ON CLOSED DIFFERENTIABLE CURVES OF ORDER n IN n-SPACE

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1. Introduction. Let C_n be a closed curve in real projective n-space S_n whose coordinates x_i $(1 \leqq i \leqq n+1)$ are given in the parametric form

$$x_i = x_i(s)\,, \quad 1 \leqq i \leqq n+1\,, \quad q \leqq s < q+1\,,$$

where $x_i(s)$ are real continuous periodic functions of period 1, and q is any real number. The point with coordinates $x_i(s)$ $(1 \leqq i \leqq n+1)$ will be designated by its defining number s.

The curve C_n is to satisfy the following *order condition.*

No hyperplane of S_n contains more than n points of C_n.

A simple consequence of the above condition is that any $k+1$ $(0 \leqq k \leqq n)$ distinct curve points $s_1, s_2, \cdots, s_{k+1}$ span a linear k-subspace $[s_1, s_2, \cdots, s_{k+1}]$. (The square-bracket symbol $[A, B, \cdots]$ will be used throughout to designate the linear subspace spanned by the sets A, B, $\cdots$.)

The curve C_n is to satisfy the following *differentiability condition.*

For each point s of C_n and for each integer k $(0 \leqq k \leqq n-1)$ a linear k-subspace (k, s), known as the osculating k-space at s, exists for which $[s_1, s_2, \cdots, s_{k+1}]$ converges to (k, s) as $s_1, s_2, \cdots, s_{k+1}$ all approach s in any way whatsoever.

The curves C_3 were considered by A. Kneser [2] who studied properties which are invariant to certain continuous displacements. One of his results is that the set of planes of the projective space each of which contains exactly k $(k=1$ or $3)$ points of a C_3 builds a connected set. In the present paper the methods used by Kneser are adapted to study the properties of the curves C_n. All the proofs make use of those lines l each point of which is included in n distinct $(n-1, s)$. Thus the paper is, in a sense, a study of this line system. Among

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the results is a generalization of the foregoing Kneser result to n dimensions. This in turn leads to the result that those hyperplanes which contain less than n points of C_n are exactly those hyperplanes which contain at least one line l. This result is related to a result, implicit in a paper of Scherk [4], which states that the above hyperplanes are exactly those hyperplanes which contain certain limiting positions of the lines l.

2. Multiplicities. As all the critical boundary cases involve multiple intersection points, these points will have special importance. In this section we record the definition for multiplicity and note some known results which we shall use.

DEFINITION 1. A linear subspace Q is defined to intersect C_n *exactly k-fold* $(0<k\leqq n-1)$ at s if $(k-1, s)\subseteqq Q$, $(k, s)\nsubseteqq Q$, and *n-fold* if $(n-1, s)=Q$.

A point P is defined to be included in $(n-1, s)$ *exactly k-fold* $(0<k\leqq n-1)$ if $P\in(n-k, s)$, $P\notin(n-k-1, s)$, and *n-fold* if $P=(0, s)$.

The following *multiplicity convention* will be assumed throughout. Let $s_1, s_2, \cdots, s_j$ be any point system, and let s_i occur k_i-times $(1\leqq i\leqq j)$ in this system. A linear subspace Q is said to contain this system provided $(k_i-1, s_i)\subseteqq Q$ $(1\leqq i\leqq j)$. A point P is said to be included in the system $(n-1, s_1)$, $(n-1, s_2), \cdots, (n-1, s_j)$ provided $P\in(n-k_i, s_i)$ $(1\leqq i\leqq j)$. Unless otherwise stated the points of any given set are not necessarily all distinct.

For reference we state the easily proved:

LEMMA 1. *For $n\geqq 2$, the projection of C_n from one of its curve points s' is a C_{n-1}. The space (k, s), $s\neq s'$, $0\leqq k\leqq n-2$, projects into the space (k, s) of the projected C_{n-1} and the space (k, s'), $1\leqq k\leqq n-1$, into the space $(k-1, s')$ of C_{n-1}.*

By use of Lemma 1, it can be proved by induction that C_n satisfies the *sharpened order condition*, that no hyperplane cuts C_n in more than n curve points where multiple intersections are now counted with their proper multiplicity. This leads to the fact that the system s_1, $s_2, \cdots, s_{k+1}$ $(0<k\leqq n-1)$ is included in a unique k-space which we designate by $[s_1, s_2, \cdots, s_{k+1}]$. We note without proof that C_n satisfies the *sharpened differentiability condition* that $[s_1, s_2, \cdots, s_{k+1}]$ converges to (k, s) as $s_1, s_2, \cdots, s_{k+1}$ all approach s.

Use will be made of the duality theorem of Scherk [3] which states that all the $(n-1, s)$ build the dual of a C_n. This implies that no point P is contained within more than n $(n-1, s)$ and also that the intersection of $(n-1, s_1)$, $(n-1, s_2)$, $\cdots$, $(n-1, s_k)$ $(1\leqq k\leqq n)$ approaches $(n-k, s)$ as $s_1, s_2, \cdots, s_k$ all approach s in any way whatsoever.

3. **Notation.** Throughout the paper the symbols l, l^μ will be tacitly assumed to represent lines each of the points of which is within n distinct $(n-1, s)$ of a given C_n; L, L^μ will be assumed to represent the $(n-2)$-spaces with the property that every hyperplane through such a space cuts C_n in n distinct points.

Where a proof involves both C_n and C_{n-1} the symbol $(k, s)_{n-1}$ will be used to designate the osculating k-space of the curve C_{n-1}.

4. **A construction for the lines l.**

THEOREM 1. *If, for $n\geqq 2$, A and B are any two distinct points of a given line l, then curve points s_i, t_i of C_n exist so that $A\in(n-1, s_i)$, $B\in(n-1, t_i)$ $(1\leqq i\leqq n)$ and $s_1<t_1<s_2<\cdots<s_n<t_n<s_1+1$ $(=s_{n+1})$.*

Conversely if A and B are points for which $A\in(n-1, s_i)$, $B\in(n-1, t_i)$, $s_1<t_1<s_2<t_2<\cdots<s_n<t_n<s_1+1$ $(=s_{n+1})$, then AB is a line l.

PROOF. Let $P(s)$ be the intersection $l\cap(n-1, s)$. Note that $l\not\subseteq(n-1, s)$; for otherwise l would contain a point of $(n-2, s)$, which point would be within $(n-1, s)$ at least twice contrary to the definition of l. Therefore $P(s)$ is defined uniquely for all s. As s moves continuously on C_n in a fixed direction, $P(s)$ moves continuously on l because $(n-1, s)$ is continuous. Also, $P(s)$ moves continuously in a fixed direction; for if $P(s)$ were to experience a reversal of direction at $P(s_0)$ then, in every curve neighborhood of s_0, points s_L, s_R would exist so that $s_L<s_0<s_R$, $P(s_L)=P(s_R)$. Then, as $P(s)$ is continuous,

$$P(s_0)\in\lim_{s_L\to s_0,\ s_R\to s_0}(n-1, s_L)\cap(n-1, s_R)=(n-2, s_0)$$

and l would contain a point not in n distinct $(n-1, s)$ contrary to the hypothesis. Let $(n-1, s_i)$ $(1\leqq i\leqq n;\ s_1<s_2<\cdots<s_n<s_1+1\ (=s_{n+1}))$ be the complete set of $(n-1, s)$ which contain A. As s increases continuously from s_1 to s_2, $P(s)$ makes one complete circuit of l in a fixed direction. Consequently it crosses the point B exactly once. Hence t_1 exists on C_n so that $B\in(n-1, t_1)$ $(s_1<t_1<s_2)$. Likewise within each arc $s_i<s<s_{i+1}$ $(2\leqq i\leqq n)$, a point t_i exists on C_n so that $s_i<t_i<s_{i+1}$, $B\in(n-1, t_i)$. Thus the theorem is proved.

To prove the converse, let C be any interior point of one of the segments AB of the line through A and B, and D any interior point of the other segment. As $P(s)$ is continuous and

$$P(s_1)=A\ , \quad P(t_1)=B\ ,$$

at least one solution $P(s)=C$, or $P(s)=D$ must exist for which $s_1<s<t_1$. Likewise each of the $2n$ arcs $s_i<s<t_i$, $t_i<s<s_{i+1}$ $(1\leqq i\leqq n)$ contains at least one solution $P(s)=C$ or $P(s)=D$. But as C is contained in at most n $(n-1, s)$ there must be exactly n solutions $P(s)=C$. As these are all distinct and C is arbitrary, AB is a line l. The proof is now complete.

This proof of the converse, due to Dr. P. Scherk, replaces a more complicated one of my own. I should like to take the opportunity to thank him for many helpful suggestions which have contributed to the readability of the paper.

5. Hyperplanes with a given number of curve points.

LEMMA 2. *If, for $n\geqq 3$, C_{n-1} is the projection of C_n from one of its points s, then a line l of C_n is projected into a line l of C_{n-1}.*

This is proved in [1].

LEMMA 3. *For $n\geqq 3$, the projection of a C_n from a line l is a C_{n-2}.*

PROOF. No hyperplane through l can cut C_n in more than $n-2$ points. This is true for $n=2$ as it is equivalent to the fact that a line l of C_2 cannot contain any curve points. Assume the assertion is true for C_{n-1} $(n>2)$. Let H be a hyperplane which contains l. The result is clear if H contains no points of C_n. Let $\bar{s}$ be a point of C_n within H. Project from s. Then C_n is projected into a C_{n-1} by Lemma 1, and l into a line l of C_{n-1}, by Lemma 2, which is within the projection $\overline{H}$ of H. By the induction assumption $\overline{H}$ contains at most $n-3$ points of C_{n-1}. Therefore H, which contains the points C_n into which these are projected together with $\bar{s}$ contains at most $n-2$ points of C_n.

The space of all 2-spaces through l is an $(n-2)$-space S_{n-2} whose hyperplanes are the hyperplanes of the original space which contain l. The elements $[l, s]$ of S_{n-2} build a curve C, and C has order $n-2$ by the result of the previous paragraph. This implies

$$[l, s']\neq[l, s''] \qquad \text{if } s'\neq s''\ .$$

Thus there is a one-to-one correspondence between the points of C_n and those of C. Where $0 \leqq k \leqq n-2$, let

$$[l,\ s_1],\ [l,\ s_2], \cdots,\ [l,\ s_{k+1}]$$

be given curve points of C. Because of the order condition these points span a $(k+2)$-space Q which contains l. If $s_1, s_2, \cdots, s_{k+1}$ all approach s, then $Q \to [l,\ (k,\ s)]$ because of the differentiability condition. Thus the set of elements $[l,\ s]$ of S_{n-2} is a C_{n-2} with osculating k-spaces $[l,\ (k,\ s)]$. As this set is equivalent to the projection of C_n from l, the lemma is established.

Most induction proofs for the curves C_n make use of Lemma 1; in the following proof Lemma 3 is used for this purpose.

THEOREM 2. *Where* $0 \leqq k \leqq n$, $k \equiv n$ (mod 2), *let* $s_1, s_2, \cdots, s_k$; $t_1, t_2, \cdots, t_k$ *be any points of* C_n; *then*:

(a) *If, for* $n \geqq 1$, H_1, H_2 *be hyperplanes which contain* $s_1, s_2, \cdots, s_k$; $t_1, t_2, \cdots, t_k$ *respectively, and no additional points of* C_n, *then hyperplanes* $H(p)$ $(0 \leqq p \leqq 1)$ *exist, continuously dependent on* p, *each of which contains exactly* k *points of* C_n *and for which* $H(0)=H_1$, $H(1)=H_2$;

(b) *If* $s_i=t_i$ $(1 \leqq i \leqq k)$, *then* $H(p)$ *can be chosen so that it contains exactly the points* s_i $(1 \leqq i \leqq k,\ 0 \leqq p \leqq 1)$;

(c) *if* $n \geqq 2$, $0 \leqq k \leqq n-2$, *for a given line* l, *a hyperplane* H^l *exists so that it contains exactly the points* $s_1, s_2, \cdots, s_k$, *together with the line* l.

PROOF. We first prove (c). If $n=2$ then $k=0$ and the result is equivalent to the fact that $H'=l$ does not cut C_2. Assume the result for for all curves C_{n-1} $(n>2)$. Project from l. Thus C_n is projected into a C_{n-2}, by Lemma 3, and $s_1, s_2, \cdots, s_k$ into points of C_{n-2} with the same numerical coordinates. If $k=n-2$, a unique hyperplane

$$H'=[s_1,\ s_2,\ \cdots,\ s_k]$$

exists in the projected $(n-2)$-space through these points. If $k<n-2$, then by the induction assumption a hyperplane H' exists in the projected space which contains exactly the points $s_1, s_2, \cdots, s_k$ of C_{n-2}. Consequently, if H^l is defined to be the hyperplane of the original space which is projected into H', this hyperplane contains exactly the points $s_1, s_2, \cdots, s_k$ of C_n. As $l \subseteqq H^l$, (c) is proved for C_n. The proof

can now be completed by induction.

To prove (a) and (b), consider first the case $k=0$. With this restriction neither H_1 nor H_2 contains points of C_n. As the curve is connected, it lies entirely within one of the two open regions of the projective space whose boundary is the set of points of H_1 and H_2. Hence an affine coordinate system exists so that the equations of H_1, H_2 are $x_1=0$, $x_1=1$, respectively, and C_n contains no points for which $0\leqq x_1\leqq 1$. Now (a) and (b) follow for $k=0$ if $H(p)$ is defined to be the hyperplane with the equation $x_1=p$, $0\leqq p\leqq 1$.

Now let $k=n$; (b) is trivial in this case. Let $f_i(p)$ $(0\leqq p\leqq 1,\ 1\leqq i\leqq n)$ be any real-valued continuous functions for which $f_i(0)=s_i$, $f_i(1)=t_i$. Then (a) follows if $H(p)$ is defined to be the hyperplane spanned by the points with coordinates $f_i(p)$ $(1\leqq i\leqq n)$.

In particular this establishes (a) and (b) for C_1 and C_2. Assume both results for all C_{n-1} $(n>2)$. We may assume $0<k\leqq n-2$. Let l be arbitrary. By (c), hyperplanes H_1^l, H_2^l exist which contain exactly the points $s_1, s_2, \cdots, s_k$; $t_1, t_2, \cdots, t_k$, respectively, together with the line l. Let $\overline{H}_1$, $\overline{H}_1^l$, C_{n-1} be the projections of H_1, H_1^l, C_n, respectively, from s_1. By the induction assumption (b), hyperplanes $\overline{H}(p)$ $(0\leqq p\leqq 1)$ exist in the projected space, continuously dependent on p, each of which contains exactly the points $s_2, \cdots, s_k$ of C_{n-1}, and for which

$$\overline{H}(0)=\overline{H}_1,\quad \overline{H}(1)=\overline{H}_1^l\,.$$

Let $H(p)$ $(0\leqq p\leqq (1/3))$ be the hyperplane of the original space which is projected into $\overline{H}(3p)$. Then $H(p)$ depends continuously on p, contains exactly the points $s_1, s_2, \cdots, s_k$ of C_n, and $H(0)=H_1$, $H(1/3)=H_1^l$. Likewise $H(p)$ $((2/3)\leqq p\leqq 1)$ exists so that it depends continuously on p, contains exactly the points $t_1, t_2, \cdots, t_k$ of C_n, and for which

$$H(2/3)=H_2^l,\quad H(1)=H_2\,.$$

After a projection from l, a similar argument can be used to construct a hyperplane $H(p)$ $((1/3)\leqq p\leqq (2/3))$ which depends continuously on p, contains exactly k points of C_n, and for which

$$H(1/3)=H_1^l\,,\quad H(2/3)=H_2^l\,.$$

This proves (a) for C_n. Also (b) is clear if $H(p)$ is defined as above with the additional conditions that

$$H_1^l=H_2^l=H(p)\qquad ((1/3)\leqq p\leqq (2/3))\,.$$

The proof can now be completed by induction.

6. Hyperplanes which do not contain n points of C_n.

DEFINITION 2. $\sum(C_n)$ is the set of all points included in at least one space L of the curve C_n (cf. § 3).

LEMMA 4. *If, for $n \geqq 3$, $\overline{P} \in \sum(C_{n-1})$, where $\overline{P}$ is the projection of a point P from a point s' of C_n, $P \neq s'$, and C_{n-1} that of C_n, then $P \in \sum(C_n)$.*

Proof. If $\overline{P} \in \sum(C_{n-1})$, then points $s_1, s_2, \cdots, s_{n-1}$; $t_1, t_2, \cdots, t_{n-1}$ of the projection C_{n-1} exist so that

$$\overline{P} \in [s_1, s_2, \cdots s_{n-1}] \cap [t_1, t_2, \cdots, t_{n-1}] = L$$

and

$$s_1 < t_1 < s_2 < \cdots < t_{n-1} < s_1 + 1 ,$$

by the dual of Theorem 1. Moreover,

$$[s_1, s_2, \cdots, s_{n-1}], [t_1, t_2, \cdots, t_{n-1}]$$

may be chosen to be any two distinct hyperplanes through L within the projected $(n-1)$-space. Therefore these hyperplanes may be chosen so that $t_{n-1} < s' < s_1 + 1$. Let the numbers

$$s_1, s_2, \cdots, s_{n-1}, t_1, t_2, \cdots, t_{n-1}, s'$$

now represent points of C_n. Then $P \in [t_1, t_2, \cdots, t_{n-1}, s']$. As $t_1, t_2, \cdots, t_{n-1}, s'$ are represented by linearly independent vectors the intersection

$$\prod_{i=1}^{i=n-1} [t_1, t_2, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{n-1}, s'] = s'.$$

Hence, because $P \neq s'$, at least one value i exists with

$$P \notin [t_1, t_2, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{n-1}, s'] \qquad (1 \leqq i \leqq n-1).$$

For such a value i

$$[t_1, t_2, \cdots, t_{i-1}, P, t_{i+1}, \cdots, t_{n-1}, s'] = [t_1, t_2, \cdots, t_{n-1}, s'].$$

Let t_n be a point of C_n with $t_n > s'$. Then

$$[t_1, t_2, \cdots, t_{i-1}, P, t_{i+1}, \cdots, t_{n-1}, t_n]$$

approaches $[t_1, t_2, \cdots, t_{n-1}, s']$ as t_n approaches s'. Because of the continuity of the curve points of C_n, $[t_1, t_2, \cdots, t_{i-1}, P, t_{i+1}, \cdots, t_n]$ will contain a point t_i' of C_n for which $s_i < t_i' < s_{i+1}$ provided t_n is sufficiently close to s'. If t_n is such a point, and s_n is defined as s', then

$$P \in [s_1, s_2, \cdots, s_n] \cap [t_1, t_2, \cdots, t_{i-1}, t_i', t_{i+1}, \cdots, t_n]$$

and

$$s_1 < t_1 < s_2 < \cdots < s_i < t_i' < s_{i+1} < s_n < t_n < s_1 + 1 .$$

It follows from the dual of Theorem 1 and Definition 2 that $P \in \sum(C_n)$. The lemma is thus established.

COROLLARY. *If, for $n \geq 3$, P is a point for which $P \in [(k, s_1), s_2]$ $(s_1 \neq s_2,\ 0 \leq k \leq n-3,\ P \neq s_2)$ $P \notin (k, s_1)$, then $P \in \sum(C_n)$.*

PROOF. If $n=3$ then $P \in [s_1, s_2]$ $(s_1 \neq s_2,\ P \neq s_1,\ P \neq s_2)$. Let t_1, t_2 be points of C_3 for which $s_1 < t_1 < s_2 < t_2 < s_1 + 1$. Then $P \notin [t_1, t_2]$; for otherwise t_1, t_2, s_1, s_2 would be coplanar in contradiction to the order condition. Hence $[P, t_1, t_2]$ is a plane. This plane must contain a third point t of C_3, as C_3 is closed. Now $P \neq t$ because $[s_1, s_2]$ cannot contain a third curve point. If $\overline{P}$ is the projection of P from t then

$$\overline{P} \in [s_1, s_2] \cap [t_1, t_2],$$

where s_1, s_2, t_1, t_2 now represent curve points of the projection C_2 of C_3 from t. This implies, by the dual of Theorem 1, that $\overline{P} \in \sum(C_2)$, and so by the Lemma that $P \in \sum(C_3)$. Thus the corollary is true for $n=3$. Assume it to be true for all C_{n-1}, $n>3$. The result for C_n then follows from the Lemma by a projection from s_1 if the least possible $k=n-3$ and otherwise by a projection from a point of C_n different from s_1 and s_2.

LEMMA 5. (a) *For $n \geq 2$, $\sum(C_n)$ is open.* (b) *If a boundary point $\overline{P}$ of $\sum(C_n)$ is approached by a sequence P^μ of points interior to $\sum(C_n)$, and $\overline{L}$ is the limit of a space sequence L^μ for which $P^\mu \in L^\mu$, then (k, s) $(0 \leq k \leq n-2)$ exists for which $\overline{P} \in (k, s) \subseteqq \overline{L}$.*

PROOF. If $P \in \sum(C_n)$ then a space L exists for which $P \in L$. By the dual of Theorem 1, $s_1, s_2, \cdots, s_n$; $t_1, t_2, \cdots, t_n$ exist so that

$$L \subseteqq [s_1, s_2, \cdots, s_n] \cap [t_1, t_2, \cdots, t_n] \text{ and } s_1 < t_1 < s_2 < \cdots < t_n < s_1 + 1.$$

If P' is sufficiently close to P then it is contained within an $(n-2)$-space L' which is so close to L that it has the form

$$[s_1', s_2', \cdots, s_n'] \cap [t_1', t_2', \cdots, t_n'] \qquad (s_1' < t_1' < s_2' < \cdots < t_n' < s_1' + 1).$$

By the dual of Theorem 1, $P' \in \sum(C_n)$, and so (a) is proved.

To prove (b), let H_1^μ, H_2^μ be any two hyperplane sequences with $L^\mu \subseteqq H_1^\mu$, $L^\mu \subseteqq H_2^\mu$, which converge to two distinct limits H_1 and H_2, respectively. By the dual of Theorem 1, $s_1^\mu, s_2^\mu, \cdots, s_n^\mu; t_1^\mu, t_2^\mu, \cdots, t_n^\mu$ exist so that $s_1^\mu < t_1^\mu < s_2^\mu < \cdots < t_n^\mu < s_1^\mu + 1$ and

$$H_1^\mu = [s_1^\mu, s_2^\mu, \cdots, s_n^\mu], \quad H_2^\mu = [t_1^\mu, t_2^\mu, \cdots, t_n^\mu].$$

As H_1^μ, H_2^μ converge, the sequences s_i^μ, t_i^μ $(1 \leqq i \leqq n)$ also converge. If s_i, t_i are the respective limits of these sequences,

$$\overline{L} = [s_1, s_2, \cdots, s_n] \cap [t_1, t_2, \cdots, t_n] \text{ and } s_1 \leqq t_1 \leqq s_2 \leqq \cdots \leqq t_n \leqq s_1 + 1.$$

At least one equality sign must occur in this system, for otherwise $\overline{P} \in \overline{L}$ and so $\overline{P} \in \sum(C_n)$; this is impossible as $\overline{P}$ is a boundary point of the open set $\sum(C_n)$. We may suppose, after a possible adjustment in the notation, $s_1 = t_1$. Hence $s_1 \in \overline{L}$. If $n=2$ this proves the Lemma, as

$$\overline{P} = \overline{L} = s_1 = (0, s_1).$$

Assume it holds for all curves C_{n-1}, $n>2$. If $\overline{P} = s_1$, then it is already true for C_n. If $\overline{P} \neq s_1$, project from s_1. Let C_{n-1} be the projection of C_n and $\overline{P}'$ that of $\overline{P}$. Then $\overline{P}' \notin \sum(C_{n-1})$, for otherwise, by Lemma 4, $\overline{P} \in \sum(C_n)$. Moreover,

$$\overline{P}' \in [s_2, s_3, \cdots, s_n] \cap [t_2, t_3, \cdots, t_n] = \overline{L}'$$

and this space is approached by the system

$$[s_2^\mu, s_3^\mu, \cdots, s_n^\mu] \cap [t_2^\mu, t_3^\mu, \cdots, t_n^\mu],$$

where all the numbers now represent points of C_{n-1}. Thus $\overline{P}'$ is a boundary point of $\sum(C_{n-1})$. Therefore by the induction assumption $(k, s)_{n-1}$ exists so that

$$\overline{P}' \in (k, s)_{n-1} \subseteqq \overline{L} \qquad (0 \leqq k \leqq n-3).$$

Consequently, $\overline{P} \in [s_1, (k, s)] \subseteqq \overline{L}$. Because $\overline{P} \neq s_1$, it now follows from the Corollary to Lemma 4 that $\overline{P} \in (k, s)$, or $s = s_1$ and $P \in (k+1, s)$. Either of these possibilities shows the lemma to be true and so the proof is complete.

LEMMA 6. *If, for $n \geqq 3$, l^μ is a sequence which converges to $\overline{l}$, and p an integer for which $\overline{l} \subseteqq (p, s)$, $\overline{l} \not\subseteqq (p-1, s)$ $(0<p<n-1)$ then $[l^\mu, (q, s)] \rightarrow (q+2, s)$ $(p-1 \leqq q \leqq n-3)$.*

PROOF. The space $[l^\mu, (q, s)]$ is a $(q+2)$-space because $q<n-1$ while l^μ and (q, s) have no common points. Consider first the case for which $q=n-3, p=n-2$. If the lemma were false then a convergent subsequence of $[l^\mu, (n-3, s)]$ would exist whose limit would be a hyperplane Q for which $Q\neq(n-1, s)$. As $\bar{l}^\mu\to l$,

$$[\bar{l}, (n-3, s)]=(n-2, s)\leqq Q.$$

Consequently Q would cut C_n in s at least $(n-1)$-fold. As C_n is closed, Q would cut C_n in one additional point s', and $s'\neq s$ as $Q\neq(n-1, s)$. Hence, if l^μ is sufficiently close to $\bar{l}$, $[l^\mu, (n-3, s)]$ would cut C_n in a point s'' so close to s' that $s''\neq s$. Therefore the hyperplane $[l^\mu, (n-3, s)]$ would cut C_n in more than $n-2$ points in contradiction to Lemma 3. Thus $[l^\mu, (n-3, s)]$ must approach $(n-1, s)$, and the lemma is proved in this case. In particular, it is completely proved for $n=3$. Assume it is established for all C_{n-1}, $n>3$.

Consider next the case for which $q<n-3$. Project from any point t of C_n different from s. As $t\notin(p, s)$, $\bar{l}$ is projected into a line $\bar{l}'$, and l^μ is projected into a line l'^μ defined for the projection C_{n-1} of C_n by Lemma 2, Clearly

$$\bar{l}'\leqq(p, s)_{n-1} \text{ and } \bar{l}'\nleqq(p-1, s)_{n-1},$$

for otherwise

$$l\leqq[(p-1, s), t]\cap(p, s)=(p-1, s).$$

Therefore, by the induction assumption, $[l'^\mu, (q, s)_{n-1}]\to(q+2, s)_{n-1}$. This implies $[l^\mu, (q, s), t]\to[(q+2, s), t]$, and, because t is arbitrary, that $[l^\mu, (q, s)]\to(q+2, s)$. Thus the lemma is proved in this case.

Finally let $q=n-3$, $p<n-2$. If $[l^\mu, (n-3, s)]$ does not converge to $(n-1, s)$ this set contains a convergent subsequence with limit Q, $Q\neq(n-1, s)$. Now $1\leq p<n-2$, and so $n\geq4$. Hence by the result of the previous paragraph $[l^\mu, (n-4, s)]\to(n-2, s)$. Consequently $(n-2, s)\leqq Q$. This leads to the contradiction encountered in the first paragraph. Thus $[l^\mu, (n-3, s)]\to(n-1, s)$, and the lemma is proved for C_n. The proof can now be completed by induction.

DEFINITION 3. $\sigma(C_n)$ is the set of all hyperplanes each of which contains at least one line l of the curve C_n.

$\sigma(C_n)$ is the dual of the space $\sum(C_n)$.

THEOREM 3. *For $n\geq2$, $\sigma(C_n)$ consists of all the hyperplanes which do not contain n points of C_n.*

PROOF. By Lemma 3 each member of $\sigma(C_n)$ contains less than n

points of C_n. It remains to show that every hyperplane which contains less than n points of C_n contains at least one line l. Let H be a hyperplane and $s_1, s_2, \cdots, s_h$ be the points of C_n contained in H, where $0 \leqq h < n$. As C_n is closed, $h \equiv n \pmod 2$. By Theorem 2 (c), for a given line l, a hyperplane H^l exists which contains l and exactly the points $s_1, s_2, \cdots, s_h$ of C_n. By Theorem 2 (b), a system $H(p)$ $(0 \leqq p \leqq 1)$ of hyperplanes exists, continuously dependent on p, each of which contains exactly the points $s_1, s_2, \cdots, s_h$ of C_n and for which

$$H(0) = H^l, \ H(1) = H.$$

By Definition 3, $H(0) \in \sigma(C_n)$. Assume $H \notin \sigma(C_n)$. By the dual of Lemma 5 (a), $\sigma(C_n)$ is open. Therefore a least value $\bar{p}$ of p exists for which $H(\bar{p}) \notin \sigma(C_n)$. Let p^μ be a sequence for which $p^\mu \to \bar{p}$, $p^\mu < \bar{p}$. As $H(p^\mu) \in \sigma(C_n)$, l^μ exists for which $l^\mu \subseteqq H(p^\mu)$. By replacing p^μ by an appropriate subsequence if necessary we may assume l^μ converges. If $\bar{l}$ be the limit of l^μ then, by the dual of Lemma 5 (b), (k, s) exists so that

$$\bar{l} \subseteqq (k, s) \subseteqq H(\bar{p}) \qquad (1 \leqq k < n-1).$$

We may assume $(k+1, s) \notin H(\bar{p})$; for otherwise (k, s) may be replaced by an osculating space of a greater dimension so that this relation holds. Consequently s occurs exactly $(k+1)$-fold in the set $s_1, s_2, \cdots, s_h$, and $k+1 \leqq h \leqq n-2$. This is impossible if $h \leqq 1$ in which case $H \in \sigma(C_n)$. In particular this proves the theorem for $h \leqq 3$. We assume therefore $n > 3$. As $k \leqq n-3$ and $\bar{l} \subseteqq (k, s)$, the number q of Lemma 6 may be specialized to k. It follows then from this Lemma that $[l^\mu, (k, s)] \to (k+2, s)$. Hence, as $[l^\mu, (k, s)] \subseteqq H(p^\mu)$, $(k+2, s) \subseteqq H(\bar{p})$. This contradicts the fact that $s_1, s_2, \cdots, s_h$ are the only points of C_n in $H(\bar{p})$ among which s occurs exactly $(k+1)$-fold. Therefore $H \in \sigma(C_n)$. Thus the theorem is established.

7. A characterization of the lines l.

THEOREM 4. *For $n \geqq 2$, a straight line is a line l if, and only if, every hyperplane through l contains less than n points of C_n.*

PROOF. Let m be a straight line which is not a line l. Then at least one point P exists on m which is not within n distinct $(n-1, s)$. A sequence of points P^μ exists with $P^\mu \to P$ for which each P^μ is within less than n $(n-1, s)$. (This can be conveniently proved by induction in the dual formulation.) If A is a point of m for which $A \neq P$ then $[A, P^\mu] \to m$. By the dual of Theorem 3, L^μ (cf. § 3) exists for which $P^\mu \in L^\mu$. Now $[A, L^\mu]$ contains $[A, P^\mu]$ and also n points of C_n by the definition of L^μ. The limit of a convergent subsequence of $[A, L^\mu]$ is a hyperplane which contains m together with n points of C_n. This proves that if every hyperplane through a straight line contains less than n points of C_n then every point of the straight line is within n distinct $(n-1, s)$ and so must be a line l.

No hyperplane through a line l can contain n points of C_n by Lemma 3. Thus the proof of the theorem is complete.

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