SOME ERGODIC THEOREMS INVOLVING TWO OPERATORS

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1. Introduction. The object of the present note is to indicate how the ergodic theorem of W. Hurewicz [3] and E. Hopf [2] can be extended to theorems involving two operators. While for a finite measure space, the Hopf theorem for two operators is readily seen to be the consequence of the theorem for one operator and the Birkhoff ergodic theorem, in the general case the theorem for two operators is established via the extended form of the Hurewicz theorem. An application is made to the theory of Markov chains in § 4.

Let (S, Ω, μ) be a fixed measure space which is assumed to be σ-finite unless otherwise stated. Capital letters are reserved for elements of Ω. For a measure ξ and for point functions we write $f(x)=g(x)[\xi]$ for equality almost everywhere $[\xi]$.

We consider two one-to-one transformations of S onto itself, t and u, each of which is measurable in the sense that for $v=t$ and $v=u$, $M\in\Omega$ implies $vM\in\Omega$ and $v^{-1}M\in\Omega$, and if $\mu(M)=0$ then $\mu(v^{-1}M)=0$. We suppose throughout that neither t nor u has wandering sets of positive measure, that is,

(1) *For* $v=t$ *and* $v=u$, *if* $A\cap v^kA=0$, $k=1, 2, \cdots$, *then* $\mu(A)=0$.

2. The Hurewicz theorem. For any finite valued countably additive set function φ defined on Ω and absolutely continuous with respect to μ, form the set functions

$$\varphi_n(X)=\sum_{k=0}^{n}\varphi(t^kX), \qquad n=0, 1, \cdots, \tag{2}$$

and

$$\nu_n(X)=\sum_{k=0}^{n}\mu(t^kX), \qquad n=0, 1, \cdots. \tag{3}$$

Then φ_n and ν_n are countably additive set functions and φ_n is absolutely continuous with respect to ν_n so admits the representation

$$\varphi_n(X)=\int_X g_n(x)\mu_n(dx), \qquad n=0, 1, \cdots. \tag{4}$$

The Hurewicz theorem then asserts that $g_n(x)$ has a limit at all points except for a nullset with respect to t, that is for all points except a t-invariant set of μ measure zero.

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To formulate the theorem for two operators we introduce

$$\mu_n(X)=\sum_{k=0}^{n}\mu(u^kX), \qquad n=0, 1, \cdots. \tag{5}$$

The set function μ_n is countably additive but φ_n is no longer automatically absolutely continuous with respect to μ_n. In order to have this absolute continuity for any countable additive set function φ absolutely continuous with respect to μ with the consequent representation

$$\varphi_n(X)=\int_X f_n(x)\mu_n(dx), \qquad n=0, 1, \cdots \tag{6}$$

it is necessary for ν_n to be absolutely continuous with respect to μ_n. To see this simply take $\varphi=\mu$, whence $\varphi_n=\nu_n$. We therefore take as a basic hypothesis

(7) *ν_n is absolutely continuous with respect to μ_n*

with the consequent representation

$$\nu_n(X)=\int_X c_n(x)\mu_n(dx), \qquad n=0, 1, \cdots. \tag{8}$$

We also assume that the operators t and u satisfy the Birkhoff ergodic theorem, that is,

(9) *For $v=t$ and $v=u$, if $f(x)\in L^1(S)$, $\lim_{n\to\infty}\sum_{k=0}^{n} f(v^kx)/n$ exists almost everywhere $[\mu]$.*

THEOREM 1. *Let t and u be one-to-one measurable transformations of S onto itself which have no wandering sets of positive measure. Let φ be a finite valued countably additive set function defined on Ω and absolutely continuous with respect to μ. If* (7) *and* (9) *are satisfied, then the "averaging sequence" $f_n(x)$ of point functions defined by* (2), (5) *and* (6) *converges everywhere except for the union of a t- and u-nullset as $n\to\infty$.*

Proof. We suppose first that $\mu(S)<\infty$. From the representations (4) and (8) we deduce that

$$\varphi_n(X)=\int_X g_n(x)c_n(x)\mu_n(dx).$$

The comparison with (6) yields $f_n(x)=g_n(x)c_n(x)[\mu_n]$. The Hurewicz theorem implies that $g_n(x)$ has a finite limit except for a t-nullset. A result of C. Ryll-Nardzewski [4] shows that the hypothesis (9) that t satisfies the Birkhoff ergodic theorem implies the existence of a countably additive measure α with the additional properties:

(10.1) $0 \leq \alpha(X) \leq k\mu(X)$.

(10.2) *If* $X = t^{-1}X$, *then* $\alpha(X) = \mu(X)$.

(10.3) $\alpha(t^{-1}X) = \alpha(X)$.

Likewise, since u satisfies (9), there is a countably additive measure with the additional properties:

(11.1) $0 \leq \beta(X) \leq k\mu(X)$

(11.2) *If* $X = u^{-1}X$, *then* $\beta(X) = \mu(X)$

(11.3) $\beta(u^{-1}X) = \beta(X)$.

From (10.1) we note that α is absolutely continuous with respect to μ. Hence if

$$\alpha_n(X) = \sum_{k=0}^{n} \alpha(t^k X) , \qquad n = 0, 1, \cdots \tag{12}$$

then α_n is absolutely continuous with respect to ν_n and we may write

$$\alpha_n(X) = \int_X a_n(x) \nu_n(dx) , \qquad n = 0, 1, \cdots . \tag{13}$$

Likewise if

$$\beta_n(X) = \sum_{k=0}^{n} \beta(u^k X) , \qquad n = 0, 1, \cdots, \tag{14}$$

β_n is absolutely continuous with respect to μ_n and

$$\beta_n(X) = \int_X \rho_n(x) \mu_n(dx) \qquad n = 0, 1, \cdots . \tag{15}$$

If $\beta(A) = 0$, then (11.3) implies $\beta\left(\bigcup_{-\infty}^{\infty} u^k A \right) = 0$, and since $\bigcup_{-\infty}^{\infty} u^k A$ is u-invariant (11.2) implies $\mu\left(\bigcup_{-\infty}^{\infty} u^k A \right) = 0$ and thus $\mu(A) = 0$. Hence we also have the representation

$$\mu_n(X) = \int_X b_n(x) \beta_n(dx) , \qquad n = 0, 1, \cdots . \tag{16}$$

If we combine (13), (8) and (16) we obtain

$$\alpha_n(X) = \int_X a_n(x) c_n(x) b_n(x) \beta_n(dx) . \tag{17}$$

By the use of (10.3) and (11.3), (17) simplifies to

$$\alpha(X) = \int_X a_n(x) c_n(x) b_n(x) \beta(dx), \qquad n = 0, 1, \cdots . \tag{18}$$

Since $c_0(x) = 1[\mu]$, we find that

(19) $$a_n(x)c_n(x)b_n(x)=a_0(x)b_0(x)[\mu]\,, \qquad n=0,1,\cdots$$

Since we are supposing at present that $\mu(S)<\infty$, the Hurewicz theorem can be applied to (13) and (16), and thus $a_n(x)$ has a limit $a(x)$ as $n\to\infty$, except for a t-nullset and $b_n(x)$ has a limit $b(x)$ as $n\to\infty$, except for a u-nullset. By a further conclusion of the Hurewicz theorem, not already stated, we know that $a(x)$ is t-invariant and that

$$\int_X a(x)\mu(dx)=\int_X a_0(x)\mu(dx)$$

for every invariant set X. Hence for $Z=\{x|a(x)=0\}$, $\alpha(Z)=0$ and since Z is t-invariant, $\mu(Z)=0$ by (10.2). The identical argument shows that $b(x)$ is not zero except for a u-nullset. If we also observe that the sets where $a_0(x)=\infty$ and $b_0(x)=\infty$ are t- and u-nullsets respectively, as are the sets where $a_0(x)=0$ and $b_0(x)=0$, we conclude that for all x except the union of a t- and u-nullset $c_n(x)$ has a finite limit as $n\to\infty$. Thus $f_n(x)$ has a finite limit excepting the union of a t- and u-nullset.

If the measure space (S,Ω,μ) is σ-finite, let $k(x)$ be a bounded positive function integrable over S. Let

$$\lambda(X)=\int_X k(x)\mu(dx)$$

and form

$$\lambda_n(X)=\sum_{j=0}^{n}\lambda(u^jX)\,.$$

The measure space (S,Ω,λ) is a finite measure space, and φ is absolutely continuous with respect to λ. Hence by the first part of the proof, if

$$\varphi_n(X)=\int_X h_n(x)\lambda_n(dx)\,,$$

then $h_n(x)$ has a finite limit at all points other then the union of a t- and u-nullset in the λ measure and hence also in the μ measure. Thus if we let

(20) $$\lambda_n(X)=\int_X k_n(x)\mu_n(dx)\,, \qquad n=0,1,\cdots$$

we have

$$\varphi_n(X)=\int_X h_n(x)k_n(x)\mu_n(dx)\,,$$

and consequently $f_n(x)=h_n(x)k_n(x)[\mu_n]$. The Hurewicz theorem applied to (20) asserts that $k_n(x)$ has a finite limit except for a u-nullset, which implies the conclusion of the theorem.

THEOREM 2. *If in addition to the hypotheses of Theorem* 1 $\mu(S)<\infty$ *and t and u commute, then* $f(x)=\lim_{n\to\infty} f_n(x)$ *has the properties*

(i) $f(tx)=f(x)$

(ii) $\int_X f(x)\mu(dx)=\int_X f_0(x)\mu(dx)$ *for any t-invariant set X.*

Proof. We use the same notation as in the proof of Theorem 1. From (10.1) we see that any function integrable with respect to α is also integrable with respect to μ. Hence the counterpart of (9) is satisfied with $v=u$ and μ replaced by α. By a further use of the results of C. Ryll-Nardzewski we find the existence of a countably additive measure γ, defined as a Banach-Mazur limit

$$\gamma(X)=\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}(u^{-j}X)$$

and having the additional properties:

(21.1) $\quad 0\leq\gamma(X)\leq k_1\alpha(X)$

(21.2) $\quad$ *If* $X=u^{-1}X,\ \gamma(X)=\alpha(X)$

(21.3) $\quad \gamma(u^{-1}X)=\gamma(X)\,.$

Since α is t-invariant and t and u commute we have $\alpha(u^{-j}tX)=\alpha(u^{-j}X)$, and thus the definition of $\gamma(X)$ shows

(2.14) $\quad \gamma(X)=\gamma(t^{-1}X)\,.$

We similarly obtain a countably additive measure with the properties:

(22.1) $\quad 0\leq\delta(X)\leq k_1\beta(X)$

(22.2) $\quad$ *If* $X=t^{-1}X$, *then* $\delta(X)=\beta(X)$

(22.3) $\quad \delta(t^{-1}X)=\delta(X)$

(22.4) $\quad \delta(u^{-1}X)=\delta(X)\,.$

From (21.1) we obtain

(23) $$\gamma(X)=\int_X m(x)\alpha(dx)\,.$$

An earlier argument showed that $\delta(X)=0$ implies $\beta(X)=0$, hence

(24) $$\beta(X)=\int_X n(x)\delta(dx)\,.$$

The combination of (23), (18), (19) and (24) then yields

$$\gamma(X)=\int_X m(x)a_0(x)b_0(x)n(x)\delta(dx)\,.$$

Since γ and δ are both t- and u-invariant, the integrand must be both t- and u-invariant. With the aid of (10.2), (11.2), (21.2) and (22.2) it is then seen that $m(x)a_0(x)b_0(x)n(x)=1[\delta]$.

Likewise the t-invariance of γ and α shows that $m(x)=1[\alpha]$, and the u-invariance of β and δ shows that $n(x)=1[\beta]$. Since a set of measure zero in any of the measures α, β, δ, and μ is also of measure zero in any of the other measure, we conclude that $a_0(x)b_0(x)=1[\mu]$.

The Hurewicz theorem, applied to (13), implies that for any t-invariant set X, if we let $a(x)=\lim_{n\to\infty} a_n(x)$

$$\int_X a(x)\mu(dx)=\int_X a_0(x)\mu(dx)=\alpha(X)\,. \tag{25}$$

If we combine (10.2) with (25) we find

$$\mu(X)=\int_X a(x)\mu(dx)\,.$$

The t-invariance of $a(x)$ then yields $a(x)=1[\mu]$. A repetition of the argument shows that $\lim_{n\to\infty} b_n(x)=1[\mu]$, consequently $\lim_{n\to\infty} c_n(x)=1[\mu]$. The conclusions of the theorem now follow from the corresponding conclusions of the Hurewicz theorem applied to (4).

3. The Hopf theorem.

THEOREM 3. *Let t and u be one-to-one measure preserving transformations of S onto itself. Let $f(x)\in L^1(S)$ and $g(x)>0$, then for almost all x the quotient*

$$\sum_{j=0}^{n} f(t^j x) \Big/ \sum_{j=0}^{n} g(u^j x)$$

has a limit as $n\to\infty$.

Proof. Let

$$\lambda(X)=\int_X g(x)\mu(dx)\,,\qquad \lambda_n(X)=\sum_{j=0}^{n}\lambda(u^j X)\,,\qquad \rho_n(X)=\sum_{j=0}^{n}\lambda(t^j X)\,.$$

Then ρ_n is absolutely continuous with respect to λ_n and

$$\varphi(X)=\int_X f(x)\lambda(dx)$$

is a finite valued countably additive set function absolutely continuous with respect to λ. We form $f_n(x)$ according to (4) and (5) with μ replaced by λ. Now

$$\lambda_n(X)=\int_X \sum_{j=0}^{n} g(u^j x)\mu(dx)\,,$$

so

$$\varphi_n(X)=\int_X f_n(x)\,\lambda_n(dx)=\int_X f_n(x)\sum_{j=0}^{n} g(u^j x)\mu(dx)\,.$$

But by definition

$$\varphi_n(x)=\int_X \sum_{j=0}^{n} f(t^j x)\mu(dx)\,.$$

Thus

$$f_n(x)=\sum_{j=0}^{n} f(t^j x)\Big/\sum_{j=0}^{n} g(u^j x)$$

and the conclusion follows from Theorem 1.

4. An application. In a recent note [1] T. E. Harris and Herbert Robbins used the Hopf ergodic theorem to obtain results concerning Markov chains admitting an infinite invariant measure. We indicate below the corresponding results that are obtainable by the use of Theorem 3.

Consider the real valued Markov chain $\cdots, x_{-1}, x_0, x_1, \cdots$ with a stationary transition probability function

$$h(u, B)=\text{prob}\,(x_{n+1}\in B|x_n=u)\,.$$

It is assumed that there is a measure $\amalg$ on the real Borel sets, which does not vanish identically, is finite for bounded Borel sets and satisfies

$$\amalg(B)=\int_{-\infty}^{\infty} h(u, B)\amalg(du)\,.$$

Let Φ be the class of real Borel sets, S the space of sequences of real numbers $x=(\cdots, x_{-1}, x_0, x_1, \cdots)$ and Ω the Borel extension of the cylinder sets, in S. If $A\in\Omega$ is determined by the coordinates $x_k, x_{k+1}, \cdots, x_r$ then $q(A|x_k=u)$ will denote the probability of A relative to the Markov chain starting with $x_k=u$, as specified by h.

A measure is established [1] in Ω by the relation

$$m(A)=\int q(A|x_j=u)\amalg(du) \qquad j\leq k$$

for cylinder sets determined by $x_k, \cdots, x_r$.

We shall apply Theorem 3, with t the ath shift transformation, $(tx)_i=x_{i+a}$, and u the bth shift transformation. If $\Gamma\in\Phi$, let R_Γ be the event that $x_n\in\Gamma$ infinitely often. The assumption

(26) *If* $\Gamma\in\Phi$, *then* $q(R_\Gamma|x_0=u)=1[\amalg]$,

then yields [1] that t and u are m measure preserving and that neither t nor u has wandering sets of positive measure.

THEOREM 4. *If* (26) *is satisfied,* $h(u)$ *is* II *summable and* $k(u)>0$, *then for almost all* x_0[II]

$$\lim_{n\to\infty} \frac{h(x_c)+h(x_{a+c})+\cdots+h(x_{na+c})}{k(x_d)+h(x_{b+d})+\cdots+k(x_{nb+d})}$$

exists with probability one.

THEOREM 5. *Let* $y_1, y_2, \cdots$ *be independent random variables with a common distribution function. Suppose that for any interval* I

$$\text{prob}\,(y_c+y_{a+c}+\cdots+y_{na+c}\in I \text{ infinitely often})=1$$

and

$$\text{prob}\,(y_d+y_{b+d}+\cdots+y_{nb+d}\in I \text{ infinitely often})=1.$$

Then for $h(u)$ *Lebesgue integrable* $k(u)>0$ *and almost all* m

$$\lim_{n\to\infty} \frac{\sum_{p=0}^{n} h\left(m+\sum_{j=0}^{p} y_{ja+c}\right)}{\sum_{p=0}^{n} k\left(m+\sum_{j=0}^{p} y_{jb+d}\right)}$$

exists with probability one.

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