ON A THEOREM OF S. BERNSTEIN

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1. Introduction and proof of the main theorem. A result of S. Bernstein [4] is the following.

THEOREM A. *If* $p(z)$ *is a polynomial of degree* n *such that* $[\max|p(z)|,\ |z|=1]=1$, *then*

$$[\max|p(z)|,\ |z|=R>1]\leq R^n, \tag{1}$$

with equality only for $p(z)=\lambda z^n$, *where* $|\lambda|=1$.

We propose to show here that if we restrict ourselves to polynomials of degree n having no zero within the unit circle the right hand member of (1) can be made smaller. In particular we have the following result.

THEOREM 1. *If* $p(z)$ *is a polynomial of degree* n *such that* $[\max|p(z)|,\ |z|=1]=1$, *and* $p(z)$ *has no zero within the unit circle, then*

$$[\max|p(z)|,\ |z|=R>1]\leq\frac{1+R^n}{2},$$

with equality only for $p(z)=(\lambda+\mu z^n)/2$, *where* $|\lambda|=|\mu|=1$.

In order to prove Theorem 1 we use a conjecture of Erdös first proved by Lax [2] (See also [1]).

THEOREM B. *If* $p(z)$ *is a polynomial of degree* n *such that* $[\max|p(z)|,\ |z|=1]=1$, *and* $p(z)$ *has no zero within the unit circle, then*

$$[\max|p'(z)|,\ |z|=1]\leq\frac{n}{2}.$$

Turning now to Theorem 1, let us assume that $p(z)$ does not have the form $(\lambda+\mu z^n)/2$. In view of Theorem B

$$|p'(e^{i\varphi})|\leq\frac{n}{2},\qquad 0\leq\varphi<2\pi, \tag{2}$$

from which we may deduce that

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(3) $$|p'(re^{i\varphi})|<\frac{n}{2}r^{n-1}\,,\qquad 0\leq\varphi<2\pi,\qquad r>1\,,$$

by applying Theorem A to the polynomial $p'(z)/(n/2)$ and observing that we have the strict inequality in (3) because $p(z)$ does not have the form $(\lambda+\mu z^n)/2$. But for each φ, $0\leq\varphi<2\pi$, we have

$$p(Re^{i\varphi})-p(e^{i\varphi})=\int_1^R e^{i\varphi}p'(re^{i\varphi})dr\,.$$

Hence

$$|p(Re^{i\varphi})-p(e^{i\varphi})|\leq\int_1^R|p'(re^{i\varphi})|dr<\frac{n}{2}\int_1^R r^{n-1}dr=\frac{R^n-1}{2}\,,$$

and

$$|p(Re^{i\varphi})|<\frac{R^n-1}{2}+|p(e^{i\varphi})|\leq\frac{1+R^n}{2}\,.$$

Finally, if $p(z)=(\lambda+\mu z^n)/2$, $|\lambda|=1$, then

$$[\max|p(z)|,\ |z|=R>1]=\frac{1+R^n}{2}\,.$$

As a corollary of Theorem 1 we may deduce

THEOREM 2. *If $p(z)$ is a polynomial of degree n with real coefficients having all zeros of nonpositive real part and if for some $R>1$*

$$p(R)>p(1)\left(\frac{R^k+R^n}{2}\right)\,,$$

k a nonnegative integer, then $p(z)$ has at least $(k+1)$ zeros in $|z|<1$.

Proof. Suppose $p(z)$ has m zeros in $|z|<1$ and $m\leq k$. Let

$$p(z)=(z-z_1)\cdots(z-z_m)(z-z_{m+1})\cdots(z-z_n)\,,$$

and suppose $|z_j|<1$, $(j=1,\cdots,m)$. Put

$$g(z)=(z-z_1)\cdots(z-z_m)$$

and

$$h(z)=(z-z_{m+1})\cdots(z-z_n)\,.$$

The polynomials $p(z)$, $g(z)$ and $h(z)$ have positive coefficients, hence for all $R>1$

$$g(R)\leq g(1)R^m$$

and

$$h(R) \leq h(1)\left(\frac{1+R^{n-m}}{2}\right)$$

according to Theorems A and 1 respectively.
Thus

$$p(R) = h(R)g(R) \leq p(1)\left(\frac{R^m+R^n}{2}\right) \leq p(1)\left(\frac{R^k+R^n}{2}\right),$$

a contradiction, establishing Theorem 2.

2. The converse problem. The converse of Theorem 1 is false as the simple example $p(z)=(z+\frac{1}{2})(z+3)$ shows. However, the following result in the converse direction is valid.

THEOREM 3. *If $p(z)$ is a polynomial of degree n such that*

$$p(1) = [\max |p(z)|,\ |z|=1] = 1$$

and

$$[\max |p(z)|,\ |z|=R>1] \leq \frac{1+R^n}{2}$$

for $0<R-1<\delta$, where δ is any positive number, then $p(z)$ does not have all its roots within the unit circle.

For the proof we need the following

LEMMA. *If*

$$q(z) = (z-z_1)\cdots(z-z_m)$$

where $|z_j|<1$, $(j=1, \cdots, m)$, then if $|a|=1$ we have

$$\left|\frac{q'(a)}{q(a)}\right| > \frac{m}{2}.$$

Proof. According to Laguerre's Theorem [3, p. 38]

$$\frac{q'(a)}{q(a)} = \frac{m}{a-w},$$

where $|w|<1$, hence $|a-w|<2$ and

$$\left|\frac{q'(a)}{q(a)}\right| > \frac{m}{2}.$$

We turn now to the proof of Theorem 3. Suppose $p(z)$ has all its zeros in $|z|<1$. Let

$$p(z)=a_0+a_1z+\cdots+a_nz^n\ ,$$

put

$$\bar{p}(z)=\bar{a}_0+\bar{a}_1z+\cdots+\bar{a}_nz^n$$

and consider the polynomial $g(z)=p(z)\bar{p}(z)$ of degree $2n$. $g(z)$ is real for real z,

$$[\max|g(z)|,\ |z|=1]=g(1)=1\ ,$$

$$|g(Re^{i\varphi})|\leq\left(\frac{1+R^n}{2}\right)^2\leq\frac{1+R^{2n}}{2}$$

and $g(z)$ has all its zeros in $|z|<1$. Now $g'(1)$ is not only real but positive. This is so since, given any $\eta>0$, we have $g(1-\eta)<g(1)$. Hence

$$g'(1)=\lim_{\eta\to 0}\frac{g(1-\eta)-g(1)}{-\eta}\geq 0.$$

Now $g'(1)\neq 0$, as all of the roots of $g(z)=0$ are inside the unit circle, hence, by Lucas' Theorem all roots of $g'(z)=0$ are within the convex closure of the unit circle namely the unit circle itself.

Given any $\varepsilon>0$, sufficiently small,

$$|g(1+\varepsilon)-g(1)|=g(1+\varepsilon)-g(1)\leq\frac{(1+\varepsilon)^{2n}+1}{2}-1=\frac{(1+\varepsilon)^{2n}-1}{2}\ ,$$

or

$$|g(1+\varepsilon)-g(1)|\leq n\varepsilon+O(\varepsilon^2)\ ,\qquad\text{as } \varepsilon\to 0$$

and $g'(1)\leq n$. Therefore $g'(1)/g(1)\leq n$ contradicting the lemma. Theorem 3 is established.

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