitta, and Suleyman Ulusoy

Abstract

We describe a duality method to prove both existence and uniqueness of solutions to nonlocal problems like

$$(-\Delta)^s v = \mu \quad \text{in } \mathbb{R}^N,$$

with vanishing conditions at infinity. Here μ is a bounded Radon measure whose support is compactly contained in $\mathbb{R}^N$, $N \geq 2$, and $-(\Delta)^s$ is the fractional Laplace operator of order $s \in (1/2, 1)$.

1. Introduction

The study of elliptic equations with measure data (or with L^1-data) is motivated by some engineering problems. See for instance [**3**], [**7**], [**8**], [**12**] for applications in electromagnetic induction heating, modeling of wells in porous media flow, and and the $k - \varepsilon$ model of turbulence.

For $N > 2$, where N denotes the dimension, the main mathematical difficulty in such kind of problems is that, it is not always possible to employ the classical variational methods for these problems.

Integro-partial differential equations, also referred to as fractional or Lévy partial differential equations, appear frequently in many different areas of research and find many applications in engineering and finance, including nonlinear acoustics, statistical mechanics, biology, fluid flow, pricing of financial instruments, and portfolio optimization. Though different approaches to these kind of problems are possible (e.g., Harmonic Analysis tools), in this note we propose a purely PDE method which is easy and flexible enough to be utilized in a variety of different contexts involving linear nonlocal operators and irregular data.

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Motivated by the applications mentioned above, for the sake of simplicity, here we focus on the study of the fractional Laplace problem

$$(1.1)\qquad \begin{aligned} &(-\Delta)^s u = \mu \quad \text{in } \mathbb{R}^N, \\ &u(x) \to 0 \qquad \text{as } |x| \to +\infty, \end{aligned}$$

where μ is a bounded, compactly supported Radon measure whose support is compactly contained in $\mathbb{R}^N$, $N \geq 2$.

In (1.1), $(-\Delta)^s$ is the fractional Laplacian of order $s \in (\frac{1}{2}, 1)$ defined, up to constants, as

$$(1.2)\qquad (-\Delta)^s v = P.V. \int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x-y|^{N+2s}}\, dy.$$

A proof of (1.2) for the case $N = 2$ can be found in [**4**]. The formula still remains valid in the general case $N > 2$ (see [**5**]).

Notice that, due to the singularity of the kernel, the right hand side of the previous expression is not well defined in general. Due to this fact we have to restrict ourselves to functions v belonging to some fractional Sobolev space $W^{\eta,p}_{\mathrm{loc}}(\mathbb{R}^N)$ (see Definition (2.1) below). In this context, as we will see, it is possible to give a precise meaning to the previous expression.

Due to the nonlocal character of the operator, we set the problem in the whole $\mathbb{R}^N$, so that some of the arguments turn out to have their own, not surprising, interpretation, also in this limit case $s = 1$.

Our main result is the following:

Theorem 1.1. *Let μ be a Radon measure with compact support in $\mathbb{R}^N$. Then there exists a unique duality solution u (see Definition 2.4 below) for problem* (1.1)*. Moreover, $u \in W^{1-\frac{2-2s}{q},q}_{\mathrm{loc}}(\mathbb{R}^N)$, for any $q < \frac{N+2-2s}{N+1-2s}$.*

We emphasize that the value of q is always less than 2, and that this is consistent with the classical theory of Dirichlet problems concerning linear elliptic equations with measure data on bounded domains (see for instance [**16**]). Furthermore, the choice to restrict ourselves to the case $1/2 < s < 1$ is made simply to clarify the statements and the proofs. Notice that, with our method, it is possible to treat the case of values s that are somewhat smaller than $\frac{1}{2}$. Indeed, we can allow for $s > \frac{1}{4}(2 + N - \sqrt{4 + N^2})$, that in particular implies $1 - \frac{2-2s}{q} > 0$.

On the other hand, the case $s = 1$ fall into the classical framework in the case of Dirichlet boundary value problems on bounded domains (see [**16**] again). Concerning the data, we will only consider bounded measures which are compactly supported in $\mathbb{R}^N$. This is crucial since our methods will rely on the use of auxiliary functions solving associated *dual problems* (namely the Riesz Potentials for the Fractional Laplace operator), so that the key role will be played by local estimates in suitable fractional Sobolev spaces gathered together with vanishing condition at infinity for these functions.

The paper is organized as follows: In Section 2 we introduce some preliminary facts and prove a first existence result. In Section 3 we establish a result regarding the existence of a duality solution in a suitable Lebesgue space. We conclude the proof of our main result, Theorem 1.1, in Section 4.

2. Preliminary facts and dual problem

We start this section by recalling some basic facts about *fractional Sobolev spaces*. For further details see [**6**], [**17**] (see also [**18**]).

Definition 2.1. For $0<\eta<1$ and $1\le p<\infty$, let Ω be an open domain of $\mathbb{R}^N$. We define the *fractional Sobolev space* $W^{\eta,p}(\Omega)$ as the set of all functions u in $L^p(\Omega)$ such that

$$\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+\eta p}}\,dx\,dy<\infty.$$

This space, endowed with the norm

$$\|u\|_{W^{\eta,p}(\Omega)}=\|u\|_{L^p(\Omega)}+\left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+\eta p}}\,dx\,dy\right)^{\frac{1}{p}}$$

is a Banach space. Moreover, we will say that u is in $W^{\eta,p}_{\mathrm{loc}}(\mathbb{R}^N)$ if $u\in W^{\eta,p}(B_R)$, for any ball B_R in $\mathbb{R}^N$.

Let us observe that, keeping an eye on the definition of fractional derivatives using the Fourier transform, we formally have

$$|D|^\eta u\in L^p(\Omega)\iff \iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+\eta p}}\,dx\,dy<\infty.$$

In our analysis, we shall also use the following Sobolev embedding theorem for fractional order spaces which we recall here in the form we need it, see for instance [**1**], [**17**].

Theorem 2.2 (Fractional Sobolev Embedding Theorem)**.** *Let B_R be a ball of radius R in $\mathbb{R}^N$. Then, there exists a constant C depending only on η and N, such that*

$$\|v\|_{L^{\overline{\gamma}}(B_R)}\le C\|v\|_{W^{\eta,p}(B_R)},\quad v\in C_0^\infty(B_R), \tag{2.1}$$

where $\overline{\gamma}=\frac{Np}{N-\eta p}$, with $p>1$ and $0<\eta<1$. Moreover, $W^{\eta,p}(B_R)$ is compactly embedded in $L^\gamma(B_R)$ for any $1\le\gamma<\overline{\gamma}$.

Let us come back to our nonlocal problem. We fix $\frac{1}{2} < s < 1$. We focus on the following elliptic nonlocal problem

$$(2.2)\qquad \begin{aligned} &(-\Delta)^s u = \mu \quad \text{in } \mathbb{R}^N \\ &u(x) \to 0, \qquad \text{as } |x| \to \infty. \end{aligned}$$

If μ is a smooth function, then a vast amount of theory has been developed for this type of problems [**9**], [**17**], [**11**], [**15**] and we refer to those for further details. Let us just recall that, if $\mu = f$ is sufficiently regular, for instance if f belongs to the Schwartz class $\mathcal{S}$, then a representation formula is available for solutions of (2.2), through the convolution with the Poisson kernel, namely there exists a constant $C_{N,s}$, such that

$$(2.3)\qquad w(x) = (-\Delta)^{-s} f = C_{N,s} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}}\, dy,$$

or, equivalently

$$(2.4)\qquad (-\Delta)^s w = f \quad \text{in } \mathbb{R}^N.$$

The formulation in (2.3), as in (1.2), can be obtained following again [**4**] and [**5**]. The function w is also called the Riesz potential of order s associated to the function f.

Moreover, it is easy to check that, if for instance f is compactly supported in $\mathbb{R}^N$, then $w(x) \to 0$ as $|x| \to \infty$.

For our aims, we need to specify a bit this general fact. In particular, we need a preliminary result concerning the regularity of functions w defined through (2.3) with less smooth f's, since these functions will be involved as auxiliary functions in our methods.

In the next lemma we collect some further properties of the Riesz potentials w defined in (2.3) in the case of $f \in L^\sigma(\mathbb{R}^N)$, with $\sigma > \frac{N}{2s}$, and compactly supported in $\mathbb{R}^N$.

Lemma 2.3. *Let w be defined as in* (2.3)*, and f in $L^\sigma(\mathbb{R}^N)$, with $\sigma > \frac{N}{2s}$. Moreover, assume that there exists $R > 0$ such that $f \equiv 0$ a.e. in B_R^c, the complement of B_R in $\mathbb{R}^N$. Then w satisfy*

i) $\|w\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,s,R,\sigma} \|f\|_{L^\sigma(\mathbb{R}^N)}$;

ii) *w is continuous on $\mathbb{R}^N$;*

iii) $\lim_{|x| \to +\infty} w(x) = 0$.

Proof: From the definition of w, using Hölder's inequality, we have

$$(2.5)\qquad |w(x)| \leq \|f\|_{L^\sigma(\mathbb{R}^N)} \left(\int_{B_R} \frac{dy}{|x-y|^{(N-2s)\sigma'}} \right)^{\frac{1}{\sigma'}} \leq C \|f\|_{L^\sigma(\mathbb{R}^N)}$$

since $\sigma > \frac{N}{2s}$ implies $(N-2s)\sigma' < N$, and so i) is proved.

To prove ii) just let $x_n \to x$ in $\mathbb{R}^N$. Then, it is easy to check that, since the Poisson Kernel is radially symmetric, the sequence $\frac{f(y)}{|x_n-y|^{N-2s}}$ is equiintegrable on B_R and so we can pass to the limit in n proving the continuity of w.

Now, observe that for a given $\varepsilon > 0$, we can choose $|x|$ large enough such that

$$\max_{y\in B_R} |x-y|^{-(N-2s)\sigma'} \leq \varepsilon.$$

This clearly implies, from (2.5)

$$|w(x)| \leq C\varepsilon,$$

for $|x| \gg R$ that is iii). □

Now we are in position to come back to the original problem (2.2). As we said, in order to deal with rough data, we will develop a duality argument as the one introduced by Stampacchia in [**16**] in order to treat linear elliptic operators in divergence form with Dirichlet boundary conditions. Here is the definition adapted to our case:

Definition 2.4. We say that a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is a *duality solution* for problem (2.2) if

$$\int_{\mathbb{R}^N} ug\,dx = \int_{\mathbb{R}^N} w\,d\mu, \tag{2.6}$$

for any $g \in C_0^\infty(\mathbb{R}^N)$, where w is defined by

$$w(x) = C_{N,s}\int_{\mathbb{R}^N} \frac{g(y)}{|x-y|^{N-2s}}\,dy. \tag{2.7}$$

Remark 2.5. It is important to observe that w defined through (2.7) is a duality solution of problem

$$(-\Delta)^s w = g \text{ in } \mathbb{R}^N, \quad w(x) \to 0, \text{ as } |x| \to \infty.$$

Indeed, if $h \in C_0^\infty(\mathbb{R}^N)$, then one can check that

$$\begin{aligned}\int_{\mathbb{R}^N} w(x)h(x)\,dx &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} C_{N,s}\frac{g(y)h(x)}{|x-y|^{N-2s}}\,dy\,dx \\ &= \int_{\mathbb{R}^N} g(x)\left(C_{N,s}\int_{\mathbb{R}^N}\frac{h(y)}{|x-y|^{N-2s}}\,dy\right)dx,\end{aligned}$$

that is the definition of *duality solution* for w. This fact allows us to better specify the meaning of "$u \to 0$ as $|x| \to \infty$" for a duality solution with measure data. As in the classical case, even if not explicitly stated in the definition, the decay at infinity for the solution is, in some sense, hidden in the formulation through the presence of the dual functions w and it turns out to be attained in the classical sense once the data are regular enough (see iii) of Lemma 2.3).

Remark 2.6. Let us also notice that, if everything is smooth, then a duality solution turns out to fall into the classical framework. For instance, it coincides with the distributional solution (see for instance [**10**] for its definition for the same problem. In fact, if $\mu = f$ is smooth, then for every $\varphi \in C_0^\infty(\mathbb{R}^N)$ we can formally compute

$$\int_{\mathbb{R}^N} f\varphi\, dx = \int_{\mathbb{R}^N} ug = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(y)-u(x))(\varphi(y)-\varphi(x))}{|x-y|^{N+2s}},$$

where $(-\Delta)^s\varphi = g$. That is, according with the definition

$$(-\Delta)^s u = f,$$

in the distributional sense.

3. A first existence result

Let us first prove the existence of a duality solution in a suitable Lebesgue space.

Theorem 3.1. *Let μ be a compactly supported Radon measure on $\mathbb{R}^N$. Then there exists a unique duality solution u for problem* (2.2)*. Moreover, $u \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$, for any $r < \frac{N}{N-2s}$.*

Remark 3.2. Notice that if $s \to 1^-$, we recover the classical optimal summability of elliptic equations with measure data, since in this case we have $u \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$, for any $q < \frac{N}{N-2}$. Moreover, the fundamental solution for the fractional Laplacian $\frac{c}{|x|^{N-2s}}$ (see for instance [**2**]) belongs to the same space around the origin. Therefore, since this is a solution with $\mu = \delta_0$ we have that our result is optimal.

Also observe that, in view of Theorem 3.1 (and also Lemma 2.3), Definition 2.4 makes sense (by density) for test functions g not only in $C_0^\infty(\mathbb{R}^N)$ but also in $L^\sigma_{\mathrm{loc}}(\mathbb{R}^N)$ with $\sigma > \frac{N}{2s}$.

Proof of Theorem 3.1: Let $R > 0$ and consider a ball B_R of radius R such that $\mathrm{supp}(\mu) \subset B_R$. For any $g \in C_0^\infty(B_R)$, let us define the following operator $T\colon C_0^\infty(B_R) \mapsto \mathbb{R}$ through

$$T(g) := \int_{\mathbb{R}^N} w(x)\, d\mu.$$

Thanks to Lemma 2.3, T is well defined, and we can write

$$|T(g)| \le \|w(x)\|_{L^\infty(\mathbb{R}^N)}\, |\mu|(\mathbb{R}^N) \le C\|g\|_{L^\sigma(B_R)},$$

where C depends only on μ, N, R, s and σ. Then for a fixed σ, T extends to a bounded continuous linear functional on $L^{\sigma}(B_R)$, so that by Riesz Representation Theorem, there exists a unique function $u \in L^{\sigma'}(B_R)$ such that

$$\int_{\mathbb{R}^N} w\,d\mu = \int_{\mathbb{R}^N} ug. \tag{3.1}$$

Notice that both of the integrals in (3.1) are actually computed on B_R. Therefore, for any $\sigma > \frac{N}{2s}$ we find a unique $u \in L^{\sigma'}(B_R)$ with $\sigma' = \frac{\sigma}{\sigma-1}$ (and so $\sigma' < \frac{N}{N-2s}$) such that (3.1) holds.

Now, if $R' > R$, we can follow the same argument to find a function $\hat{u} \in L^{\sigma'}(B_{R'})$ for all $\sigma' < \frac{N}{N-2s}$. An easy application of the fundamental theorem in the calculus of variations shows that actually $u = \hat{u}$ a.e. on B_R and this concludes the proof. □

4. Additional regularity and proof of Theorem 1.1

We shall prove some further regularity results for the duality solution of problem (2.2). Theorem 1.1 will follow immediately by combining Theorem 3.1 with the regularity result given in Theorem 4.2 below. In the proof of Theorem 4.2 we employ Young's inequality for convolutions: for the sake of the completeness we recall it here in the form we need it. Here

$$(f * g)(x) := \int_{\mathbb{R}^N} f(x-y)g(y)\,dy.$$

Lemma 4.1 ([**13**], [**14**])**.** *If $f \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, with $1 \leq p, q, r \leq \infty$. Then*

$$\|f * g\|_{L^r(\mathbb{R}^N)} \leq \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}.$$

Observe that the previous result can be applied, as a particular case, when $p = 1$ provided $r = q$. Let us state the main result of this section.

Theorem 4.2. *Let μ be a bounded, compactly supported Radon measure on $\mathbb{R}^N$. Then the duality solution of problem* (2.2) *found in Theorem 3.1 belongs to $u \in W_{\mathrm{loc}}^{1-\frac{2-2s}{q},q}(\mathbb{R}^N)$ for any $q < \frac{N+2-2s}{N+1-2s}$.*

Proof: Let us fix $R > 0$, such that $\operatorname{supp}(\mu) \subset B_R$ and let us approximate μ by smooth functions $f_n \in C_0^{\infty}(B_R)$ in the narrow topology of measures. This can be easily obtained by standard convolution arguments. Moreover we can choose f_n such that $\|f_n\|_{L^1(B_R)} \leq C|\mu|(\mathbb{R}^N)$.

Now, consider

$$u_n(x) = \int_{B_R} \frac{f_n(y)}{|x-y|^{N-2s}}\,dy,$$

where we recall that u_n is a (duality) solution of problem (2.2) with f_n as datum (see Remark 2.5).

Now, we claim that the following inequality is valid.

$$\|u_n\|_{W^{\eta,q}(B_R)} \le C, \tag{4.1}$$

where

$$\eta = 1 - \frac{2-2s}{q}, \quad q = \frac{\sigma'(N+2-2s)}{N+\sigma'}, \tag{4.2}$$

and σ is a real number such that $\sigma > \frac{N}{2s}$ with $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$.

Indeed, let $x \in B_R$ be fixed and ψ be a cut-off function that is compactly supported in $\mathbb{R}^N$ and such that $\psi \equiv 1$ on B_{2R}.

We then multiply $|D|^\eta u_n(x)$ by $\psi(x)$ and we apply convolution theorem to obtain after some algebra, that, up to constants

$$(|D|^\eta u_n(x))\,\psi(x) = \left(f_n(x) * \frac{1}{|x|^{N-2s+\eta}} \right) \psi(x).$$

Notice that $\psi(x) = \psi(x-y)$, for any $y \in B_R$. So that,

$$\left(f_n(x) * \frac{1}{|x|^{N-2s+\eta}} \right) \psi(x) = f_n(x) * \frac{\psi(x)}{|x|^{N-2s+\eta}}.$$

Therefore, we apply Lemma 4.1 at the function

$$f_n(x) * \frac{\psi(x)}{|x|^{N-2s+\eta}},$$

with $r = \frac{\sigma'(N+2-2s)}{N+\sigma'}$, $p = 1$ and thus $q = r = \frac{\sigma'(N+2-2s)}{N+\sigma'}$. Notice that, in order to apply the Lemma 4.1, we only need to know that $\frac{\psi}{|x|^{N-2s+\eta}} \in L^q(\mathbb{R}^N)$, which is true since $(N-2s+\eta)q < N$. Thus, recalling that f_n is bounded in $L^1(B_R)$, we finally obtain

$$\||D|^\eta u_n(x)\|_{L^q(B_R)} \le C.$$

The bound on u_n in $L^q(B_R)$ follows in a similar way. Indeed, we multiply u_n by the same the cut-off function ψ as before, and we observe that $\frac{\psi}{|x|^{N-2s}} \in L^q(\mathbb{R}^N)$ where q is as in (4.2) (observe that, from the previous calculations, $(N-2s)q < (N-2s+\eta)q < N$). Hence, recalling that f_n is uniformly bounded in $L^1(B_R)$ and using Young's inequality for convolutions (now with $p = 1$ and $r = q$), we get

$$\|u_n\|_{L^q(B_R)} \le C\|f_n\|_{L^1(B_R)}.$$

This concludes the proof of the claim (4.1).

Since the constant on the right hand side of (4.1) is independent of n, we can find a function v such that u_n converges along a subsequence to v

weakly in $W^{1-(2-2s)/q,q}(B_R)$ for any $q < \frac{N+2-2s}{N+1-2s}$. Moreover, thanks to Theorem 2.2 we deduce that a subsequence of u_n converges to v strongly in $L^\gamma(B_R)$ for any $1 \leq \gamma < 1 + \frac{2}{N-2s}$ and almost everywhere. By repeating the argument for any $R > 0$ we are allowed to pass to the limit in the duality formulation for u_n to prove that the limit v solves the problem. Finally, by subtracting the formulation of v from the one of u proved in Theorem 3.1 we easily deduce that $u = v$. □

Remark 4.3. Let us emphasize that $q > 1$ is approaching $\frac{N+2-2s}{N+1-2s}$ from below as σ approaches $\frac{N}{2s}$ from above. Notice also that if $s \to 1^-$ we recover the classical optimal summability for elliptic boundary value problems with measure data (see again [**16**]), since in this case we have $u \in W^{1,q}_{\text{loc}}(\mathbb{R}^N)$, for any $q < \frac{N}{N-1}$. Finally, as before, the result is optimal since a direct computation shows that the fundamental solution for the fractional Laplacian, $\frac{c}{|x|^{N-2s}}$ belongs to the space $W^{1-\frac{2-2s}{q},q}$, for any $q < \frac{N+2-2s}{N+1-2s}$ around the origin.

In this note, as a first step to study nonlocal elliptic equations with measure data, we considered the fractional Laplace equation (1.1). Employing the duality method, as the operator is linear, first introduced by Stampacchia [**16**], we introduce a solution concept and we prove existence, uniqueness and regularity of these solutions. The approach utilized in this paper can be used for other types of nonlocal operators. Indeed, the study of similar problems involving more general nonlocal operators will be the subject of future work of the same authors.

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Kenneth H. Karlsen:
Centre of Mathematics for Applications
University of Oslo
Blindern, NO-0316, Oslo
Norway
E-mail address: kennethk@math.uio.no

Francesco Petitta:
Departamento de Análisis Matemático
Universitat de València
46100, Burjassot, Valencia
Spain
E-mail address: francesco.petitta@uv.es

Suleyman Ulusoy:
Center for Scientific Computation and Mathematical Modeling (CSCAMM)
University of Maryland
College Park, MD, 20742
USA
E-mail address: SUlusoy@cscamm.umd.edu

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