WEAK LOCALLY MULTIPLICATIVELY-CONVEX ALGEBRAS[1]

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Let E be an algebra over the reals or complex numbers, E' a total subspace of the algebraic dual E^* of vector space E. We first discuss the following natural questions: When is the weak topology $\sigma(E, E')$ defined on E by E' locally m-convex? When is multiplication continuous for $\sigma(E, E')$, that is, when is $\sigma(E, E')$ compatible with the algebraic structure of E? We then apply our results to certain weak topologies on the algebra of polynomials in one indeterminant without constant term.

1. Weak topologies.

Let K be either the reals or complex numbers, E a K-algebra. A topology $\mathcal{T}$ on E is *locally multiplicatively-convex* (which we abbreviate henceforth to "locally m-convex") if it is a locally convex topology and if there exists a fundamental system of idempotent neighborhoods of zero (a subset A of E is idempotent if $A^2 \subseteq A$). Multiplication is then clearly continuous at (0, 0) and hence everywhere, so $\mathcal{T}$ is compatible with the algebraic structure of E. If A is idempotent, so is its convex envelope, its equilibrated envelope (a subset V of E is called equilibrated if $\lambda V \subseteq V$ for all scalars λ such that $|\lambda| \leq 1$), and its closure for any topology on E compatible with the algebraic structure of E. Hence if $\mathcal{T}$ is locally m-convex, zero has a fundamental system of convex, equilibrated, idempotent, closed neighborhoods. (For proofs of these and other elementary facts about locally m-convex algebras, see §§1–3 of [8] or [1].) Henceforth, E' is a total subspace of the algebraic dual of E.

LEMMA 1. *Let W be a weak, equilibrated neighborhood of zero (that is, for the topology $\sigma(E, E')$), J a subspace of E, and $g \in E'$ such that $J \subseteq W \subseteq W \cup W^2 \subseteq \{g\}^0$. Then J, JE, and EJ are contained in the kernel of g.*

Proof. Let $x \in J$, $y \in E$. As W is equilibrated and absorbing, let $\lambda > 0$ be such that $\lambda y \in W$. For all positive integers m, $\lambda^{-1}mx \in J$, and therefore $mxy = (\lambda^{-1}mx)(\lambda y) \in JW \subseteq W^2 \subseteq \{g\}^0$; hence $|g(mxy)| \leq 1$ for all positive integers m, and therefore $g(xy) = 0$. Hence JE is contained in

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the kernel of g. Similarly for EJ. Also $|g(mx)|\leq 1$ for all $x\in J$ and all positive integers m, and therefore $g(x)=0$ for all $x\in J$.

LEMMA 2. *Let V be a weak neighborhood of zero. Then $L=\bigcap[u^{-1}(0)\,|\,u\in V^0]$ is a weakly closed subspace of finite codimension.*

Proof. L is clearly a weakly closed subspace. By definition of $\sigma(E, E')$ there exist $h_1, h_2, \cdots, h_n$ in E' such that $\{h_1, h_2, \cdots, h_n\}^0\subseteq V$. Thus if $|h_i(z)|\leq 1$ for $1\leq i\leq n$, then $|u(z)|\leq 1$ for all $u\in V^0$. Then if $x\in \bigcap_{i=1}^{n} h_i^{-1}(0)$, for any positive integer m $|h_i(mx)|=0<1$ for $1\leq i\leq n$ and hence $|u(mx)|\leq 1$, so $u(x)=0$ for all $u\in V^0$. Hence $\bigcap_{i=1}^{n} h_i^{-1}(0)\subseteq L$. Since the codimension of $\bigcap_{i=1}^{n} h_i^{-1}(0)$ is at most n, so also the codimension of L is at most n.

LEMMA 3. *Let $E_1, E_2, \cdots, E_n$ be finite-dimensional, Hausdorff topological K-vector spaces, F a topological K-vector space. Any multilinear transformation from $E_1\times E_2\times\cdots\times E_n$ into F is continuous.*

Proof. This lemma is well known, and follows from Theorem 2 of [3, p. 27] just as Corollary 2 of that theorem does.

THEOREM 1. *$\sigma(E, E')$ is a locally m-convex topology on E if and only if for all $g\in E'$, the kernel of g contains a weakly closed ideal of finite codimension.*

Proof. Necessity: Let $g\in E'$. Let V be a weakly closed, convex, equilibrated, idempotent neighborhood of zero such that $V\subseteq\{g\}^0$. Let $L=\bigcap[u^{-1}(0)\,|\,u\in V^0]$. Then clearly $L\subseteq V^{00}$, but since V is weakly closed, convex, and equilibrated, $V^{00}=V$ (see [4]). By Lemma 2 L is a weakly closed subspace of finite codimension. We assert L is an ideal: Let $x\in L$, $y\in E$. Choose $\lambda>0$ such that $\lambda y\in V$. For all positive integers m, $\lambda^{-1}mx\in L$; hence $mxy=(\lambda^{-1}mx)(\lambda y)\in LV\subseteq V^2\subseteq V$. Hence for all positive integers m and any $u\in V^0$, $|u(mxy)|\leq 1$; hence $u(xy)=0$ for all $u\in V^0$, so $xy\in L$. Similarly $yx\in L$, so L is an ideal. Now let $J=L\cap g^{-1}(0)$. Then J is a weakly closed subspace of finite codimension contained in the kernel of g. It remains to show J is an ideal. Now $J\subseteq L\subseteq V=V\cup V^2\subseteq\{g\}^0$; hence by Lemma 1 $JE\subseteq g^{-1}(0)$ and $EJ\subseteq g^{-1}(0)$. Also $JE\subseteq LE\subseteq L$ and $EJ\subseteq EL\subseteq L$. Therefore $JE\subseteq L\cap g^{-1}(0)=J$ and $EJ\subseteq L\cap g^{-1}(0)=J$, so J is an ideal.

Sufficiency: It clearly suffices to show that for all $g\in E'$ there exists an idempotent neighborhood V of zero such that $V \subseteq \{g\}^0$. Let J be a weakly closed ideal of finite codimension contained in $g^{-1}(0)$. Then $F=E/J$ is a finite-dimensional algebra with a Hausdorff topology compatible with the vector space structure of F. Multiplication is a bilinear transformation from $F \times F$ into F, and hence by Lemma 3 multiplication is continuous. But also, any finite-dimensional, Hausdorff, K-vector space has its topology defined by a norm (this follows from Theorem 2 of [3, p. 27]); and by a familiar property of normed spaces with a continuous multiplication, the norm may be so chosen that F is a normed algebra [6, p. 50]. Let φ be the continuous canonical homomorphism from E onto F, and let $g=\bar{g}\circ\varphi$. $\bar{g}$ is continuous on F, so we may select an idempotent neighborhood U of zero in F such that $v \in U$ implies $|\bar{g}(v)| \leq 1$. Then $V=\varphi^{-1}(U)$ is a neighborhood of zero for $\sigma(E, E')$. As U is idempotent and φ a homomorphism, V is idempotent. Finally, if $x \in V$ then $\varphi(x) \in U$, and therefore $|g(x)|=|\bar{g}(\varphi(x))| \leq 1$, so $x \in \{g\}^0$; hence $V \subseteq \{g\}^0$, and the theorem is completely proved.

THEOREM 2. *Multiplication in E is continuous for $\sigma(E, E')$ if and only if for all $g \in E'$, the kernel of g contains a weakly closed subspace J of finite codimension such that JE and EJ are also contained in the kernel of g.*

Proof. Necessity: Let $g \in E'$. Then since $\{g\}^0$ is a neighborhood of zero, we may choose a weakly closed, convex, equilibrated neighborhood W of zero such that $W \cup W^2 \subseteq \{g\}^0$. Let $L=\bigcap[u^{-1}(0) \mid u \in W^0]$. Then clearly $L \subseteq W^{00}=W$, since W is weakly closed, convex, and equilibrated. By Lemma 2 L is a weakly closed subspace of finite codimension. Let $J=L \cap g^{-1}(0)$. Then J is also a weakly closed subspace of finite codimension contained in the kernel of g. Also $J \subseteq L \subseteq W \subseteq W \cup W^2 \subseteq \{g\}^0$, so by Lemma 1, JE and EJ are contained in the kernel of g.

Sufficiency: It suffices to show that for any $g \in E'$ and any $a \in E$, there exist neighborhoods W and V of zero in E such that $W^2 \subseteq \{g\}^0$ and $Va \cup aV \subseteq \{g\}^0$ ([5, p. 49]). Let $I=g^{-1}(0)$ and let J be a weakly closed subspace of finite codimension contained in I such that $EJ \subseteq I$ and $JE \subseteq I$. Let φ and ψ respectively be the canonical maps from E onto E/J and from E onto E/I. Let $g=\bar{g}\circ\psi$. We assert the map $(\varphi(x), \varphi(y)) \rightarrow \psi(xy)$ is a well-defined bilinear map from $(E/J) \times (E/J)$ into E/I: If $x-x' \in J$ and $y-y' \in J$, then $xy-x'y \in JE \subseteq I$ and $x'y-x'y' \in EJ \subseteq I$; hence $xy-x'y'=(xy-x'y)+(x'y-x'y') \in I+I=I$. The map is therefore well-defined; bilinearity is easily seen. Both (E/J) and (E/I) are finite-dimensional Hausdorff topological K-vector spaces, so by

Lemma 3 the above bilinear map is continuous. Hence there exists a neighborhood U of zero in E/J such that if $\varphi(x)$, $\varphi(y)\in U$, then $\psi(xy)\in\{\bar{g}\}^0$. If $W=\varphi^{-1}(U)$, then W is a neighborhood of zero for $\sigma(E, E')$; if $x, y\in W$, then $\varphi(x)$, $\varphi(y)\in U$ and hence $|g(xy)|=|\bar{g}(\psi(xy))|\leq 1$, so $xy\in\{g\}^0$. Thus $W^2\subseteq\{g\}^0$. Now let $a\in E$. We assert the maps $\varphi(x)\to\psi(ax)$ and $\varphi(x)\to\psi(xa)$ are well-defined, linear maps from E/J into E/I: For if $x-x'\in J$, then $ax-ax'\in EJ\subseteq I$ and $xa-x'a\in JE\subseteq I$, so the maps are well-defined. Linearity is immediate. Since E/J and E/I are finite dimensional and Hausdorff, again by Lemma 3 these maps are continuous. Hence we may choose a neighborhood P of zero in E/J such that if $\varphi(x)\in P$ then $\psi(ax)$, $\psi(xa)\in\{\bar{g}\}^0$. Then $V=\varphi^{-1}(P)$ is a neighborhood of zero for $\sigma(E, E')$. If $x\in V$, then $\varphi(x)\in P$ and hence $|g(ax)|=|\bar{g}(\psi(ax))|\leq 1$ and similarly $|g(xa)|\leq 1$. Hence $aV\cup Va\subseteq\{g\}^0$, and the theorem is completely demonstrated.

Here is an example of a Banach algebra E with topological dual E' such that multiplication is not continuous for the associated weak topology $\sigma(E, E')$. Let E be the algebra of all continuous functions from the compact interval $[0, 1]$ into K with the uniform topology. If $\mu(f)=\int_0^1 f(t)dt$ (dt is the usual Lebesgue complex-valued measure if K is the complex numbers), then $\mu\in E'$. But μ does not satisfy the restrictions of Theorem 2: Let J be any weakly closed subspace contained in the kernel of μ such that $JE\subseteq\mu^{-1}(0)$. If $f\in J$, then $f\bar{f}\in JE\subseteq\mu^{-1}(0)$ ($\bar{f}=f$ if K is the reals); hence $\int_0^1|f(t)|^2dt=0$ and so, since f is continuous, $f=0$. Therefore $J=\{0\}$. But since E is infinite-dimensional, J is not of finite codimension. Hence by Theorem 2, multiplication is not continuous for $\sigma(E, E')$.

2. Algebras of polynomials. If E is any locally m-convex algebra, E' its topological dual, $\mathscr{M}(E)$ is the set of all continuous multiplicative linear forms, $\mathscr{M}^-(E)$ the set of all nonzero continuous multiplicative linear forms. $\mathscr{M}(E)$ and $\mathscr{M}^-(E)$ are topologized as subsets of E'; $\sigma(E', E)$.

In [9] Šilov proved the following theorems:

(1) If E is a normed C-algebra (C is the complex numbers) with identity e, generated by e and another element x (that is, if all elements of E are of form $\alpha_0 e+\alpha_1 x+\cdots+\alpha_n x^n$), then $\mathscr{M}^-(E)$ is homeomorphic with a compact subset of C whose complement is connected; (2) every such subset of C arises in this manner.

Here we give elementary analogues of these theorems for locally m-convex algebras.

Proposition 1. *If E is a locally m-convex Hausdorff algebra generated by a single element x, then $f \to f(x)$ is a homeomorphism from $\mathscr{M}(E)$ onto a subset of K.*

Proof. The map is surely continuous and is one-to-one since x generates E. To show $f(x) \to f$ is continuous, it suffices to show $f(x) \to f(z)$ is continuous for all $z \in E$; but as x generates E it suffices for this to show $f(x) \to f(x^n)$ is continuous for all positive integers n. But $f(x^n) = f(x)^n$, so $f(x) \to f(x^n)$ is simply a restriction of the map $\lambda \to \lambda^n$ from K into K, which is surely a continuous map. Hence $f \to f(x)$ is a homeomorphism into K.

Proposition 2. *Let E be an algebra over any field F. The set M of nonzero multiplicative linear forms is a linearly independent subset of E^*, the algebraic dual of E.*

Proof. In Theorem 12 of [2, p. 34], Artin proves that if G is a group, F a field, then the set of all nonzero homomorphisms from G into the multiplicative semi-group of F is a linearly independent subset of the vector space $\mathscr{F}(G, F)$ of all functions from G into F. The proof remains valid if "semi-group" replaces "group" in the statement of the theorem, and thus modified the theorem may be applied to the multiplicative semi-group of an algebra to yield the desired result.

Henceforth, $K[X]$ is the K-algebra of all polynomials in one indeterminant, E the subalgebra of those without constant term. $K[X]$ has a base $\{e_i\}_{i=0}^{\infty}$ with multiplication table $e_i e_j = e_{i+j}$; $\{e_i\}_{i=1}^{\infty}$ is a base for E. For $\lambda \in K$ we let f_λ be the linear form defined on E by: $f_\lambda(e_j) = \lambda^j$. Also for every positive integer i, g_i is the linear form defined on E by: $g_i(e_i) = 1$, $g_i(e_j) = 0$ for $j \neq i$.

LEMMA 4. *The set of all multiplicative linear forms on E is* $[f_\lambda \mid \lambda \in K]$.

Proof. $f_\lambda(e_j e_k) = f_\lambda(e_{j+k}) = \lambda^{j+k} = \lambda^j \lambda^k = f_\lambda(e_j) f_\lambda(e_k)$. This suffices to show f_λ is multiplicative. Conversely, if f is any multiplicative linear form, let $\lambda = f(e_1)$. Then for any positive integer i, $f(e_i) = f(e_1^i) = f(e_1)^i = \lambda^i$. Hence $f = f_\lambda$.

LEMMA 5. $\{f_\lambda\}_{\lambda \in K, \lambda \neq 0} \cup \{g_i\}_{i=1}^{\infty}$ *is a linearly independent subset of E^*.*

Proof. Suppose $\sum_{i=1}^{n} \alpha_i g_i + \sum_{j=1}^{p} \beta_j f_{\lambda_j} = 0$, where the λ_j are distinct from each other and all different from zero. Then for $m > n$, $g_i(e_m) = 0$

for $1 \leq i \leq n$, so $\sum_{j=1}^{p} \beta_j f_{\lambda_j}(e_m)=0$. The subspace of E generated by $\{e_j\}_{j=n+1}^{\infty}$ is clearly a subalgebra; the restrictions of the f_{λ_j}, $1 \leq j \leq p$, to this algebra are again clearly distinct from each other and different from zero. Hence by Proposition 2 applied to this subalgebra, all $\beta_j=0$. Hence $\sum_{i=1}^{n} \alpha_i g_i=0$; but $\alpha_i=\alpha_i g_i(e_i)=\sum_{j=1}^{n} \alpha_j g_j(e_i)=0$, so the lemma is proved.

LEMMA 6. *Let $\{\lambda_i\}_{i=1}^{\infty}$ be a denumerable family of distinct nonzero elements of K. Then $\{f_{\lambda i}\}_{i=1}^{\infty}$ separates the points of E.*

Proof. For $\lambda \neq 0$, each f_λ has a unique extension to a multiplicative linear form on $K[X]$ obtained by setting $f_\lambda(e_0)=1$. Let $x=\sum_{i=1}^{n} \alpha_i e_i \in E$. Then $x=\sum_{i=0}^{n} \alpha_i e_i$ in $K[X]$ where $\alpha_0=0$. Suppose $f_{\lambda_j}(x)=0$ for $1 \leq j \leq n+1$. Then $\sum_{i=0}^{n} \alpha_i \lambda_j^i=0$ for $1 \leq j \leq n+1$. But the determinant of the system of linear equations $\sum_{i=0}^{n} \zeta_i \lambda_j^i=0$, $1 \leq j \leq n+1$, is

$$\begin{vmatrix} \lambda_1^0 & \lambda_1^1 & \cdots & \lambda_1^n \\ \lambda_2^0 & \lambda_2^1 & \cdots & \lambda_2^n \\ \vdots & \vdots & & \vdots \\ \lambda_{n+1}^0 & \lambda_{n+1}^1 & \cdots & \lambda_{n+1}^n \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{n+1} \\ \vdots & \vdots & & \vdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_{n+1}^n \end{vmatrix} = \prod_{1 \leq i < j < n+1} (\lambda_i - \lambda_j) \neq 0$$

(this is the Vandermonde determinant). Hence the above system of linear equations has only the trivial solution, and therefore $\alpha_i=0$, $0 \leq i \leq n$, and hence $x=0$. Thus the proof is complete.

Proposition 3. If L is any subset of K containing zero, there is a Hausdorff, weak locally m-convex topology $\mathcal{T}$ on E such that the canonical map $f_\lambda \rightarrow \lambda$ maps $\mathcal{M}(E)$ homeomorphically onto L. Further if L is an infinite set, $\mathcal{T}$ may be so chosen that the completion of $E: \mathcal{T}$ is semi-simple; and if L is denumerable, $\mathcal{T}$ is metrizable.

Proof. Case 1: L is finite. Let $M=[f_\lambda \mid \lambda \in L]$, and let E' be the subspace of E^* generated by $\{g_i\}_{i=1}^{\infty} \cup M$. Clearly E' is a total subspace of E^*, and so, as E' has a denumerable linear base, $\sigma(E, E')$ is a metrizable weak topology on E. To show $\sigma(E, E')$ is locally m-convex, it clearly suffices to show that the condition of Theorem 1 holds for all members of a base of E'. The condition holds trivially for all $u \in M$, since the kernel of $u \in M$ is already a weakly closed ideal. Consider any g_i: The linear subspace generated by $\{e_j\}_{j=i+1}^{\infty}$ is clearly of finite codimension, and the multiplication table shows that it is actually an ideal. Further, it is identical with $\bigcap_{k=1}^{i} g_k^{-1}(0)$ and thus is weakly closed and contained in the kernel of g_i. Hence by Theorem 1, $\sigma(E, E')$ is locally m-convex. By Lemma 5 the set of all multiplicative linear forms in E' is M. As the topological dual of E; $\sigma(E, E')$ is E' (see [7]), M is the set of all continuous multiplicative linear forms on E; $\sigma(E, E')$, and by Proposition 1 applied to $x=e_1$, M is homeomorphic with L.

Case 2: L is infinite. Again let $M=[f_\lambda \mid \lambda \in L]$, and let E' be the subspace of E^* generated by M. By Lemma 6, E' is total. The condition of Theorem 1 is trivially satisfied by E', so $\sigma(E, E')$ is a Hausdorff, weak locally m-convex topology on E. If L is denumerable, E' has a countable base and so $\sigma(E, E')$ is metrizable. M is again the set of all continuous multiplicative linear forms on E; $\sigma(E, E')$ and is homeomorphic with L. The completion of E for this topology is E'^* ([7]), and as M generates E', M separates the points of E'^*; thus the completion of E for this topology is semi-simple by Corollary 5.5 of [8].

It is easy to see that E has no divisors of zero and that zero is the only element having an adverse; thus the Jacobson radical is $\{0\}$ and E is semi-simple. If, in Proposition 3, $L=\{0\}$ and the scalar field is the complex numbers, E is a commutative, metrizable locally m-convex algebra with no continuous nonzero multiplicative linear forms; the completion $\hat{E}$ of E then has no continuous nonzero multiplicative linear forms and hence by Corollary 5.5 of [8] is a radical algebra. Thus we have an example of a semi-simple metrizable algebra whose completion is a radical algebra. This phenomenon is also known even for normed algebras. For example, an elementary calculation shows the following is a norm on E:

$$\left\| \sum_{n=1}^{m} \alpha_n e_n \right\| = \sum_{n=1}^{m} \frac{|\alpha_n|}{n!} .$$

$\|(m-1)!e_m\|=1/m \to 0$, so $(m-1)!\, e_m \to 0$ for this norm topology. But for any $\lambda \neq 0$, $|f_\lambda((m-1)!\, e_m)|=(m-1)!\, |\lambda|^m \to \infty$, so f_λ is not continuous. Hence E has no continuous nonzero multiplicative linear forms and so, assuming the scalar field is the complex numbers, the completion of E for this norm is a radical algebra.

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