SIMILARITY OF OPERATORS AND GEOMETRY OF EIGENVECTOR BUNDLES

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Abstract

We characterize the contractions that are similar to the backward shift in the Hardy space H^2. This characterization is given in terms of the geometry of the eigenvector bundles of the operators.

Contents

Notation

$:=$	equal by definition;
$\mathbb{C}$	the complex plane;
$\mathbb{D}$	the unit disk, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$;
$\mathbb{T}$	the unit circle, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$;
$\langle \cdot, \cdot \rangle_H$	inner product on the space H;

$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ ∂ and $\overline{\partial}$ derivatives: $\frac{\partial}{\partial z} := \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right)/2$, $\frac{\partial}{\partial \bar{z}} := \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right)/2$, the symbols ∂ and $\overline{\partial}$ are sometimes used;

Δ normalized Laplacian, $\Delta = \overline{\partial}\partial = \partial\overline{\partial} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$;

$\mathfrak{S}_2$ Hilbert-Schmidt class of operators;

$\|\cdot\|, |\cdot|$ norm; since we are dealing with matrix- and operator-valued functions, we will use the symbol $\|\cdot\|$ (usually with a subscript) for the norm in a function space, while $|\cdot|$ is used for the norm in the underlying vector (operator) space. Thus, for a vector-valued function f the symbol $\|f\|_2$ denotes its L^2-norm, but the symbol $|f|$ stands for the scalar-valued function whose value at a point z is the norm of the vector $f(z)$;

H^2, H^∞ Hardy classes of analytic functions,

$$H^p := \left\{ f \in L^p(\mathbb{T}) : \hat{f}(k) := \int_{\mathbb{T}} f(z)z^{-k}\frac{|dz|}{2\pi} = 0 \text{ for } k < 0 \right\}.$$

Hardy classes can be identified with the spaces of functions that are analytic in the unit disk $\mathbb{D}$: in particular, H^∞ is the space of all functions bounded and analytic in $\mathbb{D}$;

H_E^2 vector-valued Hardy class H^2 with values in E;

H_n^2 vector-valued Hardy class H^2 with values in $\mathbb{C}^n$;

$L_{E_* \to E}^\infty$ class of bounded functions on the unit circle $\mathbb{T}$ whose values are bounded operators from a Hilbert space E_* to another one E (the spaces E and E_* are not supposed to be related in any way);

$H_{E_* \to E}^\infty$ operator Hardy class of bounded analytic functions whose values are bounded operators from E_* to E:

$$\|F\|_\infty := \sup_{z \in \mathbb{D}} |F(z)| = \operatorname*{esssup}_{\xi \in \mathbb{T}} |F(\xi)|;$$

T_Φ Toeplitz operator with symbol Φ;

$\operatorname{tr} A$ trace of the operator A.

Throughout the paper all Hilbert spaces are assumed to be separable. We always assume that in any Hilbert space, an orthonormal basis is fixed so that any operator $A\colon E \to E_*$ can be identified with its matrix. Thus, besides the usual involution $A \mapsto A^*$ (A^* is the Hilbert-adjoint of A), we have two more: $A \mapsto A^T$ (transpose of the matrix) and $A \mapsto \overline{A}$ (complex conjugation of the matrix), so $A^* = (\overline{A})^T = \overline{A^T}$. Although everything in the paper can be presented in an invariant, "coordinate-free" form, the use of transposition and complex conjugation makes the notation easier and more transparent.

0. Introduction and result

The main objects of this paper are operators with complete analytic families of eigenvectors, the backward shift being one of the simplest examples of such operators. Classification of such operators up to unitary equivalence was completely done by M. J. Cowen and R. G. Douglas in [**4**]. They had shown, in particular, that if the eigenvector bundles of such operators are equivalent as Hermitian holomorphic vector bundles, then they are unitarily equivalent. They had also introduced numerous local criteria of the equivalence of the eigenvector bundles (and so the unitary equivalence of the corresponding operators).

We are interested in the problem of classification of such operators up to similarity. Let us recall that operators T_1 and T_2 are similar if there exists a bounded invertible operator A such that $T_1 = AT_2A^{-1}$. It was shown already in [**4**] that this problem is significantly more complicated than unitary classification; in particular, it was shown in [**4**] that similarity of operators cannot be expressed as a local condition on their eigenvector bundles. So, we restrict ourselves to a particular case of the general problem. Namely, we are interested in the case when an operator is similar to the backward shift S^* in the Hardy space H^2 (scalar- or vector-valued).

Let us recall that the backward shift S^* in the Hardy space H^2 is the adjoint of the *forward shift* S, $Sf(z) = zf(z)$, $f \in H^2$, and can be expressed as $S^*f(z) = (f(z) - f(0))/z$, $f \in H^2$. The same formulas can be used to define S^* on the vector Hardy space H^2_E. Sometimes, to emphasize that we are considering S^* in the vector Hardy class H^2_E we will use the notation S^*_E (or S^*_n if $\dim E = n$).

The eigenvectors of S^* are well known. Namely, the point spectrum (the set of eigenvalues) of S^* is the open unit disc $\mathbb{D}$, and $S^* f = \lambda f$, $f \in H_E^2$, $|\lambda| < 1$, if and only if f can be represented as

$$k_{\overline{\lambda}} e, \quad e \in E;$$

here k_λ denotes the reproducing kernel[1] of the (scalar) Hardy space H^2, $k_\lambda(z) := 1/(1 - \overline{\lambda} z)$. Probably the easiest way to see that is to represent $k_{\overline{\lambda}} e$ as the geometric series, $k_{\overline{\lambda}} e = \sum_0^\infty \lambda^k z^k e$.

We will also assume that the operator T is contractive, i.e., that $\|T\| \leq 1$. As one can see from our result, one cannot expect a simple solution for the general case.

One can say that for this case the problem was solved by B. Sz.-Nagy and C. Foiaş [7, Chapter 1.5], [9, Chapter 9.1], [10], [11] who proved that an operator A, $\|A\| \leq 1$, in a separable Hilbert space is similar to an isometry if and only if its characteristic function is left invertible in (operator-valued) H^∞. Under our assumptions about T the operator T^* is completely non-unitary, so the similarity of T^* to an isometry is equivalent to the similarity to the forward shift S, $S f(z) = zf$, $f \in H_E^2$, in some (generally vector-valued) H^2 space. Taking the adjoint operators, we obtain that T is similar to the backward shift S^* in some vector-valued space H^2 if and only if the characteristic function of the operator T^* is left invertible in H^∞.

However, we are interested in the description only in terms of the geometry of the eigenvector bundle, and the result of B. Sz.-Nagy and C. Foiaş does not give such a description.[2]

We assume the following about our linear operator T on a Hilbert space H:

(1) T is contractive, i.e., $\|T\| \leq 1$;

(2) $\dim \ker(T - \lambda I)$ is constant for all $\lambda \in \mathbb{D}$;

(3) $\mathrm{span}\{\ker(T - \lambda I) : \lambda \in \mathbb{D}\} = H$; and

(4) the subspaces $\mathcal{E}(\lambda) = \ker(T - \lambda I)$ depend analytically on the spectral parameter $\lambda \in \mathbb{D}$.

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[1] The function k_λ is called the reproducing kernel because $\langle f, k_\lambda \rangle = f(\lambda)$ for $f \in H^2$. This property explains why the notation k_λ and not $k_{\overline{\lambda}}$ is used for the function $1/(1 - \overline{\lambda} z)$.

[2] When the paper was ready for the publication we have learned about a paper [18], where problem of similarity and quasisimilarity was also considered from the point of view of geometry of the eigenvector bundle. Equivalence (1)⇔(2) from Theorem 0.1 was proved there, but nothing like curvature conditions (3), (4) was considered. We would like to thank Ron Douglas and Jaydeb Sarkar for bringing our attention to this paper.

Assumption (4) means that for each $\omega \in \mathbb{D}$ there exist a neighborhood U_ω of ω and a left invertible in L^∞ analytic operator-valued function F_ω defined on U_ω, $F_\omega(\lambda)\colon E_* \to E$, such that $\operatorname{ran} F_\omega(\lambda) = \mathcal{E}(\lambda)$. It is easy to see that $\dim E_*$ must be the same for all ω, so $\dim \mathcal{E}(\lambda) = \dim E_*$ for all $\lambda \in \mathbb{D}$, which means the condition (2) is redundant.

If $\dim E_* < \infty$, then the columns of $F_\omega(\lambda)$, $\lambda \in U_\omega$, form a basis in $\mathcal{E}(\lambda)$, so the disjoint union $\coprod_{\lambda \in \mathbb{D}} \mathcal{E}(\lambda) = \{(\lambda, v_\lambda) : \lambda \in \mathbb{D},\ v_\lambda \in \mathcal{E}(\lambda)\}$ is a holomorphic vector bundle over $\mathbb{D}$ (subbundle of the trivial bundle $\mathbb{D} \times H$) with the natural projection π, $\pi(\lambda, v_\lambda) = \lambda$.

In the case $\dim E_* = \infty$, the above statement can be used as a definition of a holomorphic vector bundle of infinite rank.

We will follow the usual agreements and write v_λ instead of (λ, v_λ), which simplifies the notation. Note, that the subspaces $\mathcal{E}(\lambda)$ inherit the metric from the Hilbert space H, so our bundle $\coprod_{\lambda \in \mathbb{D}} \mathcal{E}(\lambda)$ is a Hermitian holomorphic vector bundle.

One can state some assumptions about the operator T that guarantee that condition (4) holds. For example, it is proven in [**4**] that for a bounded linear operator $T\colon H \to H$ such that for all $\lambda \in \mathbb{D}$ the operator $T - \lambda I$ is Fredholm, $\operatorname{ran}(T - \lambda I) = H$ and $\dim \ker(T - \lambda I) \equiv \operatorname{const}$, condition (4) holds.

In order to state the result of the paper, we define on the unit disk $\mathbb{D}$ a projection-valued function $\Pi\colon \mathbb{D} \to B(H)$ that assigns to each $\lambda \in \mathbb{D}$, the orthogonal projection onto $\ker(T - \lambda I)$;

$$\Pi(\lambda) := P_{\ker(T - \lambda I)}.$$

This function is clearly C^∞ and even real analytic in the operator norm topology, but in what follows we will only need the fact that it is a $\mathcal{C}^2$ function, i.e., a function twice continuously differentiable (in the operator norm topology).

Let us also recall that if $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are two holomorphic vector bundles over the same set Ω, then a map $\Psi\colon \mathcal{E} \to \widetilde{\mathcal{E}}$ is called a *bundle map* if it is holomorphic, and for each $\lambda \in \Omega$ the restriction of Ψ onto the fiber $\mathcal{E}(\lambda) := \pi^{-1}(\lambda)$ is a linear transformation from $\mathcal{E}(\lambda)$ to $\widetilde{\mathcal{E}}(\lambda) = \widetilde{\pi}^{-1}(\lambda)$.

Now we are ready to state our main result:

Theorem 0.1. *Let T be a linear operator on a Hilbert space H under assumptions (1) through (4) such that $\dim \ker(T - \lambda I) = n < \infty$ for every $\lambda \in \mathbb{D}$. Let $\Pi(\lambda)$ be the orthogonal projection onto $\ker(T - \lambda I)$. Then the following statements are equivalent:*

(1) *T is similar to the backward shift operator S_n^* on H_n^2 via an invertible operator $A\colon H_n^2 \to H$.*

(2) *The eigenvector bundles of T and S_n^* are "uniformly equivalent", i.e., there exists a holomorphic bundle map bijection Ψ from the eigenvector bundle of S_n^* to that of T such that for some constant $c > 0$,*

$$\frac{1}{c}\|v_\lambda\|_{H_n^2} \leq \|\Psi(v_\lambda)\|_H \leq c\|v_\lambda\|_{H_n^2}$$

for all $v_\lambda \in \ker(S_n^* - \lambda I)$ and for all $\lambda \in \mathbb{D}$.

(3) *There exists a bounded subharmonic function φ such that*

$$\Delta\varphi(z) \geq \left|\frac{\partial\Pi(z)}{\partial z}\right|_{\mathfrak{S}_2}^2 - \frac{n}{(1-|z|^2)^2} \quad \text{for all } z \in \mathbb{D}.$$

(4) *The measure*

$$\left(\left|\frac{\partial\Pi(z)}{\partial z}\right|_{\mathfrak{S}_2}^2 - \frac{n}{(1-|z|^2)^2}\right)(1-|z|)\,dx\,dy$$

is Carleson and the estimate

$$\left(\left|\frac{\partial\Pi(z)}{\partial z}\right|_{\mathfrak{S}_2}^2 - \frac{n}{(1-|z|^2)^2}\right)^{\frac{1}{2}} \leq \frac{C}{1-|z|}$$

holds.

Remark 0.2. We will see in Section 1 that $\left|\frac{\partial\Pi(z)}{\partial z}\right|_{\mathfrak{S}_2}^2 - \frac{n}{(1-|z|^2)^2} \geq 0$.

Remark 0.3. Treating $\ker(T-\lambda I)$ as a subbundle of the trivial bundle $\mathbb{D}\times H$, one can see that $-\frac{\partial\Pi(\lambda)}{\partial\lambda}$ is its second fundamental form, so the mean curvature of the eigenvector bundle $\ker(T-\lambda)$ is $-\left|\frac{\partial\Pi(\lambda)}{\partial\lambda}\right|_{\mathfrak{S}_2}^2$. On the other hand, $-(1-|\lambda|^2)^{-2} = \Delta\ln(1-|\lambda|^2)$ is the curvature of the eigenvector bundle of S^*, so $-n(1-|\lambda|^2)^{-2}$ is the mean curvature of the eigenvector bundle of the backward shift S_n^* of multiplicity n.

Thus, statements (3) and (4) are about the mean curvatures of the eigenvector bundles of T and S_n^*.

Remark 0.4. Statement (3) of the theorem simply means that the Green potential

$$\mathcal{G}(\lambda) := \frac{2}{\pi}\iint_{\mathbb{D}} \ln\left|\frac{z-\lambda}{1-\overline{\lambda}z}\right|\left(\left|\frac{\partial\Pi(z)}{\partial z}\right|_{\mathfrak{S}_2}^2 - \frac{n}{(1-|z|^2)^2}\right)dx\,dy$$

is uniformly bounded inside the unit disk $\mathbb{D}$. Integrating separately over a small neighborhood of λ and the rest of $\mathbb{D}$, one can easily see that $(4) \Longrightarrow (3)$.

1. Preliminaries

1.1. Inner-outer factorization and invariant subspaces. Let us recall that an operator-valued function $F \in H^\infty_{E_* \to E}$ is called *inner* if $F(z)$ is an isometry a.e. on $\mathbb{T}$, and *outer* if $FH^2_{E_*}$ is dense in H^2_E. Every $F \in H^\infty_{E_* \to E}$ can be represented as $F = F_i F_o$ for an inner function $F_i \in H^\infty_{E_\# \to E}$, an outer function $F_o \in H^\infty_{E_* \to E_\#}$, and an auxiliary Hilbert space $E_\#$.

Let $S = S_E$ be the (forward) shift operator on H^2_E, $Sf = zf$, $f \in H^2_E$. The famous Beurling-Lax Theorem says that any non-zero S-invariant subspace $\mathcal{E} \subset H^2_E$, $S\mathcal{E} \subset \mathcal{E}$, can be represented as $\Theta H^2_{E_*}$, where E_* is an auxiliary Hilbert space and $\Theta \in H^\infty_{E_* \to E}$ is an inner function. The inner function Θ is unique up to a constant unitary factor on the right.

The backward shift S^*, $S^*f = (f(z) - f(0))/z$, $f \in H^2_E$, is the adjoint of S, so any non-trivial invariant subspace $K \subset H^2_E$ of S^* admits the representation $K = K_\Theta := H^2_E \ominus \Theta H^2_{E_*}$ with some inner function $\Theta \in H^\infty_{E_* \to E}$.

1.2. Tensor structure of the eigenvector bundle of T. The following theorem [**8**, Chapter 0.2] plays a critical role in what follows.

Theorem 1.1 (Model Theorem)**.** *Every contraction T on H with the property that $\lim_n \|A^n h\| = 0$ for every $h \in H$ is unitarily equivalent to $S^*_E \big| K$ for some Hilbert space E and an S^*_E-invariant subspace K of H^2_E.*

For our operator T we trivially have $\lim_n \|T^n h\| = 0$ for linear combinations of eigenvectors, which are dense in H by assumption (3). Since T is a contraction, $\|T^n\| \le 1$, so the standard $\varepsilon/3$ argument shows that the conditions of Theorem 1.1 are satisfied. So without loss of generality, we can assume that T is the restriction of the backward shift S^* in the vector Hardy space H^2_E (where E is an auxiliary Hilbert space) onto its invariant subspace $K \subset H^2_E$. If $K = H^2_E$ the operator T is the backward shift, so we only need to consider the case when K is a proper subspace of H^2_E. In this case K can be represented as $K = K_\Theta$, where $\Theta \in H^\infty_{E_* \to E}$ is an inner function.

Clearly an eigenvector of T is an eigenvector of S^*, and the eigenvectors of S^* are well known. As it was shown before in the Introduction the eigenspace $\ker(S^* - \lambda I)$ of the backward shift S^* in the

scalar Hardy space H^2 is spanned by the reproducing kernel $k_{\overline{\lambda}}$, where $k_{\lambda}(z) = 1/(1 - \overline{\lambda}z)$. So in the case of the backward shift S_E^* in H_E^2,

$$\ker(S_E^* - \lambda I) = \{k_{\overline{\lambda}}e : e \in E\},$$

and the eigenspaces of $T = S^*\big|K$ are given by

$$\ker(T - \lambda I) = k_{\overline{\lambda}}E(\lambda) = \{k_{\overline{\lambda}}e : e \in E(\lambda)\},$$

where the $E(\lambda)$ are some subspaces of the space E. Assumption (4) that $\ker(T - \lambda I)$ is a holomorphic vector bundle implies that the subspaces $E(\lambda)$ depend analytically on the spectral parameter λ, i.e., that the family of subspaces $E(\lambda)$ is a holomorphic vector bundle as well.

The vector-valued Hardy space H_E^2 is a natural realization of the tensor product $H^2 \otimes E$, so we can write

$$\ker(T - \lambda I) = k_{\overline{\lambda}} \otimes E(\lambda).$$

Remark 1.2. While it is not essential for the proof of the main result (Theorem 0.1), it is easy to see that $E(\lambda) = \ker\Theta(\overline{\lambda})^*$, where $\Theta \in H_{E_* \to E}^{\infty}$ is the inner function such that $K = K_\Theta$. Indeed, an eigenvector $k_{\overline{\lambda}}e$ belongs to K_Θ if and only if $k_{\overline{\lambda}}e \perp \Theta H_{E_*}^2$. Using the reproducing kernel property of k_λ we get that for $h \in H_{E_*}^2$,

$$(1.1) \qquad \langle \Theta h, k_{\overline{\lambda}}e \rangle_{H_E^2} = \langle \Theta(\overline{\lambda})h(\overline{\lambda}), e \rangle_E = \langle h(\overline{\lambda}), \Theta(\overline{\lambda})^*e \rangle_{E_*}.$$

Since $\{h(\overline{\lambda}) : h \in H_{E_*}^2\} = E_*$ for each $\lambda \in \mathbb{D}$, we conclude that $k_{\overline{\lambda}}e \perp \Theta H_{E_*}^2$ iff $\Theta(\overline{\lambda})^*e = 0$. Therefore, $E(\lambda) = \ker\Theta(\overline{\lambda})^*$.

Remark 1.3. The inner function $\Theta \in H_{E_* \to E}^{\infty}$ appearing above is the characteristic function of the operator T^*, and therefore the spaces E_* and E can be identified with $\operatorname{clos}(I - TT^*)^{\frac{1}{2}}H$ and $\operatorname{clos}(I - T^*T)^{\frac{1}{2}}H$, respectively [**9**, Chapter 6.2].

1.3. Curvature of the eigenvector bundle of T. Let us compute the norm $\big|\partial\Pi/\partial z\big|_{\mathfrak{S}_2}^2$, where $\Pi(z)$ is the orthogonal projection onto $\ker(T - zI)$. As we mentioned before, this expression is (up to the sign) the mean curvature of the eigenvector bundle of T.

Using the tensor structure $\ker(T - \lambda I) = k_{\overline{\lambda}} \otimes E(\lambda)$ one can represent $\Pi(\lambda)$ as

$$(1.2) \qquad\qquad \Pi(\lambda) = \Pi_1(\lambda) \otimes \Pi_2(\lambda),$$

where $\Pi_1(\lambda)$ is the orthogonal projection in the (scalar) space H^2 onto $\operatorname{span}\{k_{\overline{\lambda}}\}$, and $\Pi_2(\lambda)$ is the orthogonal projection in E onto $E(\lambda)$.

Lemma 1.4. *In the above notation, if* $\operatorname{rank}\Pi(\lambda)(=\operatorname{rank}\Pi_2(\lambda)) = n <$ *∞, then*

$$\left|\frac{\partial\Pi(\lambda)}{\partial\lambda}\right|_{\mathfrak{S}_2}^2 = \left|\frac{\partial\Pi_1(\lambda)}{\partial\lambda}\right|_{\mathfrak{S}_2}^2 + \left|\frac{\partial\Pi_2(\lambda)}{\partial\lambda}\right|_{\mathfrak{S}_2}^2$$

$$= \frac{n}{(1-|z|^2)^2} + \left|\frac{\partial\Pi_2(\lambda)}{\partial\lambda}\right|_{\mathfrak{S}_2}^2.$$

To prove this lemma we will need a couple of well known and simple facts.

Lemma 1.5. *Let* $E(\lambda)$, $\lambda \in \mathbb{D}$, *be an analytic family of subspaces (holomorphic vector bundle), and let* $\Pi(\lambda)$ *be the orthogonal projection onto* $E(\lambda)$. *Then*

$$(I - \Pi(z))\frac{\partial\Pi(z)}{\partial z}\Pi(z) = \frac{\partial\Pi(z)}{\partial z}.$$

This lemma is a well known fact in complex differential geometry, but for the sake of completeness we present the proof.

Proof of Lemma 1.5: The fact that the family of subspaces $E(\lambda)$ is a holomorphic vector bundle means that locally the subspaces $E(\lambda)$ can be represented as $\operatorname{ran} F(\lambda)$ where F is a left invertible analytic operator-valued function. Given such a representation one can write the formula for Π, $\Pi = F(F^*F)^{-1}F^*$. Direct computation shows that

$$\frac{\partial\Pi(z)}{\partial z} = (I - \Pi(z))F'(z)(F(z)^*F(z))^{-1}F(z)^*$$

and the conclusion of the lemma follows immediately. □

Lemma 1.6. *For operators A and B in the Hilbert-Schmidt class $\mathfrak{S}_2$,*

$$\|A \otimes B\|_{\mathfrak{S}_2}^2 = \|A\|_{\mathfrak{S}_2}^2\|B\|_{\mathfrak{S}_2}^2.$$

This lemma is well known and the proof is an easy exercise, so we omit it.

Proof of Lemma 1.4: Using the product rule we get from (1.2),

$$(1.3) \qquad \frac{\partial\Pi(\lambda)}{\partial\lambda} = \frac{\partial\Pi_1(\lambda)}{\partial\lambda} \otimes \Pi_2(\lambda) + \Pi_1(\lambda) \otimes \frac{\partial\Pi_2(\lambda)}{\partial\lambda} =: X + Y.$$

The identity $\Pi_2(\lambda)\frac{\partial\Pi_2(\lambda)}{\partial\lambda} = 0$ (see Lemma 1.5) implies that $X^*Y = 0$. Therefore

$$|X + Y|_{\mathfrak{S}_2}^2 = \operatorname{tr}X^*X + \operatorname{tr}Y^*Y + 2\operatorname{Re}\operatorname{tr}(X^*Y) = \operatorname{tr}X^*X + \operatorname{tr}Y^*Y$$

$$= |X|_{\mathfrak{S}_2}^2 + |Y|_{\mathfrak{S}_2}^2.$$

Applying Lemma 1.6 to each term and recalling that for an orthogonal projection P we have $|P|^2_{\mathfrak{S}_2} = \operatorname{rank} P$, we get that

$$\left|\frac{\partial \Pi(\lambda)}{\partial \lambda}\right|^2_{\mathfrak{S}_2} = n \left|\frac{\partial \Pi_1(\lambda)}{\partial \lambda}\right|^2_{\mathfrak{S}_2} + \left|\frac{\partial \Pi_2(\lambda)}{\partial \lambda}\right|^2_{\mathfrak{S}_2}.$$

The lemma is proved modulo computation of $\left|\frac{\partial \Pi_1(\lambda)}{\partial \lambda}\right|^2_{\mathfrak{S}_2}$, which is done in the next lemma. $\qquad\square$

Lemma 1.7. *Let* $\Pi_1(\lambda)$ *be the orthogonal projection onto* $\operatorname{span}\{k_{\overline{\lambda}}\}$ *in* H^2 *(scalar valued). Then*

$$\left|\frac{\partial \Pi_1(\lambda)}{\partial \lambda}\right|^2_{\mathfrak{S}_2} = (1 - |\lambda|^2)^{-2} \quad \text{for all } \lambda \in \mathbb{D}.$$

Proof of Lemma 1.7: The proof can be a simple exercise in complex differential geometry, using the fact that the quantity in question is (up to the sign) the curvature of the eigenvector bundle of S^*; see Remark 0.3.

However, for the convenience of the reader we present a direct computation (one of the many possible).

First, recall that k_λ is the *reproducing kernel* of H^2, i.e., $\langle f, k_\lambda \rangle = f(\lambda)$ for all $f \in H^2$. Using the reproducing kernel property of k_λ we conclude that $\|k_\lambda\|^2_2 = \langle k_\lambda, k_\lambda \rangle = (1 - |\lambda|^2)^{-1}$. Therefore for $f \in H^2$,

$$\Pi_1(\lambda)f = \|k_{\overline{\lambda}}\|^{-2}_2 \langle f, k_{\overline{\lambda}} \rangle k_{\overline{\lambda}} = (1 - |\lambda|^2) f(\overline{\lambda}) k_{\overline{\lambda}}.$$

Taking $\frac{\partial}{\partial \lambda}$ and using the fact that $\frac{\partial f(\overline{\lambda})}{\partial \lambda} = 0$ we get

$$(1.4) \qquad \frac{\partial \Pi_1(\lambda)}{\partial \lambda} f = f(\overline{\lambda}) \left(-\overline{\lambda} k_{\overline{\lambda}} + (1 - |\lambda|^2) \widetilde{k}_{\overline{\lambda}} \right),$$

where

$$\widetilde{k}_{\overline{\lambda}}(z) = \frac{\partial}{\partial \lambda} k_{\overline{\lambda}}(z) = \frac{z}{(1 - \lambda z)^2}.$$

Note, that for $f \in H^2$,

$$(1.5) \qquad\qquad\qquad \langle f, \widetilde{k}_\lambda \rangle = f'(\lambda).$$

Using this identity one can get that

$$\|\widetilde{k}_\lambda\|^2_2 = \frac{1 + |\lambda|^2}{(1 - |\lambda|^2)^3} = \|\widetilde{k}_{\overline{\lambda}}\|^2_2.$$

The reproducing property for k_λ implies that

$$\langle \widetilde{k}_{\overline{\lambda}}, k_{\overline{\lambda}} \rangle = \frac{\overline{\lambda}}{(1 - |\lambda|^2)^2}.$$

Combining all these together we can conclude that

$$\| -\overline{\lambda}k_{\overline{\lambda}} + (1 - |\lambda|^2)\widetilde{k}_{\overline{\lambda}}\|_2^2 = (1 - |\lambda|^2)^{-1}.$$

Since $f(\overline{\lambda}) = \langle f, k_{\overline{\lambda}}\rangle$, and as we discussed above $\|k_{\overline{\lambda}}\|_2^2 = \|k_\lambda\|_2^2 = (1 - |\lambda|^2)^{-1}$,

$$\left|\frac{\partial\Pi_1(\lambda)}{\partial\lambda}\right|^2 = (1 - |\lambda|^2)^{-2}.$$

Lastly, by Lemma 1.5, $\frac{\partial\Pi_1(z)}{\partial z} = (I - \Pi_1(z))\frac{\partial\Pi_1(z)}{\partial z}\Pi_1(z)$ and rank $\Pi_1(z) = 1$, so $\frac{\partial\Pi_1(\lambda)}{\partial\lambda}$ is a rank one operator (one can also see it from (1.4)). Therefore the operator and Hilbert–Schmidt norms of $\frac{\partial\Pi_1(\lambda)}{\partial\lambda}$ coincide. $\square$

2. From uniform equivalence of bundles to curvature condition

In this section we are going to prove the implication (2) $\implies$ (4).
Note that according to Lemma 1.4

$$\left|\frac{\partial\Pi_2(\lambda)}{\partial\lambda}\right|^2_{\mathfrak{S}_2} = \left|\frac{\partial\Pi(\lambda)}{\partial\lambda}\right|^2_{\mathfrak{S}_2} - \frac{n}{(1 - |z|^2)^2},$$

so $\left|\frac{\partial\Pi(\lambda)}{\partial\lambda}\right|^2_{\mathfrak{S}_2} - \frac{n}{(1-|z|^2)^2}$ in statements (3) and (4) of Theorem 0.1 can be replaced by $\left|\frac{\partial\Pi_2(\lambda)}{\partial\lambda}\right|^2_{\mathfrak{S}_2}$.

Let Ψ be the uniformly equivalent bundle map bijection, as in condition (2). A bundle map means that Ψ is an analytic function of λ, maps the fiber $\ker(S_n^* - \lambda I)$ to the fiber $\ker(T - \lambda I)$ and is linear in each fiber $\ker(S_n^* - \lambda I)$.

It is easy to see from the descriptions of $\ker(S_n^* - \lambda I)$ and $\ker(T - \lambda I)$ that any such bundle map bijection is represented by

$$\Psi(k_{\overline{\lambda}}e) = k_{\overline{\lambda}} \cdot F(\lambda)e \quad \forall\, e \in \mathbb{C}^n,$$

where $F \in H^\infty_{\mathbb{C}^n \to E}$ is an operator-valued function such that

$$\operatorname{ran} F(\lambda) = E(\lambda)\ (= \ker\Theta(\overline{\lambda})^*).$$

The "uniform equivalence" property of Ψ means that

(2.1) $$c^{-1}I \le F^*F \le cI \quad \forall\, z \in \mathbb{D}.$$

Hence the orthogonal projection $\Pi_2(\lambda)$ from E onto $E(\lambda)$ can be written down as

$$\Pi_2 = F(F^*F)^{-1}F^*.$$

Differentiating we get $\frac{\partial \Pi_2(z)}{\partial z} = (I - \Pi_2(z))F'(z)(F(z)^*F(z))^{-1}F(z)^*$, and taking (2.1) into account we have

$$(2.2) \qquad \left| \frac{\partial \Pi_2(z)}{\partial z} \right| \leq C \lVert F'(z) \rVert .$$

The function F takes values in the Hilbert-Schmidt class $\mathfrak{S}_2$ which is a Hilbert space, and for bounded analytic functions with values in a Hilbert space the estimate

$$(2.3) \qquad \lVert F'(z) \rVert \leq C/(1 - |z|)$$

holds, and the measure

$$(2.4) \qquad \lVert F'(z) \rVert^2 (1 - |z|)\, dx\, dy$$

is Carleson. Combining these facts with (2.2) we conclude that the curvature condition (4) holds.

Remark 2.1. Estimates (2.3) and (2.4) are very well-known for scalar-valued analytic functions, but the fact that they hold for bounded analytic functions with values in a Hilbert space probably requires some explanation. First of all, the estimate (2.3) holds for bounded analytic functions with values in a Banach space X; it follows immediately from the estimate for the scalar-valued functions if one picks $x^* = x^*(z)$, $\lVert x^* \rVert = 1$ in the dual space X^* such that $\langle F'(z), x^* \rangle = \lVert F'(z) \rVert$.

The fact that the measure (2.4) is Carleson is not true for functions in an arbitrary Banach space: on can easily construct a bounded analytic function F with values in ℓ^∞ for which the measure $\lVert F'(z) \rVert^2 (1 - |z|)\, dx\, dy$ is not Carleson.

However, for functions F with values in a Hilbert space the measure is Carleson; any standard proof of this fact for the scalar-valued functions works for the case of functions with values in a Hilbert space.

Another way to see that is to use the so-called *Uchiyama's Lemma*, cf [**8**, p. 290, Lemma 6], which states that for a bounded subharmonic function u the measure $\Delta u(z)(1 - |z|)\, dx\, dy$ is Carleson. Applying this Lemma to the function $u(z) = \lVert F(z) \rVert^2$ and noticing that for the function F with values in a Hilbert space $\Delta \lVert F(z) \rVert^2 = \lVert F'(z) \rVert^2$ one immediately gets the desired result.

3. Curvature condition implies similarity

As we already mentioned in the Introduction (see Remark 0.4 there), it is easy to show that condition (4) implies condition (3).

As it was already discussed in the beginning of the previous section, the expression $\left|\frac{\partial \Pi(\lambda)}{\partial \lambda}\right|_{\mathfrak{S}_2}^2 - \frac{n}{(1-|z|^2)^2}$ in statements (3) and (4) of Theorem 0.1 can be replaced by $\left|\frac{\partial \Pi_2(\lambda)}{\partial \lambda}\right|_{\mathfrak{S}_2}^2$. So, the implication (3) $\Longrightarrow$ (1) follows from the theorem below, which holds even in the case $\dim E(\lambda) = \infty$.

Theorem 3.1. *Let $E(\lambda)$, $\lambda \in \mathbb{D}$, be an analytic family of subspaces of a Hilbert space E, and let $\Pi(\lambda)$ be the orthogonal projection onto $E(\lambda)$. Let*

$$T = S_E^*\big|K,$$

where K is the S^*-invariant subspace of H_E^2, $K := \operatorname{span}\{k_{\overline{\lambda}}e : \lambda \in \mathbb{D},\, e \in E(\lambda)\}$. Suppose that there exists a bounded, subharmonic function φ such that

$$\Delta\varphi(z) \geq \left|\frac{\partial \Pi(z)}{\partial z}\right|^2 \quad \forall\, z \in \mathbb{D}.$$

Then T is similar to the backward shift $S_{E_*}^*$, where E_* is an auxiliary Hilbert space and $\dim E_* = \dim E(\lambda)$.

3.1. Toeplitz operators. To prove Theorem 3.1 we will need to recall some simple facts about Toeplitz operators. Let us recall that given an operator-valued function $F \in L_{E \to E_*}^\infty$, the Toeplitz operator $T_F : H_E^2 \to H_{E_*}^2$ with symbol F is defined by the formula

$$T_F f = P_+(Ff) \quad f \in H_E^2,$$

where P_+ is the orthogonal projection in L^2 onto H^2.

If the symbol F is analytic ($F \in H_{E \to E_*}^\infty$), then the Toeplitz operator T_F is simply the multiplication operator (more precisely, its restriction onto H^2).

It is easy to see that if $F, G \in H^\infty$, then

$$(3.1) \qquad\qquad\qquad T_F T_G = T_{FG}.$$

We will also need the following well known and easy to prove facts that if $F \in H_{E \to E_*}^\infty$, then

$$(3.2) \qquad\qquad T_{F^*} k_\lambda e = k_\lambda F^*(\lambda)e \quad e \in E_*,$$

and

$$(3.3) \qquad\qquad\qquad T_{F^*} S_{E_*}^* = S_E^* T_{F^*}.$$

3.2. Proof of Theorem 3.1. We want to prove the existence of an invertible operator $A\colon H_n^2 \to K$ satisfying the intertwining relation $AS_n^* = TA$ (recall that $T = S_E^*|K$).

We will need the following theorem by S. R. Treil and B. D. Wick [**17**].

Theorem 3.2. *Let* $\Pi\colon \mathbb{D} \to B(E)$ *be a* $\mathcal{C}^2$ *function whose values are orthogonal projections in* E, *satisfying the identity* $\Pi(z)\frac{\partial \Pi(z)}{\partial z} = 0$. *Assume that for some bounded subharmonic function* φ, *we have*

$$\Delta\varphi(z) \geq \left|\frac{\partial \Pi(z)}{\partial z}\right|^2 \quad \text{for all } z \in \mathbb{D}.$$

Then there exists a bounded analytic projection onto $\operatorname{ran} \Pi(z)$, *i.e., a function* $\mathcal{P} \in H_{E \to E}^\infty$ *such that* $\mathcal{P}(z)$ *is a projection onto* $\operatorname{ran} \Pi(z)$ *for all* $z \in \mathbb{D}$.

By Lemma 1.5, the function Π from Theorem 3.1 satisfies the identity $\Pi(z)\frac{\partial \Pi(z)}{\partial z} = 0$, so applying Theorem 3.2 to it we get a bounded analytic projection $\mathcal{P}(z)$ onto $\operatorname{ran} \Pi(z)$. Consider the inner-outer factorization $\mathcal{P} = \mathcal{P}_i\mathcal{P}_o$ of $\mathcal{P}$, where $\mathcal{P}_i \in H_{E_* \to E}^\infty$ is the inner part and $\mathcal{P}_o \in H_{E \to E_*}^\infty$ is the outer part of $\mathcal{P}$. Next define a function $\mathcal{P}_i^\sharp$ by $\mathcal{P}_i^\sharp(z) := \mathcal{P}_i(\overline{z})$, and consider the Toeplitz operator $T_{\mathcal{P}_i^\sharp}$. Since $(\mathcal{P}_i^\sharp)^* \in H_{E \to E_*}^\infty$, (3.3) implies

$$(3.4) \qquad\qquad T_{\mathcal{P}_i^\sharp}S_{E_*}^* = S_E^* T_{\mathcal{P}_i^\sharp}.$$

If we show that the operator $T_{\mathcal{P}_i^\sharp}$ is left invertible and that $\operatorname{ran} T_{\mathcal{P}_i^\sharp} = K$, we are done; the operator A we want to find is simply the Toeplitz operator $T_{\mathcal{P}_i^\sharp}$ treated as an operator $H_{E_*}^2 \to K$. The left invertibility of $T_{\mathcal{P}_i^\sharp}$ together with $\operatorname{ran} T_{\mathcal{P}_i^\sharp} = K$ means that A is invertible, and the intertwining $AS_{E_*}^* = TA$ follows from (3.4).

The left invertibility of $T_{\mathcal{P}_i^\sharp}$ is a corollary of the following lemma.

Lemma 3.3. *The outer part* $\mathcal{P}_o$ *is a left inverse of* $\mathcal{P}_i$, *i.e.,* $\mathcal{P}_o\mathcal{P}_i \equiv I$ *for all* $z \in \mathbb{D}$.

This lemma immediately implies that $T_{\mathcal{P}_o^\sharp}$, where $\mathcal{P}_o^\sharp(z) := \mathcal{P}_o(\overline{z})$, is a left inverse of $T_{\mathcal{P}_i^\sharp}$. Indeed

$$(T_{\mathcal{P}_i^\sharp})^*(T_{\mathcal{P}_o^\sharp})^* = T_{(\mathcal{P}_i^\sharp)^*}T_{(\mathcal{P}_o^\sharp)^*} = T_{(\mathcal{P}_i^\sharp)^*(\mathcal{P}_o^\sharp)^*} = I.$$

The last equality holds because $\mathcal{P}_o\mathcal{P}_i \equiv I$ and $T_I = I$; the previous one follows from (3.1) because $(\mathcal{P}_i^\sharp)^*, (\mathcal{P}_o^\sharp)^* \in H^\infty$.

Proof of Lemma 3.3: It follows from (3.1) that

$$T_{\mathcal{P}_i}T_{\mathcal{P}_o} = T_{\mathcal{P}} = T_{\mathcal{P}^2} = T_{\mathcal{P}_i\mathcal{P}_o\mathcal{P}_i\mathcal{P}_o} = T_{\mathcal{P}_i}T_{\mathcal{P}_o\mathcal{P}_i}T_{\mathcal{P}_o}.$$

The operator $T_{\mathcal{P}_o}$ has dense range because $\mathcal{P}_o$ is outer, and $\ker T_{\mathcal{P}_i} = \{0\}$ because $\mathcal{P}_i$ is inner (in fact, $T_{\mathcal{P}_i}$ is an isometry). Therefore $T_{\mathcal{P}_o\mathcal{P}_i} = I$, so $\mathcal{P}_o\mathcal{P}_i \equiv I$ for all $z \in \mathbb{D}$. $\square$

To complete the proof of Theorem 3.1 it remains to show that $\operatorname{ran} T_{\mathcal{P}_i^\sharp} = K$. First of all, let us notice that

$$\operatorname{ran} \mathcal{P}_i(\lambda) = E(\lambda) \quad \forall \, \lambda \in \mathbb{D}.$$

Indeed, the inclusion $E(\lambda) = \operatorname{ran} \mathcal{P}(\lambda) \subset \operatorname{ran} \mathcal{P}_i(\lambda)$ is trivial because of the factorization $\mathcal{P} = \mathcal{P}_i\mathcal{P}_o$. To prove the other inclusion, observe that since $\mathcal{P}_o$ is outer, the set $\operatorname{ran} \mathcal{P}_o(\lambda)$ is dense in E_* for all $\lambda \in \mathbb{D}$. But $\mathcal{P}_i(\lambda)\operatorname{ran} \mathcal{P}_o(\lambda) = E(\lambda)$, so $\operatorname{ran} \mathcal{P}_i(\lambda) \subset E(\lambda)$.

It follows from (3.2) that

(3.5) $$T_{\mathcal{P}_i^\sharp}k_{\overline{\lambda}}e = k_{\overline{\lambda}}\mathcal{P}_i^\sharp(\overline{\lambda})e = k_{\overline{\lambda}}\mathcal{P}_i(\lambda)e.$$

To see that we got all the complex conjugates correctly, one can fix bases in E and E_*, consider the matrix representation of the operators, and notice that $\mathcal{P}^\sharp(z) = (\mathcal{P}(z)^T)^*$. Then a direct application of (3.2) implies (3.5).

The equality $\operatorname{ran} \mathcal{P}_i(\lambda) = E(\lambda)$ together with (3.5) imply that

$$T_{\mathcal{P}_i^\sharp}k_{\overline{\lambda}} \otimes E_* = k_{\overline{\lambda}} \otimes E(\lambda).$$

Since $\operatorname{span}\{k_{\overline{\lambda}} \otimes E_* : \lambda \in \mathbb{D}\} = H_{E_*}^2$, $\operatorname{span}\{k_{\overline{\lambda}} \otimes E(\lambda) : \lambda \in \mathbb{D}\} = K$, and the operator $T_{\mathcal{P}_i^\sharp}$ is left invertible, we conclude that $\operatorname{ran} T_{\mathcal{P}_i^\sharp} = K$. $\square$

4. Connection with a result by B. Sz.-Nagy and C. Foiaş

We already mentioned in the Introduction the following result by B. Sz.-Nagy and C. Foiaş [**7**, Chapter 1.5], [**9**, Chapter 9.1], [**10**], [**11**].

Theorem 4.1. *A contraction A ($\|A\| \le 1$) in a Hilbert space is similar to an isometry if and only if its characteristic function is left invertible in H^∞.*

We are not giving the definition of the characteristic function of a contraction here, because it is quite technical and is not relevant to our paper. The reader only needs to know that if $A = T^*$, where T is the restriction of the backward shift S^* onto an S^*-invariant subspace $K = K_\Theta \subset H_E^2$, then the inner function Θ is the characteristic function of A.

If the operator $A = T^*$ is unitarily equivalent to an isometry U, the isometry U must be unitarily equivalent to the forward shift S_{E_*} in $H^2_{E_*}$. Indeed, since $\lim_n \|T^n f\| = 0$ for all $f \in K$, the same holds for U^*; $\lim_n \|(U^*)^n x\| = 0$ for all x. But it is a well known fact (an easy corollary of the Kolmogorov-Wold decomposition of isometries) that such isometry U is unitarily equivalent to the forward shift S in $H^2_{E_*}$. So, applying Theorem 4.1 to our situation we get that T is similar to the backward shift if and only if the inner function Θ is left invertible in H^∞.

We would like to investigate the relation between this statement and our result. The following lemma due to N. K. Nikolski provides necessary connection. It was communicated to the second author by N. K. Nikolski; together with the proof (due to N. K. Nikolski) it can be found in [**15**, Lemma 6.1] and [**17**, Lemma 0.1].

Lemma 4.2. *Let $F \in H^\infty_{E_* \to E}$ satisfy*

$$F(z)^* F(z) \geq \delta^2 I \quad for \ all \ z \in \mathbb{D}.$$

Then F is left invertible in H^∞, i.e., there exists a $G \in H^\infty_{E \to E_*}$ such that $GF \equiv I$, if and only if there exists a function $\mathcal{P} \in H^\infty_{E \to E}$ whose values are projections (not necessarily orthogonal) onto $\operatorname{ran} F(z)$ for all $z \in \mathbb{D}$.

By this lemma, T is similar to a backward shift if and only if there exists a bounded analytic projection $\mathcal{P}(z)$ onto $\operatorname{ran} \Theta(z)$. Let $\mathcal{Q} = I - \mathcal{P}$ be the complementary projection. Then $\mathcal{Q}(z)^*$ is a projection onto $(\operatorname{ran} \Theta(z))^\perp = \ker \Theta(z)^*$. But as we discussed above in Section 1.2, $\ker \Theta(z)^* = E(\overline{z})$.

Notice, that the function $z \mapsto \mathcal{Q}(z)^*$ is antianalytic, so T is similar to the backward shift iff there exists a bounded antianalytic projection onto $\ker \Theta(z)^*$, or equivalently, a bounded analytic projection onto $E(z) = \ker \Theta(\overline{z})^*$.

Of course this is only a sketch and we leave all the details to the reader as an exercise.

5. Remark about assumption $\|T\| \leq 1$

In this section we will show that the assumption $\|T\| \leq 1$ is essential for Theorem 0.1. We will show that if we omit this assumption, it is possible to construct an operator T whose eigenvector bundle is uniformly equivalent to that of S^* (in the scalar Hardy space), but such that T and S^* are not even quasisimilar.

Let us recall that operators T_1 and T_2 are called *quasisimilar* if there exist operators A and B with dense ranges and trivial kernels such that

$$AT_1 = T_2 A, \quad T_1 B = BT_2$$

(if T_1 and T_2 are similar, then $B = A^{-1}$).

Theorem 5.1. *Given $\varepsilon > 0$, there exists an operator T such that*

(1) *T satisfies assumptions (2) through (4) from the Introduction with $\dim\ker(T - \lambda I) = 1$ for all $\lambda \in \mathbb{D}$;*

(2) *T^* is "almost isometry", i.e.,*

$$(1 + \varepsilon)^{-1}\|x\| \leq \|T^* x\| \leq (1 + \varepsilon)\|x\| \quad \forall\, x \in H;$$

(3) *the eigenvector bundles of T and S^* are almost isomerically equivalent, i.e., there exists a bundle map bijection Ψ from the eigenvector bundle of S^* to that of T such that*

$$(1 + \varepsilon)^{-1}\|v_\lambda\| \leq \|\Psi(v_\lambda)\| \leq (1 + \varepsilon)\|v_\lambda\|$$

for all $\lambda \in \mathbb{D}$ and for all $v_\lambda \in \ker(S^* - \lambda I)$;

and such that the only operator A satisfying $AT = S^* A$ is $A = 0$, so A is not even quasisimilar to S^*.

We will construct the operator T as the backward shift, i.e., the adjoint of the forward shift $\mathbf{S}$, $\mathbf{S}f := zf$ in the space H^2_w, with the weight sequence $w = \{w_n\}_0^\infty$ $(w_n > 0)$:

$$H^2_w := \left\{ f = \sum_{n \geq 0} a_n z^n : \|f\|_w^2 := \sum_{n \geq 0} |a_n|^2 w_n < \infty \right\}.$$

If one assumes that $\liminf_n |w_n|^{1/n} = 1$, then the space H^2_w is a space of functions analytic in the unit disc $\mathbb{D}$. Moreover, for all $\lambda \in \mathbb{D}$ the functional $f \mapsto f(\lambda)$ is bounded, so for each $\lambda \in \mathbb{D}$ there exists a unique function $\mathbf{k}_\lambda \in H^2_w$ (the reproducing kernel of H^2_w) such that

(5.1) $$\langle f, \mathbf{k}_\lambda \rangle = f(\lambda) \quad \forall\, f \in H^2_w.$$

The reproducing kernel $\mathbf{k}_\lambda$ can be easily computed. Namely, it is easy to see that if $\{\varphi_n\}_0^\infty$ is an orthonormal basis in H^2_w, then

$$\mathbf{k}_\lambda(z) = \sum_0^\infty \overline{\varphi_n(\lambda)}\varphi_n(z).$$

Taking the orthonormal basis $\{z^n/\sqrt{w_n}\}_0^\infty$, we get

(5.2) $$\mathbf{k}_\lambda(z) = \sum_{n=0}^\infty \frac{z^n \overline{\lambda}^n}{w_n}.$$

Note that the Hardy space H^2 is a particular case ($w_n = 1$ for all n) of the space H_w^2, and formula (5.2) in this case gives the reproducing kernel $k_\lambda(z) = 1/(1 - \bar{\lambda}z)$ of H^2.

If one assumes that $\sup_n w_{n+1}/w_n < \infty$, then the shift operator $\mathbf{S}$, $\mathbf{S}f(z) = zf(z)$ is a bounded operator in H_w^2. The adjoint $\mathbf{S}^*$ is called the *backward shift*, and it is easy to see that

$$\mathbf{S}^* \left(\sum_{n=0}^{\infty} a_n z^n \right) = \sum_{n=0}^{\infty} \frac{w_{n+1}}{w_n} a_{n+1} z^n.$$

From this formula and (5.2) one easily concludes that

$$\ker(\mathbf{S}^* - \lambda I) = \mathrm{span}\{\mathbf{k}_{\bar{\lambda}}\}.$$

Since it follows from the reproducing kernel property that $\mathrm{span}\{\mathbf{k}_\lambda : \lambda \in \mathbb{D}\} = H_w^2$, $T = \mathbf{S}^*$ satisfies assumptions (2)–(4) from the Introduction.

Condition (2) of Theorem 5.1 is satisfied if and only if

$$(5.3) \qquad (1 + \varepsilon)^{-2} \leq w_{n+1}/w_n \leq (1 + \varepsilon)^2 \quad \forall\, n \geq 0.$$

The mapping Ψ, $\Psi(ak_\lambda) = a\mathbf{k}_\lambda$, $a \in \mathbb{C}$, $\lambda \in \mathbb{D}$, is clearly a holomorphic bundle map bijection between the eigenvector bundles of S^* and $\mathbf{S}^*$. Since $\|k_\lambda\|_{H^2}^2 = \langle k_\lambda, k_\lambda \rangle = k_\lambda(\lambda)$ ($= (1 - |\lambda|^2)^{-1}$) and similarly $\|\mathbf{k}_\lambda\|_{H_w^2}^2 = \mathbf{k}_\lambda(\lambda)$, condition (3) of Theorem 5.1 is equivalent to the estimate

$$(5.4) \qquad (1 + \varepsilon)^{-2} k_\lambda(\lambda) \leq \mathbf{k}_\lambda(\lambda) \leq (1 + \varepsilon)^2 k_\lambda(\lambda) \quad \forall\, \lambda \in \mathbb{D}.$$

Lemma 5.2. *If $\sup_n w_n = \infty$, then there is no non-zero bounded operator A satisfying $A\mathbf{S}^* = S^*A$.*

Proof: Let $A\mathbf{S}^* = S^*A$ for some $A \neq 0$. Then $\mathbf{S}A^* = A^*S$ and therefore $\mathbf{S}^n A^* = A^* S^n$.

Take $f \in H^2$ such that $A^*f = \sum_0^\infty a_n z^n \neq 0$, and pick m such that $a_m \neq 0$. Then

$$\|\mathbf{S}^n A^* f\|^2 = \sum_{j=0}^{\infty} |a_j|^2 w_{j+n} \geq |a_m|^2 w_{m+n},$$

so $\sup_n \|\mathbf{S}^n A^* f\| = \infty$ because $\sup_n w_{m+n} = \infty$.

On the other hand, $\|A^* S^n f\|_{H_w^2} \leq \|A^*\| \|f\|_{H^2}$, giving us a contradiction. $\square$

Thus to prove the theorem, we need to find an unbounded sequence $\{w_n\}_0^\infty$ satisfying (5.3), and such that (5.4) holds. We define the sequence $\{w_n\}_0^\infty$ to be 1 for all n except in sparse intervals from N_j

to $N_j + 2j$; the numbers N_j will be specified later. On the intervals $[N_j, N_j + 2j]$ the sequence has "spikes": $\ln w_n$ on $[N_j, N_j + 2j]$ is the piecewise affine function with slope $\pm 2\ln(1+\varepsilon)$, increasing from 0 to $2j\ln(1+\varepsilon)$ on $[N_j, N_j+j]$ and decreasing back to 0 on $[N_j+j, N_j+2j]$; see Figure 1.

Formally, we write

$$
\ln w_n = \begin{cases} 2m\ln(1+\epsilon) & n = N_k + m, \ 0 \le m \le j, \\ 2(j-m)\ln(1+\epsilon) & n = N_k + j + m, \ 0 \le m \le j, \\ 0 & \text{otherwise.} \end{cases}
$$

We pick a sequence $\{N_j\}_{j=1}^{\infty}$ such that

$$
N_j + 2j < N_{j+1}
$$

$$
\frac{2j-1}{N_j + 2j} \le \frac{\alpha}{2^j},
$$

where α is a small number such that $1 - \alpha \ge (1+\varepsilon)^{-2}$.

The sequence $\{w_n\}_0^{\infty}$ is clearly unbounded. Because of the slope condition for $\ln w_n$ condition (5.3) is satisfied.
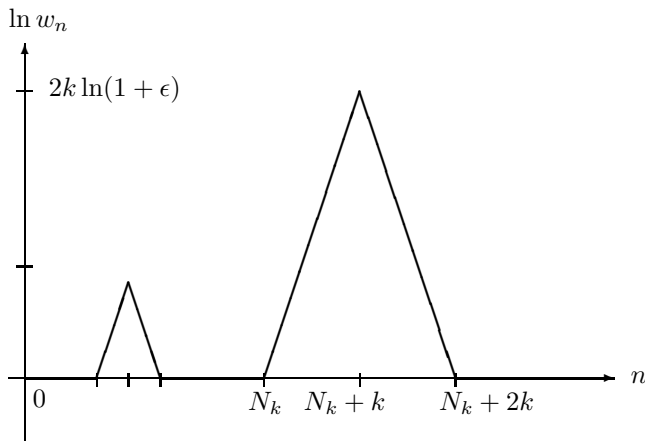

FIGURE 1. The function $\ln w_n$: two "spikes" are shown.

The lemma below completes the proof of the theorem.

Lemma 5.3. *For the weight $w = \{w_n\}_0^\infty$ constructed above, the repro-ducing kernel $\mathbf{k}_\lambda$ of H_w^2 satisfies inequality (5.4).*

Proof of Lemma 5.3: Since $w_n \geq 1$,

$$\mathbf{k}_\lambda(\lambda) = \sum_{n \geq 0} \frac{1}{w_n} |\lambda|^{2n} \leq \sum_{n \geq 0} |\lambda|^{2n} = k_\lambda(\lambda) \leq (1 + \varepsilon)^2 k_\lambda(\lambda),$$

so one estimate is obvious. To get the other one, it is enough to show that $k_\lambda(\lambda) - \mathbf{k}_\lambda(\lambda) \leq \alpha k_\lambda(\lambda)$.

Since $1 - 1/w_n \neq 0$ only for $n \in (N_j, N_j + 2j)$, we can write

$$k_\lambda(\lambda) - \mathbf{k}_\lambda(\lambda) = \sum_{n \geq 0} \left(1 - \frac{1}{w_n}\right) |\lambda|^{2n} = \sum_{j=1}^\infty \sum_{n=N_j+1}^{N_1+2j-1} \left(1 - \frac{1}{w_n}\right) |\lambda|^{2n}.$$

Then for each j,

$$\sum_{n=N_j+1}^{N_j+2j-1} \left(1 - \frac{1}{w_n}\right) |\lambda|^{2n} \leq \sum_{n=N_j+1}^{N_j+2j-1} |\lambda|^{2n}$$

$$= \frac{|\lambda|^{2(N_j+1)} - |\lambda|^{2(N_j+2j)}}{1 - |\lambda|^2} \leq \frac{A_j}{1 - |\lambda|^2},$$

where

$$A_j = \max\{x^{N_j+1} - x^{N_j+2j} : 0 \leq x \leq 1\}$$

(here $x = |\lambda|^2$). The maximum is attained at $x = \left(\frac{N_j+1}{N_k+2j}\right)^{\frac{1}{2j-1}}$, and

$$A_j = \left(\frac{N_j+1}{N_j+2j}\right)^{\frac{N_j+1}{2j-1}} \frac{2j-1}{N_j+2j} \leq \frac{2j-1}{N_j+2j} \leq \frac{\alpha}{2^j}$$

by our choice of the N_j.

Summing over all j, we finally get

$$k_\lambda(\lambda) - \mathbf{k}_\lambda(\lambda) \leq \frac{1}{1 - |\lambda|^2} \sum_{j=1}^\infty \alpha 2^{-j} = \frac{\alpha}{1 - |\lambda|^2} = \alpha k_\lambda(\lambda),$$

so the lemma is proved. $\square$

References

[1] M. ANDERSSON, The corona theorem for matrices, *Math. Z.* **201(1)** (1989), 121–130.

[2] J.-P. AUBIN, *"Applied functional analysis"*, Translated from the French by Carole Labrousse, With exercises by Bernard Cornet and Jean-Michel Lasry, John Wiley & Sons, New York-Chichester-Brisbane, 1979.

[3] L. CARLESON, Interpolations by bounded analytic functions and the corona problem, *Ann. of Math. (2)* **76** (1962), 547–559.

[4] M. J. COWEN AND R. G. DOUGLAS, Complex geometry and operator theory, *Acta Math.* **141(3–4)** (1978), 187–261.

[5] P. GRIFFITHS AND J. HARRIS, *"Principles of algebraic geometry"*, Reprint of the 1978 original, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.

[6] N. K. NIKOLSKI, *"Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz"*, Translated from the French by Andreas Hartmann, Mathematical Surveys and Monographs **92**, American Mathematical Society, Providence, RI, 2002.

[7] N. K. NIKOLSKI, *"Operators, functions, and systems: an easy reading. Vol. 2. Model operators and systems"*, Translated from the French by Andreas Hartmann and revised by the author, Mathematical Surveys and Monographs **93**, American Mathematical Society, Providence, RI, 2002.

[8] N. K. NIKOLSKI, *"Treatise on the shift operator. Spectral function theory"*, With an appendix by S. V. Hruščev and V. V. Peller, Translated from the Russian by Jaak Peetre, Grundlehren der Mathematischen Wissenschaften **273**, Springer-Verlag, Berlin, 1986.

[9] B. SZ.-NAGY AND C. FOIAŞ, *"Harmonic analysis of operators on Hilbert space"*, Translated from the French and revised North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest, 1970.

[10] B. SZ.-NAGY AND C. FOIAŞ, On the structure of intertwining operators, *Acta Sci. Math. (Szeged)* **35** (1973), 225–254.

[11] B. SZ.-NAGY AND C. FOIAŞ, Sur les contractions de l'espace de Hilbert. X. Contractions similaires à des transformations unitaires, *Acta Sci. Math. (Szeged)* **26** (1965), 79–91.

[12] S. R. TREIL, Angles between co-invariant subspaces, and the operator corona problem. The Szökefalvi-Nagy problem, (Russian), *Dokl. Akad. Nauk SSSR* **302(5)** (1988), 1063–1068; translation in: *Soviet Math. Dokl.* **38(2)** (1989), 394–399.

[13] S. R. Treil, Geometric methods in spectral theory of vector-valued functions: some recent results, in: *"Toeplitz operators and spectral function theory"*, Oper. Theory Adv. Appl. **42**, Birkhäuser, Basel, 1989, pp. 209–280.

[14] S. R. Treil, Unconditional bases of invariant subspaces of a contraction with finite defects, *Indiana Univ. Math. J.* **46(4)** (1997), 1021–1054.

[15] S. R. Treil, An operator Corona theorem, *Indiana Univ. Math. J.* **53(6)** (2004), 1763–1780.

[16] S. R. Treil, Lower bounds in the matrix Corona theorem and the codimension one conjecture, *Geom. Funct. Anal.* **14(5)** (2004), 1118–1133.

[17] S. R. Treil and B. D. Wick, Analytic projections, corona problem and geometry of holomorphic vector bundles, *J. Amer. Math. Soc.* **22(1)** (2009), 55–76.

[18] M. Uchiyama, Curvatures and similarity of operators with holomorphic eigenvectors, *Trans. Amer. Math. Soc.* **319(1)** (1990), 405–415.

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