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STRONGLY REGULAR MAPPINGS WITH COMPACT *ANR* FIBERS ARE HUREWICZ FIBERINGS

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We prove that a strongly regular map $f: E \to B$ with compact separable *ANR* fibers and complete separable finite dimensional base space B is a Hurewicz fibering. We also show that if f is a Hurewicz fiber map with locally compact *ANR* fibers and base, then the total space E is an *ANR*.

1. Statement of results. In this paper, we will be concerned with the following two questions:

Q1: When is a map between metric spaces a Hurewicz fibration?

Q2: ([2]) Is the total space of a Hurewicz fibration an *ANR* if the base space is an *ANR* and the fibers are *ANR*s?

Both of these questions have been answered in the finite dimensional case, but our interest in Hilbert cube manifolds has led us to consider the case of locally compact *ANR*s. Finite dimensional answers to Q1 may be found in [1] and [16], while the finite dimensional answer to Q2 may be found in [2]. Weaker infinite dimensional results may be found in [14] and [15].

The notion of a strongly regular map, introduced in [1], is a key ngredient in our work on Q1. Intuitively, a map is strongly regular if nearby point-inverses are homotopy equivalent via small homotopy equivalences. Here is a formal definition.

DEFINITION. A map $f: X \to B$ between metric spaces is said to be *strongly regular* if f is proper ($f^{-1}(K)$ is compact for each compact $K \subset B$) and if for each $b \in B$ and $\epsilon > 0$ there is a $\delta > 0$ such that if $d(b, b') < \delta$ then there are maps $g_{bb'}: f^{-1}(b) \to f^{-1}(b')$ and $g_{b'b}: f^{-1}(b') \to f^{-1}(b)$ and homotopies $h_t: f^{-1}(b) \to f^{-1}(b)$ and $k_t: f^{-1}(b') \to f^{-1}(b')$ such that

(i) $d(g_{bb'}(x), x) < \epsilon$ and $d(h_t(x), x) < \epsilon$ for all $x \in f^{-1}(b)$ and for all t, $0 \leqq t \leqq 1$.

(ii) $d(g_{b'b}(x), x) < \epsilon$ and $d(k_t(x), x) < \epsilon$ for all $x \in f^{-1}(b')$ and for all t, $0 \leqq t \leqq 1$.

(iii) $h_0 = g_{b'b} \circ g_{bb'}$ and $h_1 = id$.

(iv) $k_0 = g_{bb'} \circ g_{b'b}$ and $k_1 = id$.

Here are our main results.

THEOREM 1. *If $f: E \to B$ is a strongly regular map onto a complete* *finite dimensional B and $f^{-1}(b)$ is an ANR for each $b \in B$, then f is a Hurewicz fibration.*

THEOREM 2. *If $p: E \to B$ is a Hurewicz fibration with locally compact ANR fibers and locally compact ANR base B, then E is an ANR.*

Since this paper was written, the author has succeeded in proving that small homotopy equivalences between Hilbert cube manifolds are approximable by homeomorphisms. This result allows one to conclude that if $f: E \to B$ is a strongly regular map as in Theorem 1, then $f \circ \text{proj}: E \times Q \to B$ is completely regular and is therefore (under the hypotheses of Theorem 1) a locally trivial fiber bundle ([7]). This certainly implies Theorem 1. The interested reader should consult [8]. The present treatment, which motivated the results of [8], is more traditional in that it uses only standard facts about *ANR*s and Michael's selection theorem.

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2. Notation and preliminaries. All spaces in this paper are assumed to be complete, separable, and metric.

DEFINITION 2.1. Let I denote the closed unit interval, $[0, 1]$. The Hilbert cube is the product $Q = \Pi_{i=1}^{\infty} I_i$, where each I_i is a copy of I.

We will need to use Michael's selection theorem. We state the necessary definitions and the version of the theorem which we will use.

DEFINITION 2.2. Let $f: X \to B$ be a continuous map between metric spaces. The decomposition of X into point-inverses $\{f^{-1}(b)\}$ is *lower semicontinuous* (l.s.c.) if for each sequence $\{b_i\}$ with $\lim b_i = b$ and $x \in f^{-1}(b)$, there is a sequence $\{x_i\}$ with $f(x_i) = b_i$ and $\lim x_i = x$.

DEFINITION 2.3. The decomposition $\{f^{-1}(b)\}$ of X is *equilocally m-connected* (equi-LC^m) if for each $\epsilon > 0$ and $x_0 \in X$ there is a $\delta > 0$ such that if $g: S^k \to N_\delta(x_0) \cap f^{-1}(b)$, $k \leqq m$, is a continuous map then g is nullhomotopic in $N_\epsilon(x_0) \cap f^{-1}(b)$.

Theorem 2.4 below is a very weak version of Michael's selection theorem adapted from Theorem M of [7].

THEOREM 2.4 (Michael [13], [7]). *Let $f: X \to B$ be a continuous map such that*

(i) *X is a complete metric space*
(ii) *B is a finite dimensional metric space*
(iii) *the decomposition $\{f^{-1}(x)\}$ is lower semicontinuous*
(iv) *the decomposition $\{f^{-1}(x)\}$ is equi-LC^n for all n.*

Then f admits local sections, i.e., for each $b \in B$ there is a neighborhood U of b in B and a map $s: U \to X$ such that $f \circ s = id \mid U$.

We will also need the following:

PROPOSITION 2.5 (*Estimated homotopy extension theorem*). *Let X and A be ANRs with A a closed subset of X. Let $f: X \to Y$ and let $G_t: A \to Y$ be a homotopy such that $G_0 = f \mid A$. Then G extends to a homotopy $F_t: A \to Y$ with $F_0 = f$. If* $\operatorname{diam}\{G_t(a) \mid 0 \leqq t \leqq 1\} < \epsilon$ *for each $a \in A$, then F can be chosen so that* $\operatorname{diam}\{F_t(x) \mid 0 \leqq t \leqq 1\} < \epsilon$ *for each $x \in X$.*

Proof. This follows easily from the usual proof of the homotopy extension theorem.

DEFINITION 2.6. Let $f: X \to Y$ and $g: Z \to Y$ be continuous functions. The *pullback* of f and g is the set $E = \{(x, z) \in X \times Z \mid f(x) = g(z)\}$. Note that the pullback is a subset of $X \times Z$. Restricting the projection maps to E yields maps f': $E \to Z$ and g': $E \to X$ such that $g \circ f' = f \circ g'$. It is easy to show that if $g: Z \to Y$ is a Hurewicz fibration, then g': $E \to X$ is also a Hurewicz fibration.

DEFINITION 2.7. Let $p: E \to B$ be a Hurewicz fibration and let B^I be the space of continuous functions from I to B with the compact-open topology. Let $\alpha: B^I \to B$ be the evaluation map $\alpha(\omega) = \omega(0)$. Let Δ be the pullback of $p: E \to B$ and $\alpha: B^I \to B$. The elements of Δ are ordered pairs $(e, \omega) \in E \times B^I$ such that $p(e) = \omega(0)$.

A *lifting function* for (E, p, B) is a continuous function $\lambda: \Delta \to E^I$ such that $\lambda(e, \omega)[0] = e$ and $p \circ \lambda(e, \omega)[t] = \omega(t)$. Thus, the path $\lambda(e, \omega)[t]$ $0 \leqq t \leqq 1$ is a lifting of ω which starts at e. λ is said to be *regular* if $\lambda(e, \omega)$ is a constant path whenever ω is a constant path. A theorem of Hurewicz (**[10]**, [**6**, p. 397]) asserts that if B is metric, then regular lifting functions exist.

We will also need Hanner's characterization of ANRs.

DEFINITION 2.8. Let X and Y be metric spaces and let $\epsilon > 0$ be given. A map $f: X \to Y$ is an ϵ*-domination* if there is a map $g: Y \to X$ such that the composition fg is ϵ-homotopic to the identity.

PROPOSITION 2.9 (*Hanner* [9]). *A compact metric space Y is an ANR if and only if for each $\epsilon > 0$ there is an ANR X which ϵ-dominates Y.*

3. The proof of Theorem 1. This section contains the heart of our argument. The following proposition is also a key step in the proof of Theorem 1(1) of [5]. Let $f: E \to B$ satisfy the hypotheses of Theorem 1.

PROPOSITION 3.1. *For each $b \in B$ there exist a neighborhood U of b in B, an ANR (actually, a Q-manifold) M, a fiber preserving imbedding $i: E \mid U \to M \times U$, and a fiber preserving retraction of $M \times U$ onto $i(E \mid U)$.*

This proposition implies Theorem 1 and the special case of Theorem 2 in which the fibers are compact because a fiber-preserving retract of a fibration is a fibration and a retract of an *ANR* is an *ANR*. The properties of being an *ANR* or a fibration are local properties ([9], [6, p. 400]), so a proof over neighborhoods of points suffices to prove the general theorem.

Proof of 3.1. Let Q be the Hilbert cube and let $b_0 \in B$ be given. E is a separable metric space and can be imbedded in Q. We identify E with this subset of Q. Since $f^{-1}(b_0)$ is an *ANR*, there is a compact *ANR* (Q-manifold, see [3, p. 105]) neighborhood M of $f^{-1}(b_0)$ in Q which retracts to $f^{-1}(b_0)$. Since f is proper, there is an open neighbourhood W of b_0 in B such that $f^{-1}(W) \subset M$.

Let $\mathscr{C}(M, f^{-1}(W))$ be the space of continuous functions from M to $f^{-1}(W)$ in the sup norm and define $H \subset \mathscr{C}(M, f^{-1}(W))$ to be $\{g \mid g$ retracts M onto some $f^{-1}(b)\}$. The space E is a complete metric space, so $\mathscr{C}(M, f^{-1}(W))$ is also complete. H is a closed subspace of $\mathscr{C}(M, f^{-1}(W))$, so H is also complete. Let $q: H \to B$ be the map $q(r) = b$, where r retracts M onto $f^{-1}(b)$. We will show that q satisfies the hypotheses of Michael's selection theorem.

Step I. The decomposition $\{q^{-1}(b)\}_{b \in W}$ of H is lower semicontinuous.

Proof. We must show that if $r: M \to f^{-1}(b)$ is a retraction and $\lim b_i = b$, then there are retractions $r_i: M \to f^{-1}(b_i)$ so that $\{r_i\}_{i=1}^{\infty}$ converges uniformly to r.

Let $r: M \to f^{-1}(b)$ and $\epsilon > 0$ be given. Choose

(1) $\delta_1 > 0$ so that if $d(b, b') < \delta_1$ then there are homotopy equival-

ences $g_{bb'}: f^{-1}(b) \to f^{-1}(b')$ and $g_{b'b}: f^{-1}(b') \to f^{-1}(b)$ which move points less than ϵ and such that $g_{bb'}g_{b'b}$ and $g_{b'b}g_{bb'}$ are ϵ-homotopic to $id \mid f^{-1}(b')$ and $id \mid f^{-1}(b)$, respectively.

(2) $\delta_2 > 0$ such that $d(b, b') < \delta_2$ implies that $r \mid f^{-1}(b')$ and $g_{b'b}$ are ϵ-homotopic in $f^{-1}(b)$. δ_2 exists because $f^{-1}(b)$ is a compact *ANR*.

(3) $\delta = \min(\delta_1, \delta_2)$.

Consider the map $g_{bb'} \circ r: M \to f^{-1}(b')$. If $d(b, b') < \delta$, $g_{bb'} \circ r \mid f^{-1}(b')$ is 3ϵ-homotopic to $g_{bb'} \circ g_{b'b}$. This is ϵ-homotopic to the identity. Thus, by the estimated homotopy extension theorem, $g_{bb'} \circ r$ is 4ϵ-homotopic to a retraction of M onto $f^{-1}(b')$. The distance from r to this retraction is no more than 5ϵ. This completes Step I.

Step II. The decomposition $\{q^{-1}(b)\}$ is equi-LC^m for all m.

Proof. Let $\epsilon > 0$ and $r: M \to f^{-1}(b)$ be given. We must find $\delta > 0$ such that if $k: S^m \to N_\delta(r) \cap q^{-1}(b')$ then k is homotopic to a constant $k(z_0)$ in $N_{3\epsilon}(r) \cap q^{-1}(b')$. Choose

(1) $\epsilon_1 > 0$ such that if $d(x, f^{-1}(b)) < \epsilon_1$, then $d(r(x), x) < \epsilon$. Note that $\epsilon_1 < \epsilon$.

(2) $\epsilon_2 > 0$ such that maps into M which are ϵ_2-close are canonically $\epsilon_1/2$-homotopic. If two maps agree on a set A, the homotopy is stationary on A.

(3) $\delta_1 > 0$ such that if $N_\delta(r) \cap q^{-1}(b') \neq \emptyset$, then $d(x, f^{-1}(b)) < \epsilon_1/2$ for all $x \in f^{-1}(b')$.

(4) $\delta = \min(\delta_1, \epsilon_1/2, \epsilon_2/2)$.

k maps S^m into a δ-neighborhood of the fixed retraction r. By (4), the retractions $k(z)$, $z \in S^m$, are ϵ_2-close. By (2), there is an $\epsilon_1/2$-homotopy K_t from $k(z)$ to the constant map $K_0(z) = k(z_0)$. K_t is a homotopy through maps into M (not $f^{-1}(b')$) but $K_t(z) \mid f^{-1}(b')$ is the identity for each z and t.

Applying the retraction $k(z_0)$ to K_t, we get a homotopy $k(z_0)K_t$ from $k(z)$ to $k(z_0)$ in $q^{-1}(b')$. We will show that $d(k(z_0)K_t, r) < 3\epsilon$ for all z and t.

$$d(k(z_0)K_t(z), r) \leqq d(k(z_0)K_t(z), r \circ K_t(z)) + d(r \circ K_t(z), K_t(z)) + d(K_t(z), r).$$

We estimate each of these summands.

(i) $d(k(z_0)K_t(z), r \circ K_t(z)) < d(k(z_0), r) < \delta < \epsilon/2$.

(ii) $d(r \circ K_t(z), K_t(z)) < \epsilon$. This will follow from (1) once it is established that $d(K_t(z)(x), f^{-1}(b)) < \epsilon_1$ for all x. Since K_t is an $\epsilon_1/2$-homotopy, $d(K_t(z)(x), f^{-1}(b')) < \epsilon_1/2$ for all x. The result now follows from (3).

(iii) $d(K_t(z), r) < d(K_t(z), k(z_0)) + d(k(z_0), r) < \epsilon_1/2 + \delta < \epsilon_1 < \epsilon$. This completes the proof of Step II.

By Michael's theorem, there is an open neighborhood U of b_0 in W and a map $s: U \to H$ so that $q \circ s = id$. If B is an ANR, we can choose U to be an ANR. Imbed $f^{-1}(U)$ into $M \times U$ by the map $j(e) = (e, f(e))$.

Define $R: M \times U \to M \times U$ by the formula $R(m, b) = (s(b)(m), b)$. R is a fiber preserving retraction of $M \times U$ onto $j(f^{-1}(U))$. This completes our proof of Theorem 1.

4. The proof of Theorem 2.

We proceed with the proof of Theorem 2. Recall that Proposition 3.1 implies a special case of Theorem 2, namely the case in which B is locally finite dimensional and the fibers are compact. Our plan is to use Proposition 2.9 (Hanner's theorem) to prove the case where B is infinite dimensional and the fibers are compact. We will then use a cone construction to prove Theorem 2 in case the fibers are merely locally compact.

Case I. $f: E \to B$ is a proper map and B is any locally compact ANR.

Proof. Let $i: B \to Q \times [0,1)$ be a proper imbedding and let $r: U \to B$ be a retraction of a neighborhood of B onto B. Let r^*E be the pullback of E over O. Note that the natural map $f': r^*E \to U$ is a Hurewicz fibration with compact ANR fibers and that $(f')^{-1}(B)$ may be identified with E.

The natural map from r^*E to E arising from the pullback construction is a retraction covering r. Therefore, to show that E is an ANR, it suffices to show that r^*E is an ANR.

Every point $u \in U$ has a neighborhood homeomorphic to Q. Thus, to prove that r^*E is an ANR, it suffices to establish the following claim.

Claim. If $f: X \to Q$ is a Hurewicz fibration with compact ANR fibers, then X is an ANR.

Proof of Claim. Let λ be a regular lifting function for f. Note that if $\epsilon > 0$ is given then there exists a $\delta > 0$ such that the lift of a path of diameter $< \delta$ has diameter $< \epsilon$.

For each n, let $I_n \subset Q$ be $\Pi_{i=1}^n I_i \times 0$ and let $X_n = f^{-1}(I_n)$. Note that by Theorem 1, X_n is an ANR. Let $\epsilon > 0$ be given. We show that X_n ϵ-dominates X for large n. Choose $\delta > 0$ as above. Let $p_n: Q \to I_n$ be the projection map and choose n so large that $d(p_n, id) < \delta$. Let

$\alpha_s\colon Q \to Q^I$ be defined by

$$\alpha_s(q)[t] = (1-t)q + t[(1-s)p_n(q) + s \cdot q] \qquad 0 \leqq t \leqq 1.$$

This is a "straight line" path from q to $p_n(q)$ together with its natural homotopy back to a constant map.

Define $(\tilde{p}_n)_s\colon X \to X$ by $(\tilde{p}_n)_s(x) = \lambda(x, \alpha_s(f(x)))$ [1]. The map $(\tilde{p}_n)_0$ is a retraction from X to X_n coveriing p_n. $(\tilde{p}_n)_1$ is the identity map on X. Each of the paths $\alpha_s(q)$ has diameter $< \delta$, so the homotopy $(\tilde{p}_n)s$, $0 \leqq s \leqq 1$ is a 2ϵ-homotopy. Thus, $X \xrightarrow[(\tilde{p}_n)_0]{} X_n \hookrightarrow X$ is a 2ϵ-domination. We invoke Hanner's theorem and the proof of Case I is complete.

Case II. The fibers of f are locally compact *ANR*s and B is a locally compact *ANR*.

Our strategy in this case is to define a new fibration $C(E) \to B$ whose fibers are the one-point compactifications of $\{f^{-1}(b) \times [0, 1)\}_{b \in B}$. These fibers are compact *ANR*s and Case I applies. Since E is a retract of an open subset of $C(E)$, we see that E is also an *ANR*.

We proceed with the proof of Case II, which requires some definitions and preliminary lemmas.

DEFINITION 4.1. The map $p\colon E \to B$ is a *proper fibration* if for each commutative diagram of continuous maps

$$\begin{array}{ccc} X \times 0 & \xrightarrow{g} & E \\ \big\downarrow & & \big\downarrow p \\ X \times I & \xrightarrow{f} & B \end{array}$$

with g proper and X locally compact there is a continuous proper map $G\colon X \times I \to E$ such that G extends g and $p \circ G = f$.

DEFINITION 4.2 (Seidman [14] and Kim [12]). Let $f\colon E \to B$ be a surjective map. Let B^* be a copy of B and let $\tilde{E}$ be the space $E \cup B^*$ with the topology generated by open subsets of E and sets of the form $U^* \cup (f^{-1}(U) \cap E - K)$ where U is an open subset of B, U^* is the corresponding open subset of B^*, and $K \subset E$ is compact. $\tilde{E}$ is the fiberwise one-point compactification of E. Let $\tilde{f}\colon \tilde{E} \to B$ be the natural map.

With $f\colon E\to B$ as above, we define $C(f)\colon C(E)\to B$, the cone, by taking $C(E)=E\times[0,1)$ and $C(f)=f\circ\text{proj}$ where $f\circ\text{proj}$ is the composition $E\times[0,1)\xrightarrow{\text{proj}}E\xrightarrow{f}B$.

PROPOSITION 4.3. *If* $p\colon E\to B$ *is a proper fibration then* $\tilde{p}\colon\tilde{E}\to B$ *has the lifting property for compacta.*

Proof. Consider a commutative diagram of continuous maps

$$\begin{array}{ccc} X\times 0 & \xrightarrow{g} & \tilde{E} \\ \big\downarrow & & \big\downarrow \tilde{p} \\ X\times I & \xrightarrow{f} & B \end{array}$$

with X compact. Let $X'=g^{-1}(E)$. X' is locally compact. Let $f'=f|X'\times I$, $g'=g|X'\times 0$. g' is proper, so there is a proper map $G'\colon X'\times I\to E$ so that $p\circ G'=f'$.

Let $G\colon X\times I\to\tilde{E}$ be defined by

$$G(x,t)=\begin{cases} G'(x,t) & x\in X' \\ (f(x,t))^{*} & x\in X-X'. \end{cases}$$

This map is continuous and solves the original lifting problem.

PROPOSITION 4.4. *If* $p\colon E\to B$ *is a Hurewicz fibration, then* $p\circ\text{proj}\colon E\times[0,\infty)\to B$ *is a proper fibration.*

Proof. Let

$$\begin{array}{ccc} X\times 0 & \xrightarrow{g} & E\times[0,\infty) \\ \big\downarrow & & \big\downarrow p\circ\text{proj} \\ X\times I & \xrightarrow{f} & B \end{array}$$

be a proper lifting problem and let $\rho\colon X\to[0,\infty)$ be a proper map. Let $G\colon X\times I\to E\times[0,\infty)$ be a solution to the ordinary lifting problem. G is proper on a neighborhood N of $X\times\{0\}$. Let $\sigma\colon X\times[0,1]\to[0,1]$ be a continuous function which is 0 on $X \times \{0\}$ and 1 on $X \times I - N$. Define $\bar{G}: X \times [0,1] \to E \times [0,\infty)$ by

$$\bar{G}(x,t) = (\mathrm{proj}_E(g(x,t)), \mathrm{proj}_{[0,\infty)}(g(x,t)) + \sigma(x,t)\rho(x)) \in E \times [0,\infty).$$

$\bar{G}(x,t)$ is the desired proper lifting.

PROPOSITION 4.5. *If $p: E \to B$ has the lifting property for compact spaces, E is compact, and B is an ANR, then $p: E \to B$ is a Hurewicz fibration.*

Proof. Since the pullback of a map with a lifting property has that same lifting property, it suffices, as in the proof of Case I of Theorem 2, to prove this proposition in case $B = Q$.

For each point $(q_1, q_2) \in Q \times Q$, let $\alpha(q_1, q_2) \in Q^I$ be the path

$$\alpha(q_1,q_2)[t] = \begin{cases} (1 - t/d(q_1,q_2))q_1 + (t/d(q_1,q_2))q_2 & t < d(q_1,q_2) \\ q_2 & d(q_1,q_2) \leqq t \leqq 1. \end{cases}$$

Note that $\alpha: Q \times Q \to Q^I$ is continuous. Consider the lifting problem

$$\begin{array}{ccc} E \times Q \times \{0\} & \xrightarrow{p_E} & E \\ \downarrow & & \downarrow p \\ E \times Q \times I & \xrightarrow{\gamma} & B \end{array}$$

where $\gamma(e,q,t) = \alpha(p(e),q)[t]$. Let $\bar{\lambda}: E \times Q \times I \to E$ be a solution to this lifting problem. Define $\lambda: E \times Q \times I \to E$ by $\lambda(e,q,t) = \bar{\lambda}(e,q,d(p(e),q) \cdot t)$. We have

(i) $\lambda(e,q,0) = p(e)$

(ii) $p \circ \lambda(e,q,t) = \alpha(p(e),q)[t]$

(iii) $\lambda(e,q,t) = e$ if $p(e) = q$, $0 \leqq t \leqq 1$.

Thus, if X is any space and

$$\begin{array}{ccc} X \times 0 & \xrightarrow{f} & E \\ \downarrow & & \downarrow p \\ X \times I & \xrightarrow{g} & Q \end{array}$$

is a lifting problem, $G(x,t) = \lambda(f(x,0), g(x,0), g(x,t))[1]$ is a solution. Hence, $p: E \to Q$ is a Hurewicz fibration and the proof of Proposition 4.5 is complete.

REMARK. The above argument incorporates a well known proof of the existence of regular lifting functions. See [**6**, p. 397].

PROPOSITION 4.6. *If E is a locally compact separable metric ANR, then the one-point compactification of $E \times [0, 1)$ is a compact AR.*

Proof. A theorem of Hyman [**11**] states that if $X - *$ is an ANR and X is strongly locally contractible at $*$, then X is an ANR.

We now complete the proof of Theorem 2. If $p: E \to B$ is a Hurewicz fibration satisfying the hypotheses, then $p \circ \text{proj}: E \times [0, 1) \to B$ is a proper fibration and $C(p)$: $C(E) \to B$ is a Hurewicz fibration with compact AR fibers. By Case I, $C(E)$ is an ANR. Since $E \times [0, 1)$ is an open subset of $C(E)$, it is an ANR and, since E is a retract of $E \times [0, 1)$, it is also an ANR.

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