ABSTRACT RIEMANN SUMS

PAUL CIVIN

1. Introduction. A theorem of B. Jessen [5] asserts that for $f(x)$ of period one and Lebesgue integrable on [0, 1]

$$(1)\qquad \lim_{n\to\infty} 2^{-n}\sum_{k=0}^{2^n-1} f(x+k2^{-n})=\int_0^1 f(t)dt \text{ almost everywhere.}$$

We show that the theorem of Jessen is a special case of a theorem analogous to the Birkhoff ergodic theorem [1] but dealing with sums of the form

$$(2)\qquad 2^{-n}\sum_{k=0}^{2^n-1} f(T^{k/2^n}x).$$

In this form T is an operator on a σ-finite measure space such that $T^{1/2^n}$ exists as a one-to-one point transformation which is measure preserving for $n=0, 1, \cdots$, and $f(x)$ is integrable with $f(x)=f(Tx)$. We also obtain in § 3 the analogues for abstract Riemann sums of the ergodic theorems of Hurewicz [4] and of Hopf [3].

We might remark that there is no use, due to the examples of Marcinkiewicz and Zygmund [6] and Ursell [8], in considering sums of the form

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^{k/n}x)$$

without further hypothesis on $f(x)$. However we may replace 2^n throughout by $m_1m_2\cdots m_n$ with m_j integral and $m_j\geq 2$ without altering any argument.

In § 4 necessary and sufficient conditions are obtained on a transformation T in order that the sums (2) have a limit as $n\to\infty$ for almost all x. These conditions are analogous to those of Ryll-Nardzewski [7] in the ergodic case. We use the necessary conditions to establish an analogue of a form of the Hurewicz ergodic theorem for two operators [2].

2. Notation. Let (S, Ω, μ) be a fixed σ-finite measure space. We consider throughout point transformations T which have measurable square roots of all orders, that is,

(3.1) *There exist one-to-one point transformations T_n so that*

Received June 2, 1954.

$$T_0 = T\,; \quad T_n^2 = T_{n-1} \qquad n=1, 2, \cdots .$$

(3.2) *If* $X \in \Omega$, *then* $T_n X \in \Omega$ *and* $T_n^{-1} X \in \Omega$, $\qquad n=0, 1, \cdots .$

No requirement is made of the uniqueness of the sequence T_n. For example in the theorem of Jessen, T is the identity transformation while $T_n x = x + 2^{-n}$ (mod 1). We also suppose throughout that T is measure preserving

$$\mu(TX) = \mu(X) \quad for \quad X \in \Omega. \tag{3.3}$$

3. **Limit theorems.** Let Φ be a finite valued set function defined on Ω and absolutely continuous with respect to μ. Form the sums

$$\Phi_n(X) = \sum_{k=0}^{2^n-1} \Phi(T_n^k X) \qquad n=0, 1, \cdots , \tag{4}$$

and

$$\mu_n(X) = \sum_{k=0}^{2^n-1} \mu(T_n^k X) \qquad n=0, 1, \cdots . \tag{5}$$

Then Φ_n is absolutely continuous with respect to μ_n and there exists an *averaging sequence of point functions* $f_n(x)$ so that

$$\Phi_n(X) = \int_X f_n(x) \mu_n(dx)\,, \qquad n=0, 1, \cdots . \tag{2}$$

THEOREM 1. *Let T be a transformation such that* (3.1), (3.2) *and* (3.3) *are satisfied. Let Φ be a finite valued set function defined on Ω, absolutely continuous with respect to μ and such that $\Phi(TX)=\Phi(X)$. Then for almost all $x[\mu]$ the averaging sequence of point functions defined by* (4), (5) *and* (6) *has a limit as $n \to \infty$. The limit function $F(x)$ has the following properties*:

(i) *$F(T_n x) = F(x)$ almost everywhere $[\mu]$,* $\qquad n=0, 1, \cdots .$
(ii) *$F(x)$ is integrable over S.*
(iii) *For any set X with $T_n X = X$, $n=0, 1, \cdots$ and $\mu(X) < \infty$*

$$\int_X F(x) \mu(dx) = \int_X f(x)\, \mu(dx).$$

Proof. Note first that since $\Phi(TX) = \Phi(X)$,

$$\Phi_n(T_n X) = \sum_{k=0}^{2^n-1} \Phi(T_n^{k+1} X) = \Phi(X). \tag{7}$$

Likewise

(8) $$\mu_n(T_n X)=\mu_n(X)\,.$$

Therefore for all X

$$\int_X f_n(T_n x)\mu_n(dx)=\int_{T_n X} f_n(x)\,\mu_n(dx)=\int_X f_n(x)\,\mu_n(dx)$$

and consequently

(9) $$f_n(T_n x)=f_n(x) \qquad \text{almost everywhere } [\mu_n].$$

Relation (3.1) then implies

(10) $$\begin{cases} \overline{\lim_{n\to\infty}}\, f_n(T_m^j x)=\overline{\lim_{n\to\infty}}\, f_n(x) \\ \underline{\lim_{n\to\infty}}\, f_n(T_m^j x)=\underline{\lim_{n\to\infty}}\, f_n(x) \end{cases} \text{almost everywhere } [\mu] \quad \begin{matrix} j=1,\cdots,2^m-1 \\ m=1,2,\cdots \end{matrix}$$

Let

(11) $$A=\{x|\sup_{0\le n} f_n(x)\ge 0\}\,.$$

It is asserted that

(12) $$\int_A f_0(x)\mu(dx)\ge 0\,.$$

We define the following sets:

$$P_j=\{x|f_j(x)\ge 0\} \qquad j=0,1,\cdots$$

$$A_N=\{x|\sup_{0\le n\le N} f_n(x)\ge 0\} \qquad N=0,1,\cdots$$

$$C_{N,j}=P'_N\cap\cdots\cap P'_{j+1}\cap P_j \qquad j=0,\cdots,N.$$

Now (9) together with (3.1) imply that $T_k P_j=P_j$ for $k\le j$. Consequently

$$T_j C_{N,j}=C_{N,j} \text{ and } \Phi(C_{N,j})=\Phi(T_j^k C_{N,j}).$$

Therefore

$$2^j\Phi(C_{N,j})=\sum_{k=0}^{2^j-1}\Phi(T_j^k C_{N,j})=\Phi_j(C_{N,j})$$

and

$$2^j\Phi(C_{N,j})=\int_{C_{N,j}} f_j(x)\mu_j(dx)\ge 0\,, \qquad j=0,\cdots,N.$$

Since the $C_{N,j}$ are disjoint for $j=0,\cdots,N$, we have $\Phi(A_N)\ge 0$ and by a limiting process we obtain (12).

Likewise if

$$B=\{x|\inf_{0\leq n} f_n(x)\geq 0\}, \tag{13}$$

then

$$\int_B f_0(x)\mu(dx)\geq 0 . \tag{14}$$

Inasmuch as the preceding argument made no use of the finiteness of Φ, we may apply the result to the set function $\Psi=\Phi-c\mu$ for any real c. Since

$$\Psi_n(X)=\int_X (f_n(x)-c)\mu_n(dx)$$

we deduce that for

$$A^c=\{x|\sup_{0\leq n} f_n(x)\geq c\} \tag{15}$$

we have

$$\Phi(A^c)\geq c\mu(A^c) \tag{16}$$

and for

$$A_d=\{x|\inf_{0\leq n} f_n(x)\leq d\} \tag{17}$$

we have

$$\Phi(A_d)\leq d\mu(A_d). \tag{18}$$

Let now for $r>s$

$$L_s^r=\{x|\overline{\lim_{n\to\infty}} f_n(x)>r \text{ and } \underline{\lim_{n\to\infty}} f_n(x)<s\} . \tag{19}$$

From (10) we obtain

$$T_m^j L_s^r=L_s^r \qquad j=0,\ 1,\ \cdots,\ 2^m-1\ ;\ m=0,\ 1,\ \cdots . \tag{20}$$

Since L_s^r is invariant under each T_m we may consider it as a new space. The sets A^r and A_s relative to the new space are now the full space L_s^r. Hence if we apply (16) and (18) we obtain

$$\Phi(L_s^r)\geq r\mu(L_s^r)\ ;\quad \Phi(L_s^r)\leq s\mu(L_s^r) .$$

The finiteness of Φ together with the assumption $r>s$ implies $\mu(L_s^r)=0$. Thus $\lim_{n\to\infty} f_n(x)$ exists almost everywhere $[\mu]$.

Property (i) of the limit function $F(x)$ follows immediately from (10). Utilizing (i) the proofs of (ii) and (iii) are now identical with the corresponding proofs by Hurewicz [4, p. 201] in the ergodic case.

The theorem for abstract Riemann sums analogous to the Hopf ergodic theorem is now deducible as a corollary.

COROLLARY 1. *Let T be a transformation such that* (3.1) *and* (3.2) *are satisfied and in addition*

$$\mu(T_n X) = \mu(X) \qquad n = 0, 1, \cdots. \tag{21}$$

Then for any integrable $f(x)$ with $f(Tx) = f(x)$ and any $g(x) > 0$ with $g(Tx) = g(x)$

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{2^n-1} f(T_n^k x)}{\sum_{k=0}^{2^n-1} g(T_n^k x)} \tag{22}$$

exists for almost every x $[\mu]$. The limit function $h(x)$ is integrable, satisfies $h(T_n x) = h(x)$ for almost all x $[\mu]$, and for sets Y with $\mu(Y) < \infty$ and $T_m Y = Y$, $m = 0, 1, \cdots$

$$\int_Y h(x) g(x) \mu(dx) = \int_Y f(x) \mu(dx). \tag{23}$$

Proof. Introduce the measure

$$\nu(X) = \int_X g(x) \mu(dx),$$

and the set function

$$F(X) = \int_X f(x) \mu(dx).$$

The function F is absolutely continuous with respect to ν and is finite valued. Condition (21) implies that

$$F_n(X) = \int_X \sum_{k=0}^{2^n-1} f(T_n^k x) \mu(dx)$$

and

$$\nu_n(X) = \int_X \sum_{k=0}^{2^n-1} g(T_n^k x) \mu(dx).$$

Thus from the representation

$$F_n(X) = \int_X f_n(x) \nu_n(dx)$$

we deduce that

$$f_n(x)=\frac{\sum_{k=0}^{2^n-1} f(T_n^k x)}{\sum_{k=0}^{2^n-1} g(T_n^k x)} \quad \text{almost everywhere } [\mu].$$

The corollary is then an immediate consequence of Theorem 1.

The theorem of Jessen now follows from the version of Corollary 1 with $g(x)=1$ with the T_n as noted in § 2.

4. Invariant measure and two operators. It is possible for the conclusion of Corollary 1 to hold when $g(x)=1$ but T does not satisfy (21). If we introduce

$$R_n(A,\ Y)=2^{-n}\sum_{k=0}^{2^n-1}\mu(Y\cap T_n^{-k}A) \tag{24}$$

we obtain the following theorem.

THEOREM 2. *If T is a transformation such that* (3.1) *and* (3.2) *are satisfied, then the following statements are equivalent* :

(25.1) *For every integrable $f(x)$ with $f(Tx)=f(x)$,*

$$\lim_{n\to\infty} 2^{-n}\sum_{k=0}^{2^n-1} f(T_n^k x)$$

exists for almost every x $[\mu]$.

(25.2) *For each Y with $\mu(Y)<\infty$, $\lim_{n\to\infty} R_n(A,\ Y)\le K\mu(A)$.*

(25.3) *For each Y with $\mu(Y)<\infty$, $\overline{\lim_{n\to\infty}} R_n(A,\ Y)\le K\mu(A)$.*

(25.4) *For an increasing sequence of sets Y_j with $\bigcup_{j=1}^{\infty} Y_j=S$,*

$$\overline{\lim_{n\to\infty}} R_n(A,\ Y_j)\le K\mu(A)\ .$$

(25.5) *There exists a countably additive measure ν with the properties:*

(i) $0\le\nu(X)\le K\mu(X)$

(ii) *If* $A=T_nA$, $\quad n=1,\ 2,\ \cdots$, $\nu(A)=\mu(A)$

(iii) $\nu(A)=\nu(T_nA)$, $\quad n=1,\ 2,\ \cdots$.

The proof is almost identical with that of Ryll-Nardzewski [7] in the ergodic case, and is omitted. The existence of an invariant measure implies, as in the ergodic case [2], the following theorem with two operators (or two sequences of roots of the same operator).

THEOREM 3. *Let T and U each satisfy* (3.1), (3.2), (3.3) *and* (25.1), *and let*

$$\sum_{k=0}^{2^n-1} \mu(T_n^k X)$$

be absolutely continuous with respect to

$$\mu_n(X)=\sum_{k=0}^{2^n-1} \mu(U_n^k X), \qquad n=0,\ 1,\ \cdots.$$

For any finite valued set function Φ absolutely continuous with respect to μ and with $\Phi(TX)=\Phi(X)$ form

$$\Phi_n(X)=\sum_{k=0}^{2^n-1} \Phi(T_n X).$$

Then in the representation

$$\Phi_n(X)=\int_X f_n(x)\mu_n(dx),$$

the averaging sequence of point functions $f_n(x)$ tends to a limit as $n\to\infty$ for almost every x $[\mu]$.

As a consequence of Theorem 3 we obtain the following corollary in the same fashion as Corollary 1 was derived from Theorem 1.

COROLLARY 2. *Let T and U each satisfy* (3.1) *and* (3,2), *and in addition*

$$\mu(V_n X)=\mu(X) \qquad n=0,\ \cdots \tag{26}$$

for $V=T$ and $V=U$. Then for any integrable $f(x)$ with $f(Tx)=f(x)$ and any $g(x)>0$ with $g(Ux)=g(x)$

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{2^n-1} f(T_n^k X)}{\sum_{k=0}^{2^n-1} g(U_n^k X)}$$

exists for almost all x $[\mu]$.

REFERENCES

1. G.D. Birkhoff, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci., **17** (1932), 650.
2. P. Civin, *Some ergodic theorems involving two operators*, Pacific J. Math., **5** (1955), 869-876.
3. E. Hopf, *Ergodentheorie*, Berlin 1937.
4. W. Hurewicz, *Ergodic theorem without invariant measure*, Ann. Math., **45** (1944), 192-206.
5. B. Jessen, *On the approximation of Lebesgue integrals by Riemann sums*, Ann. Math., **35** (1934), 248-251.
6. J. Marcinkiewicz and A. Zygmund, *Mean values of trigonometric polynomials*, Fund. Math., **28** (1937), 131-166.
7. C. Ryll-Nardzewski, *On ergodic theorems I*, Studia Math., **12** (1951), 65-73.
8. H.D. Ursell, *On the behavior of a certain sequence of functions derived from a given one*, J. London Math. Soc., **12** (1937), 229-232.

UNIVERSITY OF OREGON AND
THE INSTITUTE FOR ADVANCED STUDY