PACIFIC JOURNAL OF MATHEMATICS
Vol 75, No 2, 1978

FOURIER TRANSFORMS AND THEIR LIPSCHITZ CLASSES

G. SAMPSON AND H. TUY

We define a class of functions A_α for each $\alpha > 0$. We show that the Fourier transform of every function of A_α exists and is Lipschitz of order α. We construct examples to show that the converse is not true in general. However, we show that for a certain class of function k (e.g., $k \in L_2$) if its Fourier transform $\hat{k}$ is Lipschitz of order α then $k \in A_\beta$ for all $\beta < \alpha$.

Boas ([1] and [2]) studied this problem in the case where the function k is nonnegative and gave a complete solution in this case. In connection with this question several authors (e.g. Hirschman [5]; Liang Shin Hahn [4], Drobot, Naparstek and Sampson [3]) have proved mapping properties of convolution operators with kernel k, by studying the behavior of $\hat{k}$. To be more precise, they have proved mapping theorems when $\hat{k} \in \mathrm{Lip}(\alpha)$ with additional conditions on k. In our applications we prove a similar result (see §3, Theorem 4).

Notations and Definitions.

L_{loc} shall denote the set of all Lebesgue measurable functions integrable over all finite intervals. In this paper, the functions $f, g, k, \cdots \in L_{\mathrm{loc}}$.

For $0 < \alpha \leqq 1$, a function f is Lipschitz of order α ($f \in \mathrm{Lip}(\alpha)$) if there is a positive constant A such that

$$\sup_{x \in \mathbf{R}} |f(x+h) - f(x)| \leqq A\,|h|^\alpha.$$

For $\alpha > 1$, we say that a function f is Lipschitz of order α if
(i) $f^{(m)} \in L_\infty$ for all $m < [\alpha]$ and
(ii) $f^{([\alpha])} \in \mathrm{Lip}(\alpha - [\alpha])$.
When we use the symbol

$$\int_a^b g(t,x)dt \quad \text{for} \quad -\infty \leqq a < b \leqq \infty.$$

We are assuming that $g(t,x) \in L_{\mathrm{loc}}$ as a function of t for each x and moreover the integral exists in the following sense:

$$(0) \qquad \int_a^b g(t,x)dt = \lim_{\substack{\alpha \to a \\ \beta \to b}} \int_\alpha^\beta g(t,x)dt.$$

We write $h(x_1, x_2, \cdots, x_n, y) = O(y^a)$ to mean that there exists a positive number C independent of $x_1, x_2, \cdots, x_n, y$ so that

$$\sup_{\substack{x_i \in \mathbf{R} \\ 1 \leq i \leq n}} |h(x_1, x_2, \cdots, x_n, y)| \leqq C|y|^a.$$

In particular, we say $h(x_1, x_2, \cdots, x_n) = O(1)$ to mean that there exists a C independent of $x_1, x_2, \cdots, x_n$ so that

$$\sup_{\substack{x_i \in \mathbf{R} \\ 1 \leq i \leq n}} |h(x_1, x_2, \cdots, x_n)| \leqq C.$$

We number each section independently.

1. Sufficient conditions.

LEMMA 1. *Let a and b be numbers so that $0 < a < b$. Then for each $y > 0$*

$$\int_0^y t^b f(t, x)dt = O(y^a) \Leftrightarrow \int_y^\infty f(t, x)dt = O(y^{a-b}) \tag{1}$$

$$\int_{-y}^0 |t|^b f(t, x)dt = O(y^a) \Leftrightarrow \int_{-\infty}^{-y} f(t, x)dt = O(y^{a-b}). \tag{2}$$

Proof. A similar lemma can be found in Boas's paper **[1]**. Thus we will be brief. We will prove (1); the argument for (2) is similar.

$\Rightarrow$: Let $F(x, y) = \int_0^y t^b f(t, x)dt$, then we get

$$\int_y^\infty f(t, x)dt = t^{-b}F(x, t)|_y^\infty + b\int_y^\infty t^{-b-1}F(x, t)dt.$$

Since $F(x, y) = O(y^a)$ and also $a < b$ then we are through.

$\Leftarrow$: Let $F(x, y) = \int_y^\infty f(t, x)dt$, then we get

$$\int_0^y t^b f(t, x)dt = -t^b F(x, t)|_0^y + b\int_0^y t^{b-1}F(x, t)dt.$$

Since $F(x, y) = O(y^{a-b})$, then we are through.

LEMMA 2. *Let $h > 0$.*

(3) $$\textit{If } \int_{1/h}^{\infty} f(t,x)dt = O(h^{\alpha}), \textit{ then}$$

(4) $$\int_0^{1/h} f(t,x)\sin th\,dt = O(h^{\alpha}) \quad \textit{for} \quad 0<\alpha<1$$

(5) $$\int_0^{1/h} f(t,x)(1-\cos th)dt = O(h^{\alpha}) \quad \textit{for} \quad 0<\alpha<2.$$

We get a similar result for $\int_{-\infty}^{-1/h} f(t,x)dt$.

Proof. From Lemma 1 we get,

(6) $$\int_{1/h}^{\infty} f(t,x) = O(h^{\alpha}) \Leftrightarrow \int_0^{1/h} tf(t,x)dt = O(h^{\alpha-1}) \quad \text{for} \quad 0<\alpha<1.$$

(7) $$\Leftrightarrow \int_0^{1/h} t^2 f(t,x)dt = O(h^{\alpha-2}) \quad \text{for} \quad 0<\alpha<2.$$

To see this, it suffices to take $y = 1/h$, $a = b - \alpha$ where $b = 1$ for (6) and $b = 2$ for (7).

The function $\varphi(t) = (th)^{-1}\sin th$ is decreasing and nonnegative for $t \in (0, 1/h)$. By the second-mean-value theorem for integrals we get,

$$\int_0^{1/h} tf(t,x)\varphi(t)dt = \int_0^{\xi} tf(t,x)dt \quad \text{for some} \quad \xi \in (0, 1/h).$$

Hence by hypothesis and (6) we conclude

$$\int_0^{1/h} tf(t,x)\varphi(t)dt = O(\xi^{1-\alpha}) = O(h^{\alpha-1}).$$

Consequently, we have

$$\int_0^{1/h} f(t,x)\sin th\,dt = O(h^{\alpha}).$$

The proof for (5) is similar with $\varphi(t) = (th)^{-2}(1-\cos th)$.

DEFINITION 3. Let α be a positive number. We say that $k \in A_{\alpha}$ if $k \in L_{\mathrm{loc}}$ and satisfies the following two conditions:

$$\int_{1/h}^{\infty} f(t)e^{-itx}dt = O(h^{\alpha}) \qquad \text{and}$$

(8)

$$\int_{-\infty}^{-1/h} k(t)e^{-itx}dt = O(h^{\alpha}).$$

LEMMA 4. *If* $k \in A_\alpha$ *then the Fourier transform* $\hat{k}(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} k(t)e^{-itx}dt$ *exists for all* x *and* $\hat{k} \in L_\infty$.

Proof. Since $k \in L_{\text{loc}}$ by (8) we get that $\hat{k}$ exists for each x. It also follows that $\hat{k} \in L_\infty$.

LEMMA 5. *Let* $0 < \alpha < 1$. *If* $k \in A_\alpha$ *then* $\hat{k} \in \operatorname{Lip}\alpha \cap L_\infty$.

Proof. By Lemma 4, $\hat{k}$ exists for each x and $\hat{k} \in L_\infty$.

Now we show that $\hat{k} \in \operatorname{Lip}\alpha$. Let $h > 0$,

$$\hat{k}(x+h) - \hat{k}(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} k(t)e^{-itx}(e^{-ith}-1)dt$$

$$= (2\pi)^{-1/2}\left\{\int_{-\infty}^{-1/h} + \int_{-1/h}^{1/h} + \int_{1/h}^{\infty}\right\}$$

$$\int_{-1/h}^{0} + \int_{0}^{1/h} = O(h^{\alpha}) \quad \text{by Lemma 2.}$$

By hypothesis we get,

$$\int_{-\infty}^{-1/h} + \int_{1/h}^{\infty} = O(h^{\alpha}).$$

Therefore, $\hat{k}(x+h) - \hat{k}(x) = O(h^{\alpha})$.

We are going to show that the above lemma can be extended to $\alpha > 1$ and $\alpha \notin \mathbf{N}^+$ ($\mathbf{N}^+$: set of positive integers). For $\alpha \in \mathbf{N}^+$ we will give another sufficient condition. We are able also to give a sufficient condition on k so that $\hat{k}$ is differentiable.

LEMMA 6. *If* k *satisfies*

(9) $$\left|\int_{0}^{\infty} tk(t)e^{-itx}dt\right| < \infty \quad \textit{for each } x, \textit{ then}$$

$$\lim_{h\to 0^+}\int_{0}^{1/h} k(t)e^{-itx}\frac{e^{-ith}-1}{h}\,dt = -i\int_{0}^{\infty} tk(t)e^{-itx}dt \quad \textit{for each } x.$$

For the negative side we get a similar result.

Proof. It suffices to show

$$\lim_{h\to 0^+}\int_0^{1/h} tk(t)e^{-itx}\frac{\sin th}{th}\,dt=\int_0^{\infty} tk(t)e^{-itx}dt \tag{10}$$

and

$$\lim_{h\to 0^+}\int_0^{1/h} tk(t)e^{-itx}\frac{1-\cos th}{th}\,dt=0. \tag{11}$$

Let $\epsilon>0$ be given and let x be such that (9) is true. Here, we keep x fixed throughout the entire argument.

From (9), we conclude that there exists an $N>0$ and $0<h_0<1/N$ such that for all h satisfying $0<h<h_0$

$$\left|\int_N^{1/h} tk(t)e^{-itx}dt\right|<\epsilon. \tag{12}$$

Since the function $(1-\cos th)(th)^{-1}$ is monotonic and nonnegative in $(N,1/h)$ there exists a $\xi\in(N,1/h)$ so that

$$\left|\int_N^{1/h} tk(t)e^{-itx}\frac{\cos th-1}{th}\right|\leqq\left|\int_\xi^{1/h} tk(t)e^{-itx}dt\right|. \tag{13}$$

It follows from (12) and (13) that

$$\left|\int_N^{1/h} tk(t)e^{-itx}\frac{\cos th-1}{th}\,dt\right|<\epsilon. \tag{14}$$

On the other hand, since $tk(t)\in L_{\mathrm{loc}}$, by the Lebesque dominated convergence theorem (for N fixed),

$$\lim_{h\to 0^+}\int_0^{N} tk(t)e^{-itx}\frac{\cos th-1}{th}\,dt=0.$$

Thus we get (11).

Now to show (10).

From (9), there exists N_0 such that for all N

$$N>N_0\Rightarrow\left|\int_0^{\infty} tk(t)e^{-itx}dt-\int_0^{N} tk(t)e^{-itx}dt\right|<\epsilon. \tag{15}$$

Since $(\sin th)(th)^{-1}$ is nonnegative and monotonic in $(N, 1/h)$, then by a similar argument, there exists a fixed $N > N_0$, h_0 and $h_1 < (1/N)$ such that

$$\left| \int_N^{1/h} tk(t)e^{-itx} \frac{\sin th}{th}\, dt \right| < \epsilon \quad \text{for all} \quad 0 < h < h_0 \tag{16}$$

$$\left| \int_0^N tk(t)e^{-itx} \frac{\sin th}{th}\, dt - \int_0^N tk(t)e^{-itx} dt \right| < \epsilon \tag{17}$$

$$\text{for all } h \text{ satisfying} \quad 0 < h < h_1.$$

From (15), (16) and (17) we conclude that

$$\left| \int_0^\infty tk(t)e^{itx} dt - \int_0^{1/h} tk(t)e^{-itx} \frac{\sin th}{th}\, dt \right| < 3\epsilon$$

$$\text{for all } h \text{ satisfying} \quad 0 < h < \min(h_0, h_1).$$

Therefore we get (10) and we are through.

THEOREM 7. *Let $\alpha > 1$, $m < \alpha$ and $m \in \mathbf{N}^+$. If $k \in A_\alpha$ then $\hat{k}$ is m times differentiable at each x and $\hat{k}^{(m)} \in L_\infty$. In fact*

$$\hat{k}^{(m)}(x) = (2\pi)^{-1/2}(-i)^m \int_{-\infty}^{\infty} t^m k(t)e^{-itx} dt.$$

Proof. By Lemma 1,

$$\int_{1/h}^{\infty} k(t)e^{-itx} dt = O(h^\alpha) \Rightarrow \int_0^{1/h} t^{\alpha+1} k(t)e^{-itx} dt = O(h^{-1}). \tag{18}$$

Hence by Lemma 1 again, for all $m < \alpha$

$$\int_{1/h}^{\infty} t^m k(t)e^{-itx} dt = O(h^{\alpha-m}). \tag{19}$$

Thus $t^m k(t) \in A_{\alpha-m}$.

It follows from Lemma 4 that

$$f_m(x) = (2\pi)^{-1/2}(-i)^m \int_{-\infty}^{\infty} t^m k(t)e^{-itx} dt \tag{20}$$

exists for each x and $f_m \in L_\infty$.

To prove the theorem, we first show that $\hat{k}'(x)$ exists for each x. Since $k \in A_\alpha$, by Lemma 4 $\hat{k}$ exists and we have,

$$\frac{\hat{k}(x+h)-\hat{k}(x)}{h}$$
$$= (2\pi)^{-1/2}h^{-1}\left\{\int_{-\infty}^{-1/h} k(t)e^{-itx}(e^{-ith}-1)dt + \int_{-1/h}^{0} + \int_{0}^{1/h} + \int_{1/h}^{\infty}\right\}.$$

Since $k \in A_\alpha\,(\alpha > 1)$ by (8) we have $\lim_{h\to 0} h^{-1}\left(\int_{1/h}^{\infty} + \int_{-\infty}^{-1/h}\right) = 0$. By (19) and Lemma 6 with $m = 1$, it follows that $\hat{k}'(x) = f_1(x)$.

The theorem is then true for $m = 1$. Now we suppose that the theorem is true up to $m-1$, i.e. $\hat{k}^{(m-1)}(x) = f_{m-1}(x)$.

$$\frac{\hat{k}^{(m-1)}(x+h)-\hat{k}^{(m-1)}(x)}{h} = (-1)^{m-1}\frac{\hat{g}(x+h)-\hat{g}(x)}{h}$$

where $g(t) \overline{\overline{,}}\ t^{m-1}k(t)$.

Since $\alpha - m + 1 > 1$, the above argument starting with (19) can be applied to $g(t)$ and we get,

$$\lim_{h\to 0^+}\frac{\hat{g}(x+h)-\hat{g}(x)}{h} = -i(2\pi)^{-1/2}\int_{-\infty}^{\infty} tg(t)e^{-itx}dt.$$

Thus $\hat{k}^{(m)}(x) = f_m(x)$.

THEOREM 8. *Let $\alpha > 0$ and $\alpha \notin \mathbf{N}^+$.*

If $k \in A_\alpha$ then $\hat{k} \in \operatorname{Lip}\alpha \cap L_\infty$.

Proof. For $0 < \alpha < 1$, this is Lemma 5.

Now look at the case where $\alpha > 1$. By Lemma 4, $\hat{k}$ exists for each x and $\hat{k} \in L_\infty$. Due to Theorem 7 we can conclude that for all $m \leqq [\alpha]$, $\hat{k}^{(m)}$ exists and $\hat{k}^{(m)} \in L_\infty$. Moreover

$$\hat{k}^{([\alpha])}(x) = (-i)^{[\alpha]}(2\pi)^{-1/2}\int_{-\infty}^{\infty} t^{[\alpha]}k(t)e^{-itx}dt. \tag{21}$$

From (19) we get

$$\int_{1/h}^{\infty} t^{[\alpha]}k(t)e^{-itx}dt = O(h^{\alpha-[\alpha]}).$$

Hence $t^{[\alpha]}k(t) \in A_{\alpha-[\alpha]}$. It follows from Lemma 5 that

$$\widehat{t^{[\alpha]}k(t)} \in \operatorname{Lip}(\alpha - [\alpha]).$$

Hence by (21) we get our result.

THEOREM 9. *Let* $\alpha \in \mathbf{N}^+$. *If* $k \in L_{\mathrm{loc}}$ *and satisfies*

$$\int_0^\infty t^\alpha k(\pm t)e^{\mp itx}dt = O(1) \tag{22}$$

and

$$\int_{1/h}^\infty k(\pm t)e^{\mp itx}dt = o(h^\alpha) \tag{23}$$

then $\hat{k} \in \operatorname{Lip}\alpha \cap L_\infty$.

Proof. First assume $\alpha = 1$. By (23), $k \in A_1$. By Lemma 4, $\hat{k}$ exists for each x and $\hat{k} \in L_\infty$. In (23) we are using the little "o" notation. We note

$$\frac{\hat{k}(x+h) - \hat{k}(x)}{h} = (2\pi)^{-1/2}h^{-1}\left\{\int_{-\infty}^{-1/h} + \int_{-1/h}^{1/h} + \int_{1/h}^{\infty}\right\}. \tag{24}$$

Hence by Lemma 6 and (23) we get

$$\hat{k}'(x) = -i(2\pi)^{-1/2}\int_{-\infty}^\infty tk(t)e^{-itx}dt.$$

From (22) we conclude that $\hat{k}$ is absolutely continuous. Hence $\hat{k} \in$ Lip (1).

For the case $\alpha > 1$, we use induction. The argument is similar to that given in Theorem 7 and will be omitted here.

2. Necessary conditions. We know that for each α $(0 \leqq \alpha < 1)$ there exists a function g such that $\hat{g} \in \operatorname{Lip}(\alpha)$ but $g \notin A_\alpha$. We give this example in §4. However, we have succeeded in showing that $\hat{k} \in \operatorname{Lip}(\alpha)$ implies $k \in A_\beta$ for all $\beta < \alpha$ with some other conditions placed on k. One of the conditions that k must satisfy is the following:

$$\int_{v/2}^{v} k(w)e^{-itw}dw = i(2\pi)^{-1/2}\int_{-\infty}^\infty \frac{\hat{k}(2x+t) - \hat{k}(x+t)}{x}e^{itx}dx. \tag{1}$$

Condition (1) is merely Parseval's identity, however we have only been able to show (1) holds for a certain class of functions. This result appears in Lemma 3. In the case where $k \in L_p$ $(1 \leqq p \leqq 2)$ and $\hat{k}$ is continuous, we can show (1) holds (the argument is similar to that given in Lemma 3).

Another condition that appears is,

DEFINITION 1. Let $\epsilon \geqq 0$. We say that $f \in V_\epsilon$ if there exists some constant A such that

$$\int_{|x| \geqq A} \frac{|f(2x+t) - f(x+t)|}{|x|^{1-\epsilon}} dx = O(1). \tag{2}$$

It is obvious that if $f \in L_p$ then $f \in V_\epsilon$ for $\epsilon < 1/p$. Furthermore, all constant functions belong to V_ϵ for all $\epsilon \geqq 0$.

THEOREM 2. *If some* $0 < \epsilon < 1$, $\hat{k} \in \operatorname{Lip} \alpha \cap V_\epsilon$ $(0 < \alpha \leqq 1)$ *and* k *satisfies* (1) *then* $k \in A_\beta$ *for all* $0 < \beta < \alpha$.

COROLLARY. *Let* $k \in L_p$ *for some* $1 < p \leqq 2$. *If* $\hat{k} \in \operatorname{Lip}(\alpha)$, *then* $k \in A_\beta$ *for all* $0 < \beta < \alpha$.

REMARK. In the above corollary $\hat{k}$ is defined as usual in the L_q sense $(1/p + 1/q = 1)$. We note that for $0 < n \leqq v \leqq 2^r n$ $\int_n^v k(w)e^{-itw}dw = \int_n^{2^2 n} + \cdots + \int_{cv}^v$ where $1/2 \leqq c \leqq 1$. Next we note that a formula similar to (1) holds for the term

$$\int_{cv}^v k(w)e^{-itw}dw, \qquad 1/2 \leqq c \leqq 1.$$

And now the Corollary follows.

Proof of Theorem 2. Let

$$\varphi(x,t) = \frac{\hat{k}(2x+t) - \hat{k}(x+t)}{x}.$$

From (1) it follows for $v > 1$ (note $k \in L_{\mathrm{loc}}$)

$$\begin{aligned} &2\int_{v/2}^v k(w)e^{-itw}dw \\ &= \left(\int_{-\infty}^{-2\pi/v} + \int_{-2\pi/v}^{\pi/v} + \int_{\pi/v}^{\infty}\right)(\varphi(x,t) - \varphi(x+\pi/v,t))e^{ivx}dx. \end{aligned} \tag{3}$$

For the middle term on the right hand side of (3) we note that there is some constant C such that,

$$|\varphi(x,t)| \leqq C|x|^{\alpha-1} \quad \text{and} \quad |\varphi(x+(\pi/v))| \leqq C|x+(\pi/v)|^{\alpha-1}.$$

It follows that

$$(4) \qquad \int_{-2\pi/v}^{\pi/v} = O(v^{-\alpha}).$$

For the remaining terms we write,

$$(5) \quad \varphi(x,t) - \varphi(x+\pi/v,t) = \psi(x,v,t) + \pi/v\left(\frac{\hat{k}(2x+t) - \hat{k}(x+t)}{x(x+\pi/v)}\right)$$

where

$$\psi(x,v,t) = \frac{\hat{k}(2x+t) - \hat{k}(x+t) - \hat{k}(2x+(2\pi/v)+t) + \hat{k}(x+(\pi/v)+t)}{x+(\pi/v)}.$$

To complete our argument we need to show that

$$\left(\int_{-\infty}^{-2\pi/v} + \int_{\pi/v}^{\infty}\right)\psi(x,v,t)e^{ivx}dx = O(v^{-\beta})\ \forall \beta < \alpha.$$

The second term on the right hand side of (5) can be handled in a straightforward manner. We will give the argument for $\int_{-\infty}^{-2\pi/v} \psi e^{ivx}dx$; the proof for $\int_{\pi/v}^{\infty} \psi e^{ivx}dx$ is similar. First let $\mu = \alpha/\epsilon$ and $s = -v^{\mu}$. We get

$$(6) \qquad \left|\int_{-\infty}^{-2\pi/v} \psi(x,v,t)e^{ivx}dx\right| \leqq \left(\int_{-\infty}^{s} + \int_{s}^{-2\pi/v}\right)|\psi(x,v,t)|\,dx.$$

Since $\hat{k} \in \operatorname{Lip}\alpha$ we have

$$|\psi(x,v,t)| \leqq Cv^{-\alpha}\,|x+(\pi/v)|^{-1}$$

for some constant C independent of x, v, and t. It follows that

$$(7) \quad \int_{s}^{-2\pi/v} |\psi(x,v,t)|\,dx \leqq Cv^{-\alpha}\int_{s}^{-2\pi/v} \frac{dx}{|x+(\pi/v)|} = O(v^{-\beta})\ \forall \beta < \alpha.$$

For the other term we have,

$$\int_{-\infty}^{s} |\psi(x,v,t)|\,dx \leqq \int_{-\infty}^{s} \frac{|\hat{k}(2x+t) - \hat{k}(x+t)|}{|x+(\pi/v)|^{1-\epsilon}\,|x+(\pi/v)|^{\epsilon}}\,dx$$
$$+ \int_{-\infty}^{s} \frac{|\hat{k}(2x+(2\pi/v)+t) - \hat{k}(x+(\pi/v)+t)|}{|x+(\pi/v)|^{1-\epsilon}\,|x+(\pi/v)|^{\epsilon}}\,dx.$$

Since $\hat{k} \in V_\epsilon$ by the second mean value theorem for integrals we conclude that

$$\int_{-\infty}^{s} |\psi(x, v, t)|\, dx = O(v^{-\alpha}). \tag{8}$$

Hence the proof is complete.

LEMMA 3. *If k is a real valued function such that*:

$$\hat{k} \quad \textit{is continuous at} \quad t, \tag{9}$$

$$\int_a^b k(w) e^{-iuw} dw = O(1)^1 \quad \textit{and} \tag{10}$$

$$\int_{|u| \geq A} \left| \frac{\hat{k}(u)}{u} \right| du < \infty \quad \textit{for some} \quad A > 0. \tag{11}$$

Then

$$\int_{v/2}^{v} k(w) e^{-itw} dw = i(2\pi)^{-1/2} \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\hat{k}(2x+t) - \hat{k}(x+t)}{x} e^{ivx} dx.$$

Proof. From (10) it follows that,

$$\int_a^b k(w) dw = O(1) \quad \text{and} \quad \hat{k} \in L_\infty. \tag{12}$$

We will assume $v > 0$, the proof for $v < 0$ is similar. Let $P_\delta(u) = \delta/(\delta^2 + u^2)$ which is the well-known Poisson kernel. We begin by showing that

$$\int_{v/2}^{v} k(u) e^{-itu} du = \lim_{\delta \to 0^+} 1/\pi \int_{v/2}^{v} e^{-itw} \int_{-\infty}^{\infty} k(u) P_\delta(w-u)\, du\, dw. \tag{13}$$

Using (12) we note that

$$\lim_{\delta \to 0^+} \int_{v/2}^{v} e^{itw} \left(\int_{-\infty}^{v/4} + \int_{2v}^{\infty} \right) k(u) P_\delta(w-u)\, du\, dw = 0.$$

Hence since $k \in L_{\text{loc}}$ we get

[1] Refer to page 2 with $g(u, a, b) = O(1)$.

$$\lim_{\delta\to 0^+}\int_{v/2}^{v} e^{-itw}\int_{v/4}^{2v} k(u)P_\delta(w-u)\,du\,dw$$
$$= \lim_{\delta\to 0^+} 1/\pi\int_{v/4}^{2v} k(u)\int_{v/2}^{v} e^{-itw}P_\delta(w-u)\,dw\,du.$$

Now (13) follows immediately.

By (10) and the second mean value theorem for integrals we get

$$\text{(14)}\qquad \int_{-\infty}^{\infty} e^{-\delta|u|}\hat{k}(u)e^{iwu}du = 2(2\pi)^{-1/2}\int_{-\infty}^{\infty} k(u)P_\delta(w-u)\,du.$$

From (13) and (14) and using the fact that $\hat{k}\in L_\infty$ we get

$$\text{(15)}\int_{v/2}^{v} k(w)e^{itw}dw = \lim_{\delta\to 0^+}(2\pi)^{-1/2}\int_{v/2}^{v} e^{-itw}\int_{-\infty}^{\infty} e^{-\delta|u|}\hat{k}(u)e^{iwu}du\,dw$$
$$= \lim_{\delta\to 0^+}\left(\int_{|u-t|\leqq\epsilon}+\int_{|u-t|\geqq\epsilon}\right)e^{-\delta|u|}\hat{k}(u)\rho(u-t,v)\,du$$

where $\rho(u,v)=(2\pi)^{-1/2}(iu)^{-1}(e^{iuv}-e^{iuv/2})$.

We note that there exists some constant C independent of δ and ϵ such that

$$\text{(16)}\qquad \left|\int_{|u-t|\leqq\epsilon} e^{-\delta|u|}\hat{k}(u)\rho(u-t,v)\,du\right| \leqq C\epsilon.$$

By (11) we can conclude that,

$$\lim_{\delta\to 0^+}\int_{|u-t|\geqq\epsilon} e^{-\delta|u|}\hat{k}(u)\rho(u-t,v)\,du = \int_{|u-t|\geqq\epsilon}\hat{k}(u)\rho(u-t,v)\,du.$$

After substitution we have,

$$\text{(17)}\quad i\int_{|u-t|\geqq\epsilon}\hat{k}(u)\rho(u-t,v)\,du$$
$$= (2\pi)^{-1/2}\left\{\int_{|x|\geqq\epsilon}\frac{\hat{k}(x+t)-\hat{k}(2x+t)}{x}e^{ivx}dx - \int_{\epsilon/2\leqq|x|\leqq\epsilon}\frac{\hat{k}(2x+t)}{x}e^{ivx}dx\right\}.$$

We have,

$$\left|\int_{\epsilon/2\leqq|x|\leqq\epsilon}\frac{\hat{k}(2x+t)}{x}e^{ivx}dx\right|$$
$$\leqq \int_{\epsilon/2\leqq|x|\leqq\epsilon}\frac{|\hat{k}(2x+t)-\hat{k}(t)|}{|x|}dx + \left|\hat{k}(t)\int_{\epsilon/2\leqq|x|\leqq\epsilon}\frac{e^{ivx}}{x}dx\right|.$$

Since $\hat{k} \in L_\infty$ and $\hat{k}$ is continuous at t,

$$\lim_{\epsilon \to 0^+} \int_{\epsilon/2 \leqq |x| \leqq \epsilon} \frac{\hat{k}(2x+t)}{x} e^{iwx} dx = 0. \tag{18}$$

The conclusion follows from (15), (16), (17) and (18).

3. Applications.

1. If $k(t) = e^{i|t|^a}/(|t|^b + 1)$ where $a(a-1) \neq 0$ and $b + a/2 - 1 > 0$, then $\hat{k} \in \mathrm{Lip}(b + a/2 - 1)$. This follows immediately from Theorem 8 of §1, and van der Corput's Lemma (see [6]).
2. We adopt the following definitions:

Definition 1. We say that $k \in L_p^p$ if for all $f \in L_0^\infty$ (set of L_∞ functions with compact support)

$$\int |k * f|^p \leqq C \int |f|^p$$

where C is independent of f.

Definition 2. We say that $k \in L_2^*$ if for all $f \in L_0^\infty$, there is a constant C such that

$$\left| \left\{ x : \left| \int_0^\infty k(\pm t) f(x \pm t) dt \right| > y \right\} \right| \leqq \frac{C}{y^2} \|f\|_2^2, \quad \text{for all} \quad y > 0.$$

Here, C is independent of f and y.

Lemma 3. (Jurkat and Sampson). *If $k \in L_2^*$ and $\int_s^{2s} |k(t)| dt = O(1)$ for all s, then $\int_a^b k(t) e^{-itx} dt = O(1)$.*

Proof. Let f be the characteristic function of $[0, 2b]$ with $b > 0$. For all $u \in [0, b]$ we have, for fixed x,

$$\int_0^\infty f(u+t) k(t) e^{-itx} dt = \left(\int_0^b + \int_b^{2b-u} \right) k(t) e^{-ixt} dt. \tag{1}$$

But

$$\left| \int_b^{2b-u} k(t) e^{-ixt} dt \right| \leqq \sup_{s \in R} \left| \int_s^{2s} |k(t)| dt \right| = M.$$

Therefore if $\left| \int_0^b k(t)e^{-ixt}dt \right| \leqq 2M$ the proof is over. Now suppose that $\left| \int_0^b k(t)e^{-ixt}dt \right| > 2M$. In this case, from (1) it follows that

$$\left| \left\{ u : \left| \int_0^\infty k(t)e^{-ixt}f(u+t)dt \right| > \frac{1}{2} \left| \int_0^b k(t)e^{-ixt}dt \right| \right\} \right| \geqq b. \tag{2}$$

Since $k \in L_2^*$, there exists some constant C independent of x and b such that

$$\begin{aligned} &\left| \left\{ u : \left| \int_0^\infty k(t)e^{-ix(t+u)}f(u+t) \right| > \frac{1}{2} \left| \int_0^b k(t)e^{-ixt}dt \right| \right\} \right| \\ &\qquad \leqq \frac{C}{\left| \int_0^b k(t)e^{-ixt}dt \right|^2} \int_0^\infty |f(t)|^2 dt. \end{aligned} \tag{3}$$

From (2) and (3) it follows that $\left| \int_0^{b} k(t)e^{-ixt}dt \right| \leqq \sqrt{C}$ where C is independent of b and x. A similar argument works for $b < 0$ and hence we get our result.

THEOREM 4. *If k is real-valued and satisfies the following conditions:*

$$\int_s^{2s} |k(t)|\, dt = O(1) \tag{4}$$

$$\hat{k} \in \operatorname{Lip} \alpha \quad \textit{for some} \quad 0 < \alpha < 1 \tag{5}$$

$$|x|^\lambda \hat{k}(x) = O(1) \quad \textit{for some} \quad \lambda > 0, \quad \textit{and} \tag{6}$$

$$k \in L_2^*. \tag{7}$$

Then $k \in L_p^p$ for $1 < p < \infty$.

Proof. By Lemma 3, (4) and (7) imply that $\int_a^b k(t)e^{-ixt}dt = O(1)$. By Lemma 3 in §2 we can conclude that (1) in §2 holds. Furthermore (6) implies that $\hat{k} \in V_\epsilon$ for some $\epsilon > 0$. Hence due to Theorem 2 in §2, $k \in A_\beta$ $\forall \beta < \alpha$. The conclusion follows from [3, Theorem 1.17].

4. Examples.

LEMMA 1. *Let l, m, a and b be given numbers. Set* $M = \max(|l|, |m|)$, $L = \min(|l|, |m|)$ *and* $V = \max(|a|, |b|)$. *Then,*

$$\left| \int_a^b \frac{\sin lu \cos mu}{u} du \right| = O(1) \tag{1}$$

$$\left| \int_{-b}^b \frac{\sin lue^{-imu}}{u} du \right| = O(1) \tag{2}$$

(3) *if M and V are sufficiently large,*

$$\left| \int_a^b \frac{\sin lu}{u} e^{-imu} du \right| = O(\log V + L).$$

Proof. We get (1) since $\left| \int_c^d \sin u/udu \right| \leqq A$ where A is a positive constant independent of c and d; also, (2) follows immediately from (1). For (3) it suffices to dominate

$$\left| \int_a^b \frac{\sin lu}{u} \sin mudu \right|. \tag{4}$$

Since the expression (4) is even and symmetric in l and m we can assume w.l.o.g. that $0 < l \leqq m$. Furthermore, we can assume that $0 \leqq a < b$ since the integrand is an odd function in u. Thus

$$\left| \int_a^b \frac{\sin lu}{u} \sin mudu \right| \leqq l \quad \text{if} \quad 0 \leqq a < b < 1, \tag{5}$$

$$\left| \int_a^b \right| \leqq \left| \int_a^1 \right| + \left| \int_1^b \right| \leqq l + \log b \quad \text{if} \quad 0 \leqq a \leqq 1 < b, \tag{6}$$

and

$$\left| \int_a^b \right| \leqq \log b \quad \text{if} \quad 1 < a < b. \tag{7}$$

Hence we get our result.

THEOREM 2. *For each $0 < \alpha < 1$ there exists a function $g \in L_p$ $(1 \leqq p < \infty)$ such that $\hat{g} \in \operatorname{Lip} \alpha$ but $g \notin A_\alpha$.*

Proof. It suffices to show that there exists a $g \in L_p$ such that $\hat{g} \in \mathrm{Lip}(\alpha)$ and for some sequences $\{h_n\} \to 0$, $\{x_n\}$, and $\{B_n\} \to \infty$ then

$$\left| h_n^{-\alpha} \int_{1/h_n}^{2/h_n} g(t) e^{-itx} dt \right| > B_n \quad \text{for} \quad |x - x_n| \leqq 1/a_n. \tag{8}$$

Consider

$$g(t) = \sum_{m=1}^{\infty} \frac{\sin(m(t - c_m))}{m^{\gamma-1} a_m^{\alpha} (t - c_m)} \chi_{[a_m, b_m]}^{(t)}$$

where γ is a fixed positive integer $\geqq 3$ and $\gamma \geqq (1-\alpha)^{-1}$. Also,

$$\begin{aligned} &a_m = 2^{m^\gamma}, \qquad b_m = 2a_m \quad \text{and} \quad c_m = 3/2a_m \\ &\text{and} \quad \chi_I \text{ is the characteristic function of } I. \end{aligned} \tag{9}$$

To show that $\hat{g}(x)$ exists for each $x \in \mathbf{R}$, it suffices to show that $g \in L_1$.

$$\begin{aligned} \int_{-\infty}^{\infty} |g(t)|\, dt &\leqq \sum_{m=1}^{\infty} m^{1-\gamma} a_m^{-\alpha} \int_{a_m}^{b_m} \frac{|\sin(m(t - c_m))|}{|t - c_m|} dt \\ &\leqq 2 \sum_{m=1}^{\infty} \frac{1 + \log(m a_m)}{m^{\gamma-1} a_m^{\alpha}} < \infty. \end{aligned}$$

Now we are going to show that $\hat{g} \in \mathrm{Lip}\, \alpha$ for $\gamma \geqq (1-\alpha)^{-1}$.
Given h such that $2|h| < 1$, there exists an m so that,

$$1/a_{m+1} < |h| \leqq 1/a_m \tag{10}$$

$$\begin{aligned} \hat{g}(x+h) - \hat{g}(x) = &\sum_{l=1}^{m-1} l^{1-\gamma} a_l^{-\alpha} \int_{a_l}^{b_l} \frac{\sin l(t - c_l)}{t - c_l} e^{-itx}(e^{-ith} - 1) dt, \\ &+ m^{1-\gamma} a_m^{-\alpha} \int_{a_m}^{b_m} \frac{\sin m(t - c_m)}{t - c_m} e^{-itx}(e^{-ith} - 1) dt \\ &+ \left(\sum_{l=m+1}^{\infty} l^{1-\gamma} a_l^{-\alpha} \int_{a_l}^{b_l} \frac{\sin l(t - c_l)}{t - c_l} e^{-it(x+h)} dt \right. \\ &\left. - \sum_{l=m+1}^{\infty} l^{1-\gamma} a_l^{-\alpha} \int_{a_l}^{b_l} \frac{\sin l(t - c_l)}{t - c_l} e^{-itx} dt \right) \\ = &\, I_1 + I_2 + I_3. \end{aligned} \tag{11}$$

We are going to show that each term on the right hand side of (11) is $O(h^\alpha)$ for $\gamma \geqq (1-\alpha)^{-1}$ separately.

After substituting $u = t - c_l$, by (2) and (10) $I_3 = O(h^\alpha)$.

Since $0 < h < 1/a_m$, by the second mean value theorem for integrals there exists ξ_l $(a_l < \xi_l < b_l)$ such that

$$|I_1| \leqq \sum_{l=1}^{m-1} \frac{\sin b_l h}{l^{\gamma-1} a_l^\alpha} \left| \int_{\xi_l}^{b_l} \frac{\sin l(t-c_l)}{t-c_l} e^{-itx} dt \right|.$$

After substitution $u = t - c_l$, by (3), (9) and (10) it follows

$$|I_1/h^\alpha| \leqq C \sum_{l=1}^{m-1} (a_l h)^{1-\alpha} l^{1-\gamma} (\log a_l + l) = O(1).$$

It remains to show $I_2 = O(h^\alpha)$. By (2) we have,

$$I_2/h^\alpha = O((ha_m)^{-\alpha} m^{1-\gamma}) = O(1) \tag{12}$$

if $(ha_m)^{-\alpha} m^{1-\gamma} \leqq 1$.

Since $e^{-ith} - 1 = (\cos th - 1) + i \sin th$, by the second mean value theorem for integrals and (3) we get

$$I_2/h^\alpha = O(m(a_m h)^{1-\alpha}) = O(1), \quad \text{if} \quad m(a_m h)^{1-\alpha} \leqq 1. \tag{13}$$

Hence by (12) and (13) we conclude that $I_2 = O(h^\alpha)$ if $\gamma \geqq (1-\alpha)^{-1}$.

Now we are going to show that there exist some sequences $\{h_n\} \to 0$, $\{x_n\}$, and $\{B_n\} \to \infty$ such that

$$\left| \frac{1}{h_n^\alpha} \int_{1/h_n}^{2/h_n} g(t) e^{-itx} dt \right| > B_n \quad \text{for} \quad |x - x_n| \leqq 1/a_n.$$

Consider $h_m = 1/c_m$ and $x_m = m$.

$$h_m^{-\alpha} \int_{1/h_m}^{2/h_m} g(t) e^{-itx} dt = J_1 + J_2$$

where

$$J_1 = -\frac{ie^{ic_m x}}{h_m^\alpha m^{\gamma-1} a_m^\alpha} \int_0^{1/2a_m} \frac{\sin mv}{v} \sin xv dv,$$

and

$$J_2 = \frac{e^{ic_m x}}{h_m^\alpha m^{\gamma-1} a_m^\alpha} \int_0^{1/2a_m} \frac{\sin mv}{v} \cos xv dv.$$

From (1) of Lemma 1 we can conclude that $J_2 = O(1)$. Hence it suffices to show that $|J_1(x)| \geqq m/2$ if $|x - x_m| \leqq 1/a_m$.

$$|J_1| = \left(\frac{3}{2}\right)^{\alpha} m^{1-\gamma} \left| \int_0^1 + \int_1^{\frac{1}{2}a_m} \right|.$$

It is clear that ($\gamma \geqq 3$)

$$m^{1-\gamma} \left| \int_0^1 \frac{\sin mv}{v} \sin xv dv \right| = O(1).$$

On the other hand,

$$2m^{1-\gamma} \int_1^{\frac{1}{2}a_m} \frac{\sin mv}{v} \sin xv dv$$
$$= m^{1-\gamma} \int_1^{\frac{1}{2}a_m} \left(\frac{\cos (m-x)v}{v} + \frac{\cos (m+x)v}{v} \right) dv.$$

We can easily see that for these x's,

$$m^{1-\gamma} \int_1^{\frac{1}{2}a_m} \frac{\cos (m+x)v}{v} dv = O(1).$$

For the remaining term we note that $\cos u \geqq 1 - u^2/2$. Therefore for x satisfying $|x - m| \leqq 1/a_m$,

$$m^{1-\gamma} \left| \int_1^{\frac{1}{2}a_m} \frac{\cos (m-x)v}{v} dv \right| \geqq m^{1-\gamma} \int_1^{\frac{1}{2}a_m} \left(\frac{1}{v} - (m-x)^2 v \right) dv.$$
$$\geqq m^{1-\gamma} \left(\log a_m/2 - \frac{(m-x)^2 a_m^2}{8} \right).$$

Since $(m-x)^2 a_m^2 \leqq 1$ we conclude that for m sufficiently large $|J_1| \geqq m/2$.

The proof is then complete.

REFERENCES

1. R. P. Boas, Jr., *Fourier series with positive coefficients,* J. Math. Analysis and Appl., **17** (1967), 463–483.

2. ———, *Lipschitz behavior and integrability of characteristic functions,* Ann. Math. Stat., **38** PTI (1967), 32–36.

3. V. Drobot, A. Naparstek and G. Sampson, (L_p, L_q) *mapping properties of convolution transforms*, Studia Mathematica, LV (1975), 41–70.

4. Liang Shin Hahn, *On multipliers of p integrable functions*, Trans. Amer. Math. Soc., **128** (1967), 321–335.

5. I. I. Hirschman, Jr., *On multiplier transformations*, Duke Math. J., **26** (1959), 221–242.

6. A. Zygmund, *Trigonometric series*, 2nd Edit. Vol. I and II, Cambridge Univ. Press, N. Y. (1959).

Received January 12, 1976. We wish to thank the referee for his helpful comments.

SUNY–BUFFALO
BUFFALO, NY 14214